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Mathematica Slovaca

DOI: 10.1515/ms-2022-0002
Math. Slovaca **72** (2022), No. 1, 11–34

ORDINARY GENERATING FUNCTIONS OF BINARY PRODUCTS OF THIRD-ORDER RECURRENCE RELATIONS AND 2-ORTHOGONAL POLYNOMIALS

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(*Communicated by István Gaál*)

ABSTRACT. In this paper, we introduce the new generating functions for the product of some numbers and 2-orthogonal polynomials by making use of the symmetrizing endomorphism operators $\pi_{e_2e_3}\pi_{e_1e_2}$ to the series $\sum\limits_{n=0}^{+\infty} S_n(A_3)\, e_1^n z^n$.

1. Introduction

The Tribonacci sequence $\{T_n\}_{n=0}^{+\infty}$ and Tribonacci Lucas sequence $\{K_n\}_{n=0}^{+\infty}$ are defined respectively by the third order recurrence relations

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad \text{and} \quad K_n = K_{n-1} + K_{n-2} + K_{n-3}$$

for $n \geq 3$, with the initial values $T_0 = 0$, $T_1 = T_2 = 1$, $K_0 = 3$, $K_1 = 1$, and $K_2 = 3$, respectively. The Tribonacci numbers T_n were defined in [13] for the first time and some properties for T_n have been investigated in [9, 13]. In [9: Example 3.3], the determinants expression

$$T_{n+1} = \begin{vmatrix} 1 & 1 & & & \\ -1 & 1 & 1 & & \\ 1 & -1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & 1 & -1 & 1 \end{vmatrix}_n, \qquad n \geq 1 \tag{1.1}$$

was derived. There are many generalizations of the Tribonacci sequences T_n. One of these generalizations is the generalized Tribonacci sequence $V_n(x, y, z; a, b, c)$ defined for $n \geq 3$ by

$$V_n(x, y, z; a, b, c) = xV_{n-1}(x, y, z; a, b, c) + yV_{n-2}(x, y, z; a, b, c) + zV_{n-3}(x, y, z; a, b, c), \tag{1.2}$$

where $V_0(x, y, z; a, b, c) = a$, $V_1(x, y, z; a, b, c) = b$, $V_2(x, y, z; a, b, c) = c$ are arbitrary integers and x, y, z are real numbers.

2020 Mathematics Subject Classification: Primary 06B05, 06D35; Secondary 35R30, 39A10.
Keywords: Symmetric functions, generating functions, Tribonacci polynomials, Narayana numbers.
This work was supported by Directorate General for Scientific Research and Technological Development (DGRSDT), Algeria.

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The generating function of the generalized Tribonacci sequence $V_n(x,y,z;a,b,c)$ is

$$\sum_{n=0}^{+\infty} V_n(x,y,z;a,b,c)z^n = \frac{a+(b-ax)z+(c-xb-ya)z^2}{1-xz-yz^2-z^3}. \tag{1.3}$$

There have been many studies on the generalized Tribonacci numbers $V_n(x,y,z;a,b,c)$. For more information, please refer to [10,12] and closely related references therein. Some special cases of the generalized Tribonacci sequence $V_n(x,y,z;a,b,c)$ are as follows:

1. $V_n(1,1,1;0,1,1) = T_n$, the Tribonacci sequence;
2. $V_n(1,1,1;3,1,3) = K_n$, the Tribonacci–Lucas sequence;
3. $V_n(0,1,1;3,0,2) = Q_n$, the Perrin sequence;
4. $V_n(0,1,1;1,1,1) = P_n$, the Padovan (Cordonnier) sequence;
5. $V_n(0,1,1;1,0,1) = R_n$, the Van Der Laan sequence;
6. $V_n(1,0,1;0,1,1) = N_n$, the Narayana sequence;
7. $V_n(1,1,2;0,1,1) = J_n^{(3)}$, the third order Jacobsthal sequence;
8. $V_n(1,1,2;2,1,5) = j_n^{(3)}$, the third order Jacobsthal Lucas sequence.

