

JOINT APPROXIMATION BY DIRICHLET L -FUNCTIONS

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ABSTRACT. In the paper, collections of analytic functions are simultaneously approximated by collections of shifts of Dirichlet L -functions $(L(s + i\gamma_1(\tau), \chi_1), \dots, L(s + i\gamma_r(\tau), \chi_r))$, with arbitrary Dirichlet characters χ_1, \dots, χ_r . The differentiable functions $\gamma_1(\tau), \dots, \gamma_r(\tau)$ and their derivatives satisfy certain growth conditions. The obtained results extend those of [PAŃKOWSKI, L.: *Joint universality for dependent L -functions*, Ramanujan J. **45** (2018), 181–195].

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1. Introduction

Let χ be a Dirichlet character modulo $q \in \mathbb{N}$, i.e., χ is a function on \mathbb{Z} , $\chi(m) \neq 0$, $\chi(m) = 0$ for $(m, q) > 1$, $\chi(m_1 m_2) = \chi(m_1) \chi(m_2)$ for $m_1, m_2 \in \mathbb{Z}$ and $\chi(m + q) = \chi(m)$ for all $m \in \mathbb{Z}$. The properties of Dirichlet characters can be found, for example, in [9]. The Dirichlet L -function $L(s, \chi)$, $s = \sigma + it$, is defined, for $\sigma > 1$, by

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where the product is taken over all prime numbers. If $\chi(m)$ is the principal character ($\chi(m) = 1$ if $(m, q) = 1$), then

$$L(s, \chi) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function,

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1.$$

Therefore, in this case, the function $L(s, \chi)$ has analytic continuation to the whole complex plane, except for the point $s = 1$ which is a simple pole with residue $\prod_{p|q} (1 - 1/p)$. If $\chi(m)$ is a non-principal character, then $L(s, \chi)$ has analytic continuation to an entire function.

Voronin discovered the universality property of the functions $L(s, \chi)$ concerning the approximation of analytic functions defined in the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. He proved [21] that if

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$f(s)$ is a continuous non-vanishing function on the disc $|s| \leq r$, $0 < r < 1/4$, and analytic in the interior of this disc, then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s| \leq r} |L(s + 3/4 + i\tau, \chi) - f(s)| < \varepsilon.$$

Moreover, Voronin in [22] considered the functional independence of Dirichlet L -functions with pairwise non-equivalent Dirichlet characters χ_1, \dots, χ_r , and, for this, he in fact obtained in a non-explicit form a joint universality theorem for $L(s, \chi_1), \dots, L(s, \chi_r)$. Voronin's investigations were continued by Gonek [7], Bagchi [1, 2] and the first author [11]. For the modern version of a joint universality theorem, we need some notation. Denote by \mathcal{K} the class of compact subsets of the strip D with connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ the class of continuous non-vanishing functions on K that are analytic in the interior of K . Let $\text{meas} A$ be the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the following statement is true, see, for example, [19].

THEOREM 1.1. *Let χ_1, \dots, χ_r be pairwise non-equivalent Dirichlet characters. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$, and let $f_j(s) \in H_0(K_j)$. Then for every $\varepsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + i\tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

We recall that if $\chi(m)$ for $(m, q) = 1$ has a period less than q , then the character χ is non-primitive. In opposite case, χ is primitive. Every non-primitive character χ is induced by a primitive character, i.e., there exists a primitive character χ_1 modulo q_1 , $q_1 \mid q$, such that

$$\chi(m) = \begin{cases} \chi_1(m) & \text{if } (m, q_1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Two Dirichlet characters are called non-equivalent if they are not induced by the same primitive character.

It is possible to consider the approximation of the collection $(f_1(s), \dots, f_r(s))$ by more general shifts $(L(s + i\gamma_1(\tau), \chi_1), \dots, L(s + i\gamma_r(\tau), \chi_r))$. Let $K_1 = \dots = K_r = K$. Then it follows from [8] that, under hypotheses of Theorem 1.1, for every $\varepsilon > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K} |L(s + i\gamma_j(\tau), \chi_j) - f_j(s)| < \varepsilon \right\} > 0,$$

where $\gamma_j(\tau) = \tau + \lambda_j$, with K satisfying $\hat{K}_k \cap \hat{K}_l = \emptyset$, $k \neq l$, where $\hat{K}_j = \{s + i\lambda_j : s \in K\}$, $j = 1, \dots, r$. Nakamura [14] obtained the inequality

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K} |L(s + i\gamma_j(\tau), \chi) - f_j(s)| < \varepsilon \right\} > 0,$$

where $\gamma_j(\tau) = a_j \tau$ with algebraic numbers $a_1, \dots, a_r \in \mathbb{R}$ linearly independent over the field of rational numbers \mathbb{Q} . The case $r = 2$ was studied in [15]–[17] with $a_1, a_2 \in \mathbb{R} \setminus \{0\}$, $a_1 \neq \pm a_2$.

The most general result belongs to Pańkowski [18]. He proved the following theorem.

THEOREM 1.2. *Suppose that χ_1, \dots, χ_r are Dirichlet characters, $\alpha_1, \dots, \alpha_r \in \mathbb{R}$, $a_1, \dots, a_r \in \mathbb{R}^+$, and b_1, \dots, b_r are such that*

$$b_j \in \begin{cases} \mathbb{R} & \text{if } a_j \notin \mathbb{N}, \\ (-\infty, 0] \cup (1 + \infty) & \text{if } a_j \in \mathbb{N}, \end{cases}$$

and $a_j \neq a_k$ or $b_j \neq b_k$ if $k \neq j$. Moreover, let $K \in \mathcal{K}$, $f_1, \dots, f_r \in H_0(K)$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [2, T] : \sup_{1 \leq j \leq r} \sup_{s \in K} |L(s + i\alpha_j \tau^{a_j} \log^{b_j} \tau, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

It is very important to stress that, in Theorem 1.2, χ_1, \dots, χ_r are arbitrary, not necessarily pairwise non-equivalent, Dirichlet characters. The proof is based on the uniform distribution modulo 1.

Our aim is to obtain the joint universality for Dirichlet L -functions with other functions $\gamma_j(\tau)$ without using the uniform distribution theory. Moreover, we approximate in different sets $K_1, \dots, K_r \in \mathcal{K}$.