In [9: Example 3.4], the determinants expression

$$Q_{n+1} = \begin{vmatrix} 2 & 1 & & & & \\ -3 & 0 & 1 & & & \\ 0 & -1 & 0 & 1 & & \\ & 1 & -1 & 0 & \ddots & \\ & & \ddots & \ddots & \ddots & 1 \\ & & & 1 & -1 & 0 \end{vmatrix}_n, \qquad n \geq 1 \tag{1.4}$$

was derived. In [16], the Tribonacci polynomials $T_n(x)$ were defined for $n \geq 3$ by

$$T_n(x) = x^2 T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x),$$

where $T_0(x) = 0$, $T_1(x) = 1$, and $T_2(x) = x^2$. In [9: Theorem 3.10], the determinant

$$T_{n+1}(x) = \begin{vmatrix} x^2 & 1 & & & \\ -x & x^2 & 1 & & \\ 1 & -x & x^2 & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & 1 & -x & x^2 \end{vmatrix}_n, \qquad n \geq 1 \tag{1.5}$$

was established. Now, we recall the notion of d-orthogonal polynomials. A remarkable property of the d-monic orthogonal polynomials sequence is that those sequences satisfy a $(d+1)$-order recurrence relation, written in the form

$$P_{m+d+1}(x) = (x-\beta_{m+d})P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x), \qquad m \geq 0,$$

with the initial conditions $P_0(x) = 1$, $P_{-1}(x) = 0$. So, if $d \geq 2$:

$$P_n(x) = (x-\beta_{n-1})P_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{d-1-\nu}^{n-1-\nu} P_{n-2-\nu}(x), \qquad 2 \leq n \leq d,$$

and the regularity conditions $\gamma_{m+1}^{0} \neq 0$, $m \geq 0$. The 2-orthogonal monic Chebyshev polynomials (2-classical) of the first kind $\{\widehat{T}_n\}_{n\geq 0}$ studied in [11], and defined by the following relations, where α and γ are constants (see also [19]):

$$\widehat{T}_0(x) = 1, \qquad \widehat{T}_1(x) = x, \qquad \widehat{T}_2(x) = x^2 - \alpha$$
$$\widehat{T}_{n+3}(x) = x\widehat{T}_{n+2}(x) - \alpha\widehat{T}_{n+1}(x) - \gamma\widehat{T}_n(x), \; n \geq 0, \qquad \gamma \neq 0.$$

Definition 1. A d-orthogonal polynomials sequence $\{P_n\}_{n\geq 0}$ is called d-classical d-orthogonal polynomials sequence if both $\{P_n\}_{n\geq 0}$ and its derivative $\{P'_n\}_{n\geq 0}$ are d-orthogonal. For more details, see [8].

Note that the 2-classical 2-orthogonal polynomials sequence is analogous to the Chebyshev orthogonal polynomials sequence of the first kind $\{\widehat{T}_n\}_{n\geq 0}$ (see [11]). In this contribution, we shall define a new useful operator denoted by $\pi_{e_2e_3}\pi_{e_1e_2}$ for which we can formulate, extend and prove new results based on our previous ones, (see [10], [16], [17]). In order to determine generating functions for third-order recurrence relations and 2-classical 2-orthogonal polynomials sequence, we combine between our indicated past techniques and these presented polishing approaches. The further contents of this paper are as follows: Section 1 gives introduction. In Section 2, we introduce a new symmetric function and give some properties of this symmetric function. We also give some more useful definitions which are used in the subsequent sections. In Section 3, we prove our main result which relates the symmetric function defined in the previous section with the symmetrizing operator. This main theorem unifies several previously known results about the generating functions. It is then used to find the new generating functions of products for some know numbers and polynomials, in Section 4.

2. Definitions and some properties

We need some preliminaries on symmetric functions and divided differences. We will follow the conventions of [15] rather than those of [18], so that you can conveniently use multiple alphabets at once.

Definition 2 ([1])**.** Let A and E be any sets of given variables. Then we give $S_n(A - E)$ by the following form:

$$\frac{\prod_{e\in E}(1 - ez)}{\prod_{a\in A}(1 - az)} = \sum_{n=0}^{\infty} S_n(A - E)\, z^n, \tag{2.1}$$

with the condition $S_n(A - E) = 0$ for $n < 0$.

Equation (2.1) can be rewritten in the following form:

$$\sum_{n=0}^{\infty} S_n(A - E)z^n = \left(\sum_{n=0}^{\infty} S_n(A)z^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-E)z^n\right),$$

where

$$S_n(A - E) = \sum_{i=0}^{n} S_i(A)S_{n-i}(-E).$$

Remark 1. Taking $A = \{0\}$ in (2.1) gives

$$\sum_{n=0}^{\infty} S_n(-E)\, z^n = \prod_{e\in E}(1 - ez).$$

The summation is in fact limited to a finite number of non-zero terms, in particular:

$$\prod_{e\in E}(x-e)=S_n(x-E)=x^nS_0(-E)+x^{n-1}S_1(-E)+x^{n-2}S_2(-E)+\dots \tag{2.2}$$

The $S_j(-E)$ are the coefficients of the polynomials $S_n(x-E)$ for $0\le j\le n$. These are therefore at the sign near the elementary symmetric functions of the alphabet E, who are void for $j\ge n$. In particular, when $E=\{e,e,e,\dots,e\}$ (we notice $E=ne$), we have $S_n(x-ne)=(x-e)^n$. The specialization $e=1$, i.e., $E=\{1,1,1,\dots\}$ (we notice $E=n$), gives the binomial coefficients:

$$S_j(-n)=(-1)^j\binom{n}{j},\qquad S_j(n)=\binom{n+j+1}{j},\qquad \text{and}\qquad S_j(-ne)=(-e)^j\binom{n}{j}.$$

Definition 3 ([2])**.** Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows:

$$\partial_{x_ix_{i+1}}f\left(x_1,\dots,x_i,x_{i+1},\dots,x_n\right)=\frac{f\left(x_1,\dots,x_i,x_{i+1},\dots,x_n\right)-f\left(x_1,\dots,x_{i-1},x_{i+1},x_i,\dots,x_n\right)}{x_i-x_{i+1}}.$$

Definition 4 ([3,6])**.** Let n be positive integer and $A_2=\{a_1,a_2\}$ are set of given variables. Then, the n -th symmetric function $S_n(a_1+a_2)$ is defined by

$$S_n(A_2)=S_n(a_1+a_2)=\frac{a_1^{n+1}-a_2^{n+1}}{a_1-a_2},$$

with

$$\begin{aligned}
S_0(A_2)&=S_0(a_1+a_2)=1,\\
S_1(A_2)&=S_1(a_1+a_2)=a_1+a_2,\\
S_2(A_2)&=S_2(a_1+a_2)=a_1^2+a_1a_2+a_2^2,\\
&\ \vdots
\end{aligned}$$

Definition 5 ([4,5])**.** The symmetrizing operator $\delta^k_{a_1a_2}$ is defined by

$$\delta^k_{a_1a_2}f(a_1)=\frac{a_1^kf(a_1)-a_2^kf\left(a_2\right)}{a_1-a_2}\qquad\text{for all } k\in\mathbb{N}_0. \tag{2.3}$$

If $f(a_1)=a_1$, the operator (2.3) gives us

$$\delta^k_{a_1a_2}f(a_1)=\frac{a_1^{k+1}-a_2^{k+1}}{a_1-a_2}=S_k\left(a_1+a_2\right).$$

3. Lemmas and main results

In this section, we state and prove our main results.

Definition 6. Given an alphabet $E_3=\{e_1,e_2,e_3\}$, the symmetrizing operator is defined by

$$\pi_{e_2e_3}\pi_{e_1e_2}f\left(e_1\right)=\frac{e_1^2f\left(e_1\right)+e_2e_3\pi_{e_2e_3}f\left(e_2\right)-e_1\pi_{e_2e_3}\left(e_2f\left(e_2\right)\right)}{\left(e_1-e_2\right)\left(e_1-e_3\right)}.$$

Proposition 1. *Let $E_3=\{e_1,e_2,e_3\}$ be an alphabet. Then we have*

$$S_n(E_3)=\frac{e_2S_n(e_1+e_2)-e_3S_n(e_1+e_3)}{e_2-e_3}.$$

P r o o f. We have (see [7])

$$\sum_{n=0}^{+\infty} S_n\left(e_1+e_2\right) z^n=\frac{1}{\left(1-e_1 z\right)\left(1-e_2 z\right)}. \tag{1}$$

Then

$$\sum_{n=0}^{+\infty} S_n\left(e_1+e_3\right) z^n=\frac{1}{\left(1-e_1 z\right)\left(1-e_3 z\right)}. \tag{2}$$

Multiplying the equation (1) by e_2 and subtracting it from (2) multiplying by e_3, we have

$$\sum_{n=0}^{+\infty}\left[e_2 S_n\left(e_1+e_2\right)-e_3 S_n(e_1+e_3)\right] z^n=\frac{e_2-e_3}{\left(1-e_1 z\right)\left(1-e_2 z\right)\left(1-e_3 z\right)}.$$