Suppose that, for $j = 1, \dots, r$, $\gamma_j(\tau)$ is an increasing to infinity real continuously differentiable functions on $[T_0, \infty)$, $T_0 > 0$, with derivative

$$(i) \quad \gamma_j'(\tau) = \hat{\gamma}_j(\tau)(1 + o(1)),$$

where $\hat{\gamma}_j(\tau)$ is monotonic such that

$$(ii) \quad \hat{\gamma}_1(\tau) = o(\hat{\gamma}_2(\tau)), \dots, \hat{\gamma}_{r-1}(\tau) = o(\hat{\gamma}_r(\tau))$$

and

$$(iii) \quad \gamma_j(2\tau) \max_{\tau \leq u \leq 2\tau} \frac{1}{\gamma_j'(u)} \ll \tau$$

as $\tau \rightarrow \infty$.

Denote the class of r -tuples $(\gamma_1, \dots, \gamma_r)$ satisfying the above hypotheses by U_r . Then the following joint universality theorem for Dirichlet L -functions is valid.

THEOREM 1.3. *Suppose that χ_1, \dots, χ_r are arbitrary Dirichlet characters, and $(\gamma_1, \dots, \gamma_r) \in U_r$. Let, for $j = 1, \dots, r$, $K_j \in \mathcal{K}$ and $f_j \in H_0(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T - T_0} \text{meas} \left\{ \tau \in [T_0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + i\gamma_j(\tau), \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T \rightarrow \infty} \frac{1}{T - T_0} \text{meas} \left\{ \tau \in [T_0, T] : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |L(s + i\gamma_j(\tau), \chi_j) - f_j(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For example, the system of polynomials $\gamma_1(\tau) = \tau + 1$, $\gamma_2(\tau) = \tau^2 + \tau + 1, \dots, \gamma_r(\tau) = \tau^r + \tau^{r-1} + \dots + 1$ is a member of the class U_r . Also $(\tau \log \tau, \dots, \tau^r \log \tau) \in U_r$ and $(\tau(\Gamma'(\tau)/\Gamma(\tau)), \dots, \tau^r(\Gamma'(\tau)/\Gamma(\tau))) \in U_r$, where $\Gamma(\cdot)$ is the Euler gamma-function. We note that $(\tau \log \tau, \dots, \tau^r \log \tau)$ does not satisfy hypotheses of Theorem 1.2.

Denote by $H(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacta. Theorem 1.3 can be generalized for some compositions. We will give only one theoretical example. Denote by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K . Thus, $H_0(K) \subset H(K)$. Let $S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

THEOREM 1.4. *Suppose that $(\gamma_1, \dots, \gamma_r) \in U_r$ and $F: H^r(D) \rightarrow H(D)$ is a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S^r$ is non-empty. Let $K \in \mathcal{K}$ and $f \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T - T_0} \text{meas} \left\{ \tau \in [T_0, T] : \sup_{s \in K} |F(L(s + i\gamma_1(\tau), \chi_1), \dots, L(s + i\gamma_r(\tau), \chi_r)) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T \rightarrow \infty} \frac{1}{T - T_0} \text{meas} \left\{ \tau \in [T_0, T] : \sup_{s \in K} |F(L(s + i\gamma_1(\tau), \chi_1), \dots, L(s + i\gamma_r(\tau), \chi_r)) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

It is known that the sets of values taking by zeta or L -functions are in some sense dense. First, Bohr [4] obtained that the function $\zeta(s)$ takes every non-zero value infinitely many times in the strip $\{s \in \mathbb{C} : 1 < \sigma < 1 + \delta\}$ for any $\delta > 0$. Bohr and Courant [5] obtained that, for any fixed σ , $1/2 < \sigma < 1$, the set

$$\{\zeta(\sigma + it) : t \in \mathbb{R}\}$$

is dense in \mathbb{C} . Voronin extended and generalized the above results. He proved [20] that the set

$$\{(\zeta(s_1 + i\tau), \dots, \zeta(s_n + i\tau)) : \tau \in \mathbb{R}\}$$

with any fixed different s_1, \dots, s_n , $1/2 < \text{Res}_k < 1$, $1 \leq k \leq n$, and the set

$$\left\{ \left(\zeta(\sigma + it), \zeta'(\sigma + it), \dots, \zeta^{(n-1)}(\sigma + it) \right) : t \in \mathbb{R} \right\}$$

with every fixed σ , $1/2 < \sigma < 1$, are dense in \mathbb{C}^n . Moreover, Voronin obtained a joint generalization of the later result for Dirichlet L -functions. Namely, he proved [22] that if χ_1, \dots, χ_r are pairwise non-equivalent Dirichlet characters, then the set

$$\left\{ \left(L(\sigma + it, \chi_1), L'(\sigma + it, \chi_1), \dots, L^{(n-1)}(\sigma + it, \chi_1), \dots, L(\sigma + it, \chi_r), L'(\sigma + it, \chi_r), \dots, L^{(n-1)}(\sigma + it, \chi_r) \right) : t \in \mathbb{R} \right\}$$

is everywhere dense in $\mathbb{C}^{r \times n}$ for every fixed σ , $1/2 < \sigma < 1$.

Theorem 1.3 has the following corollary.

COROLLARY 1.4.1. *Suppose that χ_1, \dots, χ_r are arbitrary Dirichlet characters, and $(\gamma_1, \dots, \gamma_r) \in U_r$. Then, for every fixed σ , $1/2 < \sigma < 1$, the set*

$$\left\{ \left(L(\sigma + i\gamma_1(t), \chi_1), L'(\sigma + i\gamma_1(t), \chi_1), \dots, L^{(n-1)}(\sigma + i\gamma_1(t), \chi_1), \dots, L(\sigma + i\gamma_r(t), \chi_r), L'(\sigma + i\gamma_r(t), \chi_r), \dots, L^{(n-1)}(\sigma + i\gamma_r(t), \chi_r) \right) : t \geq T_0 \right\}$$

is everywhere dense in $\mathbb{C}^{r \times n}$.

The proof of the corollary uses Theorem 1.3 and repeats Voronin's arguments.

Corollary 1.4.1 implies the following functional independence property of Dirichlet L -functions.

COROLLARY 1.4.2. *Suppose that χ_1, \dots, χ_r are arbitrary Dirichlet characters, $\Phi : \mathbb{C}^{r \times n} \rightarrow \mathbb{C}$ is a continuous function, and*

$$\Phi \left(L(s, \chi_1), L'(s, \chi_1), \dots, L^{(n-1)}(s, \chi_1), \dots, L(s, \chi_r), L'(s, \chi_r), \dots, L^{(n-1)}(s, \chi_r) \right) = 0$$

identically for s . Then $\Phi \equiv 0$.

For the proof of universality theorems, we apply a method different from that of [18]. This method is probabilistic, and is based on weak convergence of probability measures in the space of analytic functions, see [1, 10, 12] and [19].

2. Lemmas

Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ -field of the space \mathbb{X} , and by γ the unit circle on the complex plane. Define the set

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where \mathbb{P} is the set of all prime numbers, and $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. With product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Let

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$. Then, again, Ω^r is a compact topological Abelian group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H^r exists. This gives the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H^r)$. For $j = 1, \dots, r$, denote by $\omega_j(p)$ the p th component of an element $\omega_j \in \Omega_j$, $p \in \mathbb{P}$. Let $\omega = (\omega_1, \dots, \omega_r)$, $\omega_j \in \Omega_j$, be the elements of Ω^r .

For $A \in \mathcal{B}(\Omega^r)$, define

$$Q_T^r(A) = \frac{1}{T - T_0} \text{meas} \left\{ \tau \in [T_0, T] : \left((p^{-i\gamma_1(\tau)} : p \in \mathbb{P}), \dots, (p^{-i\gamma_r(\tau)} : p \in \mathbb{P}) \right) \in A \right\}.$$

LEMMA 2.1 (Main lemma). *Suppose that $(\gamma_1, \dots, \gamma_r) \in U_r$. Then Q_T^r converges weakly to the Haar measure m_H^r as $T \rightarrow \infty$.*

Proof. Let $g_{Q_T^r}(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \dots, r$, be the Fourier transform of Q_T^r , i.e.,

$$g_{Q_T^r}(\underline{k}_1, \dots, \underline{k}_r) = \int_{\Omega^r} \left(\prod_{j=1}^r \prod_{p \in \mathbb{P}}' \omega_j^{k_{jp}}(p) \right) dQ_T^r,$$

where the sign “ $'$ ” means that only a finite number of integers k_{jp} , $j = 1, \dots, r$, are distinct from zero. Thus, by the definition of Q_T^r ,

$$g_{Q_T^r}(\underline{k}_1, \dots, \underline{k}_r) = \frac{1}{T - T_0} \int_{T_0}^T \exp \left\{ -i \sum_{j=1}^r \gamma_j(\tau) \sum_{p \in \mathbb{P}}' k_{jp} \log p \right\} d\tau. \quad (2.1)$$

Let, for brevity,

$$a_j = \sum_{p \in \mathbb{P}}' k_{jp} \log p, \quad j = 1, \dots, r.$$

Since the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers \mathbb{Q} , $a_j = 0$ if and only if $\underline{k}_j = \underline{0}$, $j = 1, \dots, r$. Clearly, in view of (2.1),

$$g_{Q_T^r}(\underline{0}, \dots, \underline{0}) = 1. \quad (2.2)$$

Now, suppose that $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$. Since $(\gamma_1, \dots, \gamma_r) \in U_r$, we have

$$\left(\sum_{j=1}^r a_j \gamma_j(\tau) \right)' = \sum_{j=1}^r a_j \gamma_j'(\tau) = \sum_{j=1}^r a_j \hat{\gamma}_j(\tau) (1 + o(1)) = a_{j_0} \hat{\gamma}_{j_0}(\tau) (1 + o(1))$$

as $\tau \rightarrow \infty$, where $j_0 = \max(j : a_j \neq 0)$. Hence,

$$\left(\sum_{j=1}^r a_j \gamma_j'(\tau) \right)^{-1} = \frac{1}{a_{j_0} \hat{\gamma}_{j_0}(\tau) (1 + o(1))} = \frac{1}{a_{j_0} \hat{\gamma}_{j_0}(\tau)} (1 + o(1)) \quad (2.3)$$

as $\tau \rightarrow \infty$. Moreover, since $\gamma_j(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$, we have, in view of (iii) of the class U_r , that

$$\frac{1}{\hat{\gamma}_j(\tau)} = o(\tau) \quad (2.4)$$

as $\tau \rightarrow \infty$, $j = 1, \dots, r$. Let $A(\tau) = \sum_{j=1}^r a_j \gamma_j(\tau)$. Then (2.3), (2.4), the monotonicity of $\hat{\gamma}_j(\tau)$, and the second mean value theorem show that

$$\begin{aligned} \int_{T_0}^T \cos A(\tau) d\tau &= \int_{\log T}^T \cos A(\tau) d\tau + O(\log T) = \int_{\log T}^T \frac{1}{A'(\tau)} \cos A(\tau) dA(\tau) + O(\log T) \\ &= \int_{\log T}^T \frac{1}{a_{j_0} \hat{\gamma}_{j_0}(\tau)} \cos A(\tau) dA(\tau) + \int_{\log T}^T \frac{o(1)}{a_{j_0} \hat{\gamma}_{j_0}(\tau)} \cos A(\tau) dA(\tau) + O(\log T) \quad (2.5) \\ &= \int_{\log T}^T \frac{1}{a_{j_0} \hat{\gamma}_{j_0}(\tau)} d(\sin A(\tau)) + \int_{\log T}^T \frac{o(1)(1+o(1))}{A'(\tau)} \cos A(\tau) dA(\tau) + O(\log T) \\ &= o(T) + \int_{\log T}^T o(1) \cos A(\tau) d\tau + O(\log T) = o(T) \end{aligned}$$

as $T \rightarrow \infty$. Similarly, we find that

$$\int_{T_0}^T \sin A(\tau) d\tau = o(T)$$

as $T \rightarrow \infty$. This, (2.5) and (2.1) show that, in the case $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$,

$$g_{Q_T^r}(\underline{k}_1, \dots, \underline{k}_r) = o(1), \quad T \rightarrow \infty.$$

Thus, in view of (2.2),

$$\lim_{T \rightarrow \infty} g_{Q_T^r}(\underline{k}_1, \dots, \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0}). \end{cases}$$