Thus

$$\begin{aligned}\sum_{n=0}^{+\infty} \frac{e_2 S_n\left(e_1+e_2\right)-e_3 S_n(e_1+e_3)}{e_2-e_3} z^n&=\frac{1}{\left(1-e_1 z\right)\left(1-e_2 z\right)\left(1-e_3 z\right)}\\&=\sum_{n=0}^{+\infty} S_n\left(e_1+e_2+e_3\right) z^n.\end{aligned}$$

Therefore

$$\frac{e_2 S_n\left(e_1+e_2\right)-e_3 S_n(e_1+e_3)}{e_2-e_3}=S_n\left(e_1+e_2+e_3\right).$$

This completes the proof. □

Proposition 2. *Let $E_3=\{e_1,e_2,e_3\}$ be an alphabet. Then we have*

$$S_n(E_3)=\frac{e_1^{n+2}}{\left(e_1-e_2\right)\left(e_1-e_3\right)}+\frac{e_2^{n+2}}{\left(e_2-e_1\right)\left(e_2-e_3\right)}+\frac{e_3^{n+2}}{\left(e_3-e_1\right)\left(e_3-e_3\right)}.$$

If $E_3=\{1,2,3\}$ in Proposition 2 gives

$$S_n(E_3)=\frac{3^{n+2}+1}{2}-2^{n+2}.$$

Corollary 1. *The following identity holds true:*

$$\pi_{e_2e_3} S_n(e_1+e_2)=S_n(E_3).$$

In order to prove our main results, we recall several lemmas below.

Lemma 1. *Let $A_3=\{a_1,a_2,a_3\}$ and $E_3=\{e_1,e_2,e_3\}$ be two alphabets. Then action operator $\pi_{e_2e_3}\pi_{e_1e_2}$ on the series $\sum\limits_{n=0}^{+\infty} S_n(A_3)e_1^n z^n$ is given by*

$$\pi_{e_2e_3}\pi_{e_1e_2}\sum_{n=0}^{+\infty} S_n\left(A_3\right) e_1^n z^n=\sum_{n=0}^{+\infty} S_n\left(A_3\right) S_n\left(E_3\right) z^n.$$

P r o o f. We have

$$\begin{aligned}\pi_{e_2e_3}\pi_{e_1e_2}\sum_{n=0}^{+\infty} S_n(A_3)e_1^n z^n&=\pi_{e_2e_3}\left(\frac{e_1\sum\limits_{n=0}^{+\infty} S_n(A_3)e_1^n z^n-e_2\sum\limits_{n=0}^{+\infty} S_n(A_3)e_2^n z^n}{(e_1-e_2)}\right)\\&=\pi_{e_2e_3}\left(\sum_{n=0}^{+\infty} S_n(A_3)\frac{e_1^{n+1}-e_2^{n+1}}{(e_1-e_2)} z^n\right)\end{aligned}$$

$$= \pi_{e_2e_3}\left(\sum_{n=0}^{+\infty} S_n(A_3)S_n(e_1+e_2)z^n\right)$$

$$= \frac{e_2\sum\limits_{n=0}^{+\infty} S_n(A_3)S_n(e_1+e_2)z^n - e_3\sum\limits_{n=0}^{+\infty} S_n(A_3)S_n(e_1+e_3)z^n}{(e_2-e_3)}$$

$$= \sum_{n=0}^{+\infty} S_n(A_3)\frac{e_2S_n(e_1+e_2)-e_3S_n(e_1+e_3)}{(e_2-e_3)}z^n,$$

according to Proposition 1, we have then

$$\pi_{e_2e_3}\pi_{e_1e_2}\sum_{n=0}^{+\infty} S_n\left(A_3\right)e_1^nz^n = \sum_{n=0}^{+\infty} S_n\left(A_3\right)S_n\left(E_3\right)z^n.$$

This completes the proof. □

Lemma 2. *Let $A_3 = \{a_1,a_2,a_3\}$ and $E_2 = \{e_1,e_2\}$ be two alphabets. Then action operator $\pi_{e_1e_2}$ on the series $\frac{1}{\sum_{n=0}^{+\infty} S_n(-A_3)e_1^nz^n}$ is given by*

$$\pi_{e_1e_2}\frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^nz^n} = \frac{1-e_1e_2S_2(-A_3)z^2-e_1^2e_2S_3(-A_3)z^3-e_1e_2^2S_3(-A_3)z^3}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n\left(-A_3\right)e_2^nz^n\right)}.$$