Since the right-hand side of the latter equality is the Fourier transform of the Haar measure m_H^r , the lemma follows by a continuity theorem for probability measures on compact groups. \square

Lemma 2.1, by a standard way, implies a joint limit theorem in the space $H^r(D)$ for absolutely convergent Dirichlet series. Let $\sigma_0 > 1/2$ be a fixed number, $\underline{\chi} = (\chi_1, \dots, \chi_r)$, for $m, n \in \mathbb{N}$,

$$v_n(m) = \exp \left\{ - \left(\frac{m}{n} \right)^{\sigma_0} \right\},$$

and

$$\underline{L}_n(s, \underline{\chi}) = (L_n(s, \chi_1), \dots, L_n(s, \chi_r)),$$

where

$$L_n(s, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) v_n(m)}{m^s}, \quad j = 1, \dots, r,$$

and

$$\underline{L}_n(s, \omega, \underline{\chi}) = (L_n(s, \omega_1, \chi_1), \dots, L_n(s, \omega_r, \chi_r)),$$

where

$$L_n(s, \omega_j, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) \omega_j(m) v_n(m)}{m^s}, \quad j = 1, \dots, r,$$

and, for $m \in \mathbb{N}$,

$$\omega_j(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_j^l(p), \quad j = 1, \dots, r.$$

Then the series for $L_n(s, \chi_j)$ and $L(s, \omega_j, \chi_j)$ are absolutely convergent for $\sigma > 1/2$, $j = 1, \dots, r$. Define the function $u_n : \Omega^r \rightarrow H^r(D)$ by the formula

$$u_n(\omega) = \underline{L}_n(s, \omega, \underline{\chi}), \quad \omega \in \Omega^r.$$

The absolute convergence of the series for $L_n(s, \omega_j, \chi_j)$, $j = 1, \dots, r$, implies the continuity of the function u_n . Let $V_n = m_H^r u_n^{-1}$, where

$$V_n(A) = m_H^r u_n^{-1}(A) = m_H^r(u_n^{-1}A), \quad A \in \mathcal{B}(H^r(D)).$$

For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,n}(A) = \frac{1}{T - T_0} \text{meas} \{ \tau \in [T_0, T] : \underline{L}_n(s + i\gamma(\tau), \underline{\chi}) \in A \},$$

where $\underline{\gamma}(\tau) = (\gamma_1(\tau), \dots, \gamma_r(\tau))$ and

$$\underline{L}_n(s + i\gamma(\tau), \underline{\chi}) = (L_n(s + i\gamma_1(\tau), \chi_1), \dots, L_n(s + i\gamma_r(\tau), \chi_r)).$$

Then Lemma 2.1, the continuity of u_n and [3: Theorem 5.1] lead to the following statement.

LEMMA 2.2. *Suppose that $(\gamma_1, \dots, \gamma_r) \in U_r$. Then $P_{T,n}$ converges weakly to the measure V_n as $T \rightarrow \infty$.*

The family of probability measures $\{V_n : n \in \mathbb{N}\}$ is very important for the investigation of the collection

$$\underline{L}(s + i\gamma(\tau), \underline{\chi}) = (L(s + i\gamma_1(\tau), \chi_1), \dots, L(s + i\gamma_r(\tau), \chi_r)).$$

We recall that the family of probability measures $\{P\}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is tight if, for every $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset \mathbb{X}$ such that

$$P(K) > 1 - \varepsilon$$

for all $P \in \{P\}$.

LEMMA 2.3. *The family $\{V_n : n \in \mathbb{N}\}$ is tight.*

Proof. For $j = 1, \dots, r$, let $m_{H,j}$ be the probability Haar measure on $(\Omega_j, \mathcal{B}(\Omega_j))$, and $u_{n,j} : \Omega_j \rightarrow H(D)$ be given by the formula

$$u_{n,j}(\omega_j) = L_n(s, \omega_j, \chi_j).$$

Then $V_{n,j} = m_{H,j} u_{n,j}^{-1}$, $j = 1, \dots, r$, are the marginal measures of V_n . Actually, for $A \in \mathcal{B}(H(D))$,

$$\begin{aligned} & V_n \left(\underbrace{H(D) \times \dots \times H(D)}_{j-1} \times A \times H(D) \times \dots \times H(D) \right) \\ &= m_H^r u_n^{-1} \left(\underbrace{H(D) \times \dots \times H(D)}_{j-1} \times A \times H(D) \times \dots \times H(D) \right) \\ &= m_H^r \left(u_n^{-1} \left(\underbrace{H(D) \times \dots \times H(D)}_{j-1} \times A \times H(D) \times \dots \times H(D) \right) \right) \\ &= m_H^r (u_{n,j}^{-1} A) = m_{H,j} u_{n,j}^{-1} (A). \end{aligned}$$

It is easy to see using the absolute convergence of the series for $L_n(s, \chi_j)$, see, for example, the proof of Lemma 4.11 from [19] for more general functions from the Selberg class, that the families $\{V_{n,j} : n \in \mathbb{N}\}$, $j = 1, \dots, r$, are tight. Therefore, for every $\varepsilon > 0$, there exists a compact set $K_j = K_j(\varepsilon) \subset H(D)$ such that

$$V_{n,j}(K_j) > 1 - \frac{\varepsilon}{r}, \quad j = 1, \dots, r \quad (2.6)$$

for all $n \in \mathbb{N}$. The set $K = K_1 \times \dots \times K_r$ is compact in the space $H^r(D)$, and, in view of (2.6),

$$\begin{aligned} V_n(H^r(D) \setminus K) &= V_n \left(\bigcup_{j=1}^r \left(\underbrace{H(D) \times \dots \times H(D)}_{j-1} \times (H(D) \setminus K_j) \times H(D) \times \dots \times H(D) \right) \right) \\ &\leq \sum_{j=1}^r V_{n,j}(H(D) \setminus K_j) \leq \varepsilon \end{aligned}$$

for all $n \in \mathbb{N}$. The lemma is proved. \square

3. Mean square estimates

Mean square estimates play an important role in the universality theory of zeta- and L -functions. In this section, we present estimates for generalized mean squares of Dirichlet L -functions.

LEMMA 3.1. *Suppose that $(\gamma_1, \dots, \gamma_r) \in U_r$. Then, for fixed σ , $1/2 < \sigma < 1$, and $t \in \mathbb{R}$,*

$$\int_{T_0}^T |L(\sigma + it + i\gamma_j(\tau), \chi_j)|^2 d\tau \ll_{\sigma} T(1 + |t|), \quad j = 1, \dots, r.$$

Proof. It is well known that, for fixed σ , $1/2 < \sigma < 1$,

$$\int_{T_0}^T |L(\sigma + it, \chi_j)|^2 dt \ll_{\sigma} T.$$

Therefore, for all $t \in \mathbb{R}$,

$$\int_0^{|t| + \gamma_j(\tau)} |L(\sigma + iu, \chi_j)|^2 du \ll_{\sigma} (|t| + \gamma_j(\tau)).$$

Thus, for $X \geq T_0$,

$$\begin{aligned} \int_X^{2X} |L(\sigma + it + i\gamma_j(\tau), \chi_j)|^2 d\tau &= \int_X^{2X} \frac{1}{\gamma_j'(\tau)} |L(\sigma + it + i\gamma_j(\tau), \chi_j)|^2 d\gamma_j(\tau) \\ &\ll \max_{X \leq \tau \leq 2X} \frac{1}{\gamma_j'(\tau)} \int_X^{2X} d\left(\int_0^{t+\gamma_j(\tau)} |L(\sigma + iu, \chi_j)|^2 du \right) \\ &\ll_\sigma (|t| + \gamma_j(2X)) \max_{X \leq \tau \leq 2X} \frac{1}{\gamma_j'(\tau)} \ll_\sigma X(1 + |t|) \end{aligned}$$

in virtue of properties of U_r . Now, taking $X = T2^{-k-1}$ and summing over $k = 0, 1, \dots$, we get the estimate of the lemma. \square

Lemma 3.1 allows to obtain the approximation in the mean for $\underline{L}(s, \underline{\chi})$ by $\underline{L}_n(s, \underline{\chi})$. For $g_1, g_2 \in H(D)$, let

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact sets such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_l$ for some l . Then ρ is a metric in $H(D)$ inducing the topology of uniform convergence on compacta. For $\underline{g}_1 = (g_{11}, \dots, g_{1r})$, $\underline{g}_2 = (g_{21}, \dots, g_{2r}) \in H^r(D)$, define

$$\underline{\rho}(\underline{g}_1, \underline{g}_2) = \max_{1 \leq j \leq r} \rho(g_{1j}, g_{2j}).$$

Then $\underline{\rho}$ is a metric in $H^r(D)$ inducing the product topology.

LEMMA 3.2. *The equality*

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T - T_0} \int_{T_0}^T \underline{\rho}(\underline{L}(s + i\gamma(\tau), \underline{\chi}), \underline{L}_n(s + i\gamma(\tau), \underline{\chi})) d\tau = 0$$

holds.

Proof. From the definitions of the metrics ρ and $\underline{\rho}$, it follows that it suffices to prove that, for every compact set $K \subset D$ and all $j = 1, \dots, r$,

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T - T_0} \int_{T_0}^T \sup_{s \in K} |L(s + i\gamma_j(\tau), \chi_j) - L_n(s + i\gamma_j(\tau), \chi_j)| d\tau = 0. \quad (3.1)$$

Let

$$l_n(s) = \frac{s}{\sigma_0} \Gamma\left(\frac{s}{\sigma_0}\right) n^{-s},$$

where the number σ_0 is from the definition of $v_n(m)$. Then an application of the Mellin formula leads to the representation

$$L_n(s, \chi) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} L(s + z, \chi) l_n(z) \frac{dz}{z},$$

where χ is an arbitrary Dirichlet character modulo q . Let K be an arbitrary fixed compact set of the strip D . We fix $\varepsilon > 0$ such that $1/2 + 2\varepsilon \leq \sigma \leq 1 - \varepsilon$ for points $s \in K$. The residue theorem for $\hat{\sigma}_0 > 0$ implies

$$L_n(s, \chi) - L(s, \chi) = \frac{1}{2\pi i} \int_{-\hat{\sigma}_0 - i\infty}^{-\hat{\sigma}_0 + i\infty} L(s + z, \chi) l_n(z) \frac{dz}{z} + R_n(s), \quad (3.2)$$

where

$$R_n(s) = \begin{cases} 0 & \text{if } \chi \text{ is a non-principal character,} \\ \prod_{p|q} \left(1 - \frac{1}{p}\right) \frac{l_n(1-s)}{1-s} & \text{otherwise.} \end{cases}$$

Denote by $s = \sigma + iv$ the points of K , and take

$$\hat{\sigma}_0 = \sigma - \varepsilon - \frac{1}{2}, \quad \sigma_0 = \frac{1}{2} + \varepsilon.$$

Let $\gamma(\tau)$ be one of the functions $\gamma_j(\tau)$, $j = 1, \dots, r$. Then, by (3.2),

$$\begin{aligned} & |L_n(s + i\gamma(\tau), \chi) - L(s + i\gamma(\tau), \chi)| \\ & \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |L(s + i\gamma(\tau) - \hat{\sigma}_0 + it, \chi)| \frac{|l_n(-\hat{\sigma}_0 + it)|}{|-\hat{\sigma}_0 + it|} dt + |R_n(s + i\gamma(\tau))| \\ & = \frac{1}{2\pi} \int_{-\infty}^{\infty} |L(1/2 + \varepsilon + i(t + \gamma(\tau)), \chi)| \frac{|l_n(1/2 + \varepsilon - s + it)|}{|1/2 + \varepsilon - s + it|} dt + |R_n(s + i\gamma(\tau))| \end{aligned}$$

after a shift $t + v \rightarrow t$. Thus,

$$\begin{aligned} & \frac{1}{T - T_0} \int_{T_0}^T \sup_{s \in K} |L(s + i\gamma(\tau), \chi) - L_n(s + i\gamma(\tau), \chi)| d\tau \\ & \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{T - T_0} \int_{T_0}^T |L(1/2 + \varepsilon + i(t + \gamma(\tau)), \chi)| d\tau \right) \sup_{s \in K} \frac{|l_n(1/2 + \varepsilon - s + it)|}{|1/2 + \varepsilon - s + it|} dt \\ & \quad + \frac{1}{T - T_0} \int_{T_0}^T \sup_{s \in K} |R_n(s + i\gamma(\tau))| d\tau \\ & \stackrel{\text{def}}{=} I_1 + I_2. \end{aligned} \quad (3.3)$$