P r o o f. We have

$$\pi_{e_1e_2}\frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^nz^n} = \frac{1}{(e_1-e_2)}\left(\frac{e_1}{\sum\limits_{n=0}^{+\infty} S_n\left(-A_3\right)e_1^nz^n} - \frac{e_2}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^nz^n}\right)$$

$$= \frac{e_1\left(\sum\limits_{n=0}^{+\infty} S_n\left(-A_3\right)e_2^nz^n\right) - e_2\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3e_1^nz^n\right)}{(e_1-e_2)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^nz^n\right)}$$

$$= \frac{\sum\limits_{n=0}^{+\infty} S_n(-A_3)\frac{e_1e_2^n-e_2e_1^n}{(e_1-e_2)}z^n}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^nz^n\right)}$$

$$= \frac{1-e_1e_2\sum\limits_{n=2}^{+\infty} S_n(-A_3)S_{n-2}\left(e_1+e_2\right)z^n}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^nz^n\right)}$$

$$= \frac{1-e_1e_2\left(S_2(-A_3)z^2+S_3(-A_3)S_1(e_1+e_2)z^3\right)}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^nz^n\right)}$$

$$= \frac{1-e_1e_2S_2(-A_3)z^2-e_1^2e_2S_3(-A_3)z^3-e_1e_2^2S_3(-A_3)z^3}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^nz^n\right)}.$$

This completes the proof. □

Lemma 3. *Given an alphabet $E_2 = \{e_2, e_3\}$, we have*

$$\pi_{e_2e_3}\left(\frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n}\right) = \frac{1 - e_2e_3\left(S_2(-A_3)z^2 + S_3(-A_3)S_1(e_2+e_3)z^3\right)}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)}.$$

P r o o f. We have

$$\begin{aligned}
\pi_{e_2e_3}\left(\frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n}\right) &= \frac{1}{e_2-e_3}\left(\frac{e_2}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n} - \frac{e_3}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n}\right) \\
&= \frac{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)\frac{e_2e_3^n - e_3e_2^n}{e_2-e_3} z^n\right)}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)} \\
&= \frac{1 - e_2e_3\sum\limits_{n=2}^{+\infty} S_n(-A_3)S_{n-2}(e_2+e_3)z^n}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)} \\
&= \frac{1 - e_2e_3S_2(-A_3)z^2 - e_2e_3S_3(-A_3)S_1(e_2+e_3)z^3}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)}.
\end{aligned}$$

This completes the proof. □

Lemma 4. *Given an alphabet $E_2 = \{e_2, e_3\}$, we have*

$$\pi_{e_2e_3}\left(\frac{e_2}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n}\right) = \frac{(e_2+e_3) + e_2e_3S_1(-A_3)z - e_2^2e_3^2S_3(-A_3)z^3}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)}.$$

P r o o f. We have

$$\begin{aligned}
\pi_{e_2e_3}\left(\frac{e_2}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n}\right) &= \frac{1}{e_2-e_3}\left(\frac{e_2^2}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n} - \frac{e_3^2}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n}\right) \\
&= \frac{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)\frac{e_2^2e_3^n - e_3^2e_2^n}{e_2-e_3} z^n\right)}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)} \\
&= \frac{(e_2+e_3) + e_2e_3S_1(-A_3)z - e_2^2e_3^3\sum\limits_{n=3}^{+\infty} S_n(-A_3)S_{n-3}(e_2+e_3)z^n}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)} \\
&= \frac{(e_2+e_3) + e_2e_3S_1(-A_3)z - e_2^2e_3^2S_3(-A_3)z^3}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)}.
\end{aligned}$$

This completes the proof. □

Lemma 5. *Given an alphabet $E_2 = \{e_2, e_3\}$, we have*

$$\pi_{e_2e_3}\left(\frac{e_2^2}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n}\right) = \frac{\left((e_2+e_3)^2 - e_2e_3\right) + e_2e_3(e_2+e_3)S_1(-A_3)z + e_2^2e_3^2S_2(-A_3)z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)}.$$