For the function $\Gamma(s)$, the well-known estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0,$$

uniform in $\sigma_1 \leq \sigma \leq \sigma_2$ is valid. Therefore, the definition of $l_n(s)$ implies the bound

$$\frac{l_n(1/2 + \varepsilon - s + it)}{1/2 + \varepsilon - s + it} \ll \frac{n^{1/2+\varepsilon-\sigma}}{\sigma_0} \exp\left\{-\frac{c}{\sigma_0}|t-v|\right\} \ll_K n^{-\varepsilon} \exp\{-c|t|\}.$$

Thus, by Lemma 3.1,

$$I_1 \ll_{K,\varepsilon} n^{-\varepsilon} \int_{-\infty}^{\infty} \left(1 + |t|^{1/2}\right) \exp\{-c|t|\} dt \ll_{K,\varepsilon} n^{-\varepsilon}. \quad (3.4)$$

Similarly, we find that

$$\begin{aligned} I_2 &\ll_{K,q} n^{1/2-2\varepsilon} \frac{1}{T-T_0} \int_{T_0}^T \exp\{-c\gamma(\tau)\} d\tau \ll_{K,q} n^{1/2-2\varepsilon} \left(\frac{\log T}{T} + \frac{1}{T} \int_{\log T}^T \exp\{-c\gamma(\tau)\} d\tau \right) \\ &\ll_{K,q} n^{1/2-2\varepsilon} \left(\frac{\log T}{T} + \frac{1}{T} \exp\left\{-\frac{c}{2}\gamma(\log T)\right\} \int_{\log T}^T \exp\left\{-\frac{c}{2}\gamma(\tau)\right\} d\tau \right) = o(T) \end{aligned}$$

as $T \rightarrow \infty$ because $\gamma(\tau) \rightarrow \infty$ as $\tau \rightarrow \infty$. This, (3.4) and (3.3) prove (3.1). The lemma is proved. \square

4. Limit theorem

In this section, we consider the weak convergence for

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T-T_0} \text{meas} \left\{ \tau \in [T_0, T] : \underline{L}(s + i\gamma(\tau), \underline{\chi}) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)),$$

as $T \rightarrow \infty$. For this, we recall the useful property of convergence in distribution ($\xrightarrow{\mathcal{D}}$).

PROPOSITION 4.1. *Suppose that the space (\mathbb{X}, d) is separable, and X_{kn} and X_n , $k \in \mathbb{N}$, $n \in \mathbb{N}$, are \mathbb{X} -valued random elements defined on the same probability space with measure ν . If $X_{kn} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} Z_k \xrightarrow[k \rightarrow \infty]{\mathcal{D}} X$ and, for every $\varepsilon > 0$,*

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \nu \{d(X_{kn}, X_n) \geq \varepsilon\} = 0,$$

then $X_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} X$.

The proof of the proposition is given in [3].

Define

$$\underline{L}(s, \omega, \underline{\chi}) = (L(s, \omega_1, \chi_1), \dots, L(s, \omega_r, \chi_r)),$$

where

$$L(s, \omega_j, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m) \omega_j(m)}{m^s}, \quad j = 1, \dots, r,$$

and denote by $P_{\underline{L}}$ the distribution of the $H^r(D)$ -valued random element $\underline{L}(s, \omega, \underline{\chi})$, i.e.,

$$P_{\underline{L}}(A) = m_H^r \left\{ \omega \in \Omega^r : \underline{L}(s, \omega, \underline{\chi}) \in A \right\}, \quad A \in \mathcal{B}(H^r(D)).$$

THEOREM 4.2. *Suppose that $(\gamma_1, \dots, \gamma_r) \in U_r$. Then P_T converges weakly to $P_{\underline{L}}$ as $T \rightarrow \infty$.*

Proof. On a certain probability space with measure μ , define a random variable ξ_T and assume that ξ_T is uniformly distributed on $[T_0, T]$. On the above probability space, define the $H^r(D)$ -valued random element

$$\underline{X}_{T,n} = \underline{X}_{T,n}(s) = L_n(s + i\gamma(\xi_T), \underline{\chi}),$$

and denote by $\hat{X}_n = \hat{X}_n(s)$ the $H^r(D)$ -valued random element with distribution V_n , where V_n is the limit measure in Lemma 2.2. Then Lemma 2.2 implies the relation

$$\underline{X}_{T,n} \xrightarrow[T \rightarrow \infty]{\mathcal{D}} \hat{X}_n. \quad (4.1)$$

By Lemma 2.3, the family $\{V_n : n \in \mathbb{N}\}$ is tight, therefore, in view of the Prokhorov theorem [3: Theorem 6.1], it is relatively compact. Thus, there exists a subsequence $\{V_{n_k}\} \subset \{V_n\}$ weakly convergent to a certain probability measure P on $(H^r(D), \mathcal{B}(H^r(D)))$ as $k \rightarrow \infty$. This is equivalent to the relation

$$\hat{X}_{n_k} \xrightarrow[k \rightarrow \infty]{\mathcal{D}} P. \quad (4.2)$$

Define one more $H^r(D)$ -valued random element

$$\underline{X}_T = \underline{X}_T(s) = \underline{L}(s + i\gamma(\xi_T), \underline{\chi}).$$

Then, using Lemma 3.2, we find that, for every $\varepsilon > 0$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \mu \{ \rho(\underline{X}_T, \underline{X}_{T,n}) \geq \varepsilon \} \\ & \leq \lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{(T - T_0)\varepsilon} \int_{T_0}^T \rho(\underline{L}(s + i\gamma(\tau), \underline{\chi}), \underline{L}_n(s + i\gamma(\tau), \underline{\chi})) \, d\tau = 0. \end{aligned}$$

This, (4.1), (4.2) and Proposition 4.1 show that

$$\underline{X}_T \xrightarrow[T \rightarrow \infty]{\mathcal{D}} P, \quad (4.3)$$

i.e., P_T converges weakly to P as $T \rightarrow \infty$.

It remains to prove that $P = P_{\underline{L}}$. The relation (4.3) shows that the measure P is independent of the choice of the sequence $\{V_{n_k}\}$. Hence, we have that

$$\hat{X}_n \xrightarrow[n \rightarrow \infty]{\mathcal{D}} P,$$

or V_n converges weakly to P as $n \rightarrow \infty$. In [6], a discrete limit theorem for Dirichlet L -functions was discussed, and it was obtained that the limit measure P of V_n , as $n \rightarrow \infty$, is $P_{\underline{L}}$. This remark and (4.3) complete the proof of the theorem. \square

Theorem 4.2 implies a limit theorem for the compositions $F(\underline{L}(s, \underline{\chi}))$.

THEOREM 4.3. *Suppose that $(\gamma_1, \dots, \gamma_r) \in U_r$, and $F: H^r(D) \rightarrow H(D)$ is a continuous operator. Then*

$$P_{T,F}(A) \stackrel{\text{def}}{=} \frac{1}{T} \text{meas} \{ \tau \in [0, T] : F(\underline{L}(s + i\gamma(\tau), \underline{\chi})) \in A \} \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\underline{L}}F^{-1}$ as $T \rightarrow \infty$.

Proof. The theorem follows from Theorem 4.2, continuity of F and [3: Theorem 5.1]. \square

5. Support

For proving of universality, we need the explicit form of the support of the measure $P_{\underline{L}}$. Since the space $H^r(D)$ is separable, the support $S_{P_{\underline{L}}}$ of $P_{\underline{L}}$ is a minimal closed set of $H^r(D)$ such that $P_{\underline{L}}(S_{P_{\underline{L}}}) = 1$. The set $S_{P_{\underline{L}}}$ consists of all $\underline{g} \in H^r(D)$ such that, for every open neighbourhood \underline{G} of \underline{g} , the inequality $P_{\underline{L}}(\underline{G}) > 0$ is satisfied.

We recall that

$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

PROPOSITION 5.1. *The support of $P_{\underline{L}}$ is the set S^r .*

PROOF. Let, for $\omega \in \Omega$,

$$L(s, \omega, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)\omega(m)}{m^s} = \prod_p \left(1 - \frac{\chi(p)\omega(p)}{p^s}\right)^{-1},$$

and P_L be the distribution of the $H(D)$ -valued random element $L(s, \omega, \chi)$. Then it is well known, see, for example, [1], [11], that the support of P_L is the set S . We will apply this remark for the support of $P_{\underline{L}}$.

Since the space $H^r(D)$ is separable, it is known that [3]

$$\mathcal{B}(H^r(D)) = \underbrace{\mathcal{B}(H(D)) \times \cdots \times \mathcal{B}(H(D))}_r.$$

Therefore, it suffices to consider the measure $P_{\underline{L}}$ on the sets $A \in H^r(D)$ of the form

$$A = A_1 \times \cdots \times A_r, \quad A_1, \dots, A_r \in H(D).$$

The Haar measure m_H^r is the product of the Haar measures $m_{H,j}$ on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$. Therefore,

$$\begin{aligned} P_{\underline{L}}(A) &= m_H^r \{ \omega \in \Omega^r : \underline{L}(s, \omega, \underline{\chi}) \in A \} \\ &= \prod_{j=1}^r m_{H,j} \{ \omega_j \in \Omega_j : L(s, \omega_j, \chi_j) \in A_j \} = \prod_{j=1}^r P_{L_j}(A_j), \end{aligned}$$

where P_{L_j} is the distribution of the random element $L(s, \omega_j, \chi_j)$. Since, for all $j = 1, \dots, r$, the support of P_{L_j} is the set S , the minimality of the support proves the proposition. \square

PROPOSITION 5.2. *Let $F: H^r(D) \rightarrow H(D)$ be a continuous operator such that, for every open set $G \subset H(D)$, the set $(F^{-1}G) \cap S^r$ is non-empty. Then the support of the measure $P_{\underline{L}}F^{-1}$ is the whole of $H(D)$.*

PROOF. Let $g \in H(D)$ be an arbitrary element, and G its any open neighbourhood. Then the set $F^{-1}G$ is open as well, and contains an element of the set S^r . Thus, in view of Proposition 5.1, $F^{-1}G$ is an open neighbourhood of an element of the support of the measure $P_{\underline{L}}$. Hence,

$$P_{\underline{L}}F^{-1}(G) = P_{\underline{L}}(F^{-1}G) > 0.$$

Since g and G are arbitrary, this proves the proposition. \square

6. Proof of universality

We recall the Mergelyan theorem on the approximation of analytic functions by polynomials [13]. Let $K \subset \mathbb{C}$ be a compact set with connected complements, and $f(s)$ be a continuous function on K and analytic in the interior of K . Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \varepsilon.$$

Proof of Theorem 1.3. By the Mergelyan theorem, there exist polynomials $p_j(s)$ such that

$$\sup_{1 \leq j \leq r} \sup_{s \in K_j} |f_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2}. \quad (6.1)$$

Define the set

$$G_\varepsilon^r = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - e^{p_j(s)}| < \frac{\varepsilon}{2} \right\}.$$

Then G_ε^r , in view of Proposition 5.1, is an open neighbourhood of the element $(e^{p_1(s)}, \dots, e^{p_r(s)})$ of the support of the measure $P_{\underline{L}}$. Therefore

$$P_{\underline{L}}(G_\varepsilon^r) > 0. \quad (6.2)$$

Thus, Theorem 4.2 and the equivalent of weak convergence of probability measures in terms of open sets ([3: Theorem 2.1]) show that

$$\liminf_{T \rightarrow \infty} P_T(G_\varepsilon^r) > 0.$$

This, (6.1) and the definitions of P_T and G_ε^r prove the first part of the theorem.

To prove the second part of the theorem, define one more set

$$\hat{G}_\varepsilon^r = \left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| < \varepsilon \right\}.$$

Then the boundary $\partial \hat{G}_\varepsilon^r$ of \hat{G}_ε^r lies in the set

$$\left\{ (g_1, \dots, g_r) \in H^r(D) : \sup_{1 \leq j \leq r} \sup_{s \in K_j} |g_j(s) - f_j(s)| = \varepsilon \right\},$$

therefore, $\partial \hat{G}_{\varepsilon_1}^r \cap \partial \hat{G}_{\varepsilon_2}^r = \emptyset$ for different positive ε_1 and ε_2 . From this remark, it follows that the set \hat{G}_ε^r is a continuity set of the measure $P_{\underline{L}}$ for all but at most countably many $\varepsilon > 0$. Thus, Theorem 4.2 and the equivalent of weak convergence of probability measures in terms of continuity sets ([3: Theorem 2.1]) imply the equality

$$\lim_{T \rightarrow \infty} P_T(\hat{G}_\varepsilon^r) = P_{\underline{L}}(\hat{G}_\varepsilon^r) \quad (6.3)$$

for all but at most countably many $\varepsilon > 0$. In view of (6.1), $G_\varepsilon^r \subset \hat{G}_\varepsilon^r$. Therefore, $P_{\underline{L}}(\hat{G}_\varepsilon^r) > 0$ by (6.2). This, (6.3) and the definitions of P_T and \hat{G}_ε^r prove the second part of the theorem. \square

Proof of Theorem 1.4. By the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \quad (6.4)$$

Define the set

$$G_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Then, by Proposition 5.2, G_ε is an open neighbourhood of the element $p(s)$ of the support of the measure $P_{\underline{L}}F^{-1}$. Therefore,

$$P_{\underline{L}}F^{-1}(G_\varepsilon) > 0. \quad (6.5)$$

From this, Theorem 4.3 and the equivalent of weak convergence of probability measures in terms of open sets, we obtain that

$$\liminf_{T \rightarrow \infty} P_{T,F}(G_\varepsilon) \geq P_{\underline{L}}F^{-1}(G_\varepsilon) > 0,$$

and the definitions of $P_{T,F}$ and G_ε , and (6.4) prove the first part of the theorem.

Define one more set

$$\hat{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}.$$

Then we have that this set is a continuity set of the measure $P_{\underline{L}}F^{-1}$ for all but at most countably many $\varepsilon > 0$. Therefore, Theorem 4.3 and the equivalent of weak convergence of probability measures in terms of continuity sets show that

$$\lim_{T \rightarrow \infty} P_{T,F}(\hat{G}_\varepsilon) = P_{\underline{L}}F^{-1}(\hat{G}_\varepsilon) \quad (6.6)$$

for all but at most countably many $\varepsilon > 0$. Moreover, in view of (6.4), we have that $G_\varepsilon \subset \hat{G}_\varepsilon$. Therefore, by (6.5), the inequality $P_{\underline{L},F^{-1}}(\hat{G}_\varepsilon) > 0$ holds. This together with (6.6) proves the second part of the theorem. \square

REFERENCES

- [1] BAGCHI, B.: *The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and other Allied Dirichlet Series*, Ph. D. Thesis, Indian Stat. Institute, Calcutta, 1981.
- [2] BAGCHI, B.: *A joint universality theorem for Dirichlet L -functions*, Math. Z. **181** (1982), 319–334.
- [3] BILLINGSLEY, P.: *Convergence of Probability Measures*, Wiley, New York, 1968.
- [4] BOHR, H.: *Über das Verhalten von $\zeta(s)$ in der Halbebene $\sigma > 1$* , Nachr. Akad. Wiss. Göttingen II Math. Phys. Kl. (1911), 409–428.
- [5] BOHR, H.—COURANT, R.: *Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunktion*, J. Reine Angew. Math. **144** (1914), 249–274.
- [6] DUBICKAS, A.—LAURINČIKAS, A.: *Joint discrete universality of Dirichlet L -functions*, Arch. Math. **104** (2015), 25–35.
- [7] GONEK, S.M.: *Analytic Properties of Zeta and L -functions*, Ph. D. Thesis, University of Michigan, 1979.
- [8] KACZOROWSKI, J.—LAURINČIKAS, A.—STEUDING, J.: *On the value distribution of shifts of universal Dirichlet series*, Monatsh. Math. **147**(4) (2006), 309–317.
- [9] KARATSUBA, A. A.—VORONIN, S. M.: *The Riemann Zeta-Function*, Walter de Gruyter, Berlin, New York, 1992.
- [10] LAURINČIKAS, A.: *Limit Theorems for the Riemann Zeta-Function*, Kluwer Academic Publishers, Dordrecht, 1996.
- [11] LAURINČIKAS, A.: *On joint universality of Dirichlet L -functions*, Chebyshevskii Sb. **12**(1) (2011), 124–139.
- [12] LAURINČIKAS, A.—GARUNKŠTIS, R.: *The Lerch Zeta-Function*, Kluwer Academic Publishers, Dordrecht, 2002.
- [13] MERGELYAN, S.N.: *Uniform approximation to functions of a complex variable*, Usp. Matem. Nauk **7**(2) (1952), 31–122 (in Russian); [Amer. Math. Soc. Translation vol. 101, 1954, p. 99].
- [14] NAKAMURA, T.: *The joint universality and the generalized strong recurrence for Dirichlet L -functions*, Acta Arith. **138** (2009), 357–362.
- [15] NAKAMURA, T.—PAŃKOWSKI, L.: *Erratum to: The generalized strong recurrence for non-zero rational parameters*, Archivum Math. **99** (2012), 43–47.
- [16] PAŃKOWSKI, L.: *Some remarks on the generalized strong recurrence for L -functions*. In: New Directions in Value Distribution Theory of Zeta and L -Functions, Shaker Verlag, Aachen, 2009, pp. 307–315.

- [17] PAŃKOWSKI, L.: *Joint universality and generalized strong recurrence with rational parameter*, J. Number Theory **163** (2016), 61–74.
- [18] PAŃKOWSKI, L.: *Joint universality for dependent L -functions*, Ramanujan J. **45** (2018), 181–195.
- [19] STEUDING, J.: *Value-Distribution of L -Functions*. Lecture Notes in Math. 1877, Springer, Berlin, 2007.
- [20] VORONIN, S. M.: *The distribution of the non-zero values of the Riemann zeta-function*, Trudy Matem. Inst. Steklov **128** (1972), 131–150 (in Russian).
- [21] VORONIN, S. M.: *Theorem on the “universality” of the Riemann zeta-function*, Izv. Akad. Nauk SSSR, Ser. Mat. **39** (1975), 475–486 (in Russian); [Math. USSR Izv. **9** (1975), 443–453].
- [22] VORONIN, S. M.: *On the functional independence of Dirichlet L -functions*, Acta Arith. **27** (1975), 493–503 (in Russian).

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