Proof. We have

$$\begin{aligned}
&\pi_{e_2e_3}\left(\frac{e_2^2}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n}\right)\\
&= \frac{1}{e_2-e_3}\left(\frac{e_2^3}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n} - \frac{e_3^3}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n}\right)\\
&= \frac{\left(\sum\limits_{n=0}^{+\infty} S_n(-A)\frac{e_2^3e_3^n - e_3^3e_2^n}{e_2-e_3} z^n\right)}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)}\\
&= \frac{\left((e_2+e_3)^2 - e_2e_3\right) + e_2e_3(e_2+e_3)S_1(-A_3)z + e_2^2e_3^2S_2(-A_3)z^2 - e_2^3e_3^3\sum\limits_{n=4}^{+\infty} S_n(-A_3)S_{n-4}(e_2+e_3)z^n}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)}\\
&= \frac{\left((e_2+e_3)^2 - e_2e_3\right) + e_2e_3(e_2+e_3)S_1(-A_3)z + e_2^2e_3^2S_2(-A_3)z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right)}.
\end{aligned}$$

This completes the proof. □

In this part, we are now in a position to provide Theorem 1. Also, we derive the new generating functions of the products of some known numbers and polynomials.

Theorem 1. *Let A_3 and E_3 be two alphabets defined respectively by $\{a_1, a_2, a_3\}$ and $\{e_1, e_2, e_3\}$. Then we have*

$$\sum_{n=0}^{+\infty} S_n(A_3)S_n(E_3)z^n = \frac{P_1(z)}{D_1(z)}, \tag{3.1}$$

with

$$\begin{aligned}
P_1(z) = 1 &- S_2(-A_3)S_2(-E_3)z^2\\
&+ \left[S_3(-A_3)\left(S_2(-E_3)S_1(-E_3) - 2S_3(-E_3)\right) + S_1(-A_3)S_3(-E_3)\right]z^3\\
&- S_1(-A_3)S_1(-E_3)S_3(-A_3)S_3(-E_3)z^4 + S_3^2(-A_3)S_3^2(-E_3)z^6,
\end{aligned}$$

and

$$D_1(z) = \left(\sum_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)\left(\sum_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\right).$$

Proof. We have

$$\sum_{n=0}^{+\infty} S_n(A_3)e_1^n z^n = \frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n}.$$

Action the operator $\pi_{e_2e_3}\pi_{e_1e_2}$ on the identity above gives us

$$\pi_{e_2e_3}\pi_{e_1e_2}\sum_{n=0}^{+\infty} S_n(A_3)e_1^n z^n = \pi_{e_2e_3}\pi_{e_1e_2}\frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n}.$$

Lemmas 1 and 2 then allow us to write

$$\begin{aligned}
&\sum_{n=0}^{+\infty} S_n(A_3)S_n(E_3)z^n \\
&= \Bigg[\pi_{e_2e_3}\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\Big)^{-1} - (e_1S_2(-A_3)z^2 + e_1^2S_3(-A_3)z^3)\pi_{e_2e_3}e_2\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\Big)^{-1} \\
&\quad - e_1S_3(-A_3)z^3\pi_{e_2e_3}e_2^2\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\Big)^{-1}\Bigg]\Bigg[\sum_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\Bigg]^{-1},
\end{aligned}$$

after the Lemmas 3, 4, and 5, we obtain

$$\begin{aligned}
&\sum_{n=0}^{+\infty} S_n(A_3)S_n(E_3)z^n \\
&= \Big[1 - e_2e_3\left(S_2(-A_3)z^2 + S_3(-A_3)S_1(e_2+e_3)z^3\right) \\
&\quad - \left(e_1S_2(-A_3)z^2 + e_1^2S_3(-A_3)z^3\right)\left((e_2+e_3) + e_2e_3S_1(-A_3)z - e_2^2e_3^2S_3(-A_3)z^3\right) \\
&\quad - e_1S_3(-A_3)z^3\left[S_2(e_2+e_3) + e_2e_3S_1(-A_3)S_1(e_2+e_3)z + e_2^2e_3^2S_2(-A_3)z^2\right]\Big] \\
&\quad \times \left[\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\Big)\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\Big)\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\Big)\right]^{-1}
\end{aligned}$$

By reorganizing the numerator according to the powers of z, we obtain

$$\begin{aligned}
&\sum_{n=0}^{+\infty} S_n(A_3)S_n\left(E_3\right)z^n \\
&= \Big[1 - S_2(-A_3)(e_1e_2 + e_1e_3 + e_2e_3)z^2 \\
&\quad - \Big(S_3(-A_3)(e_2^2e_3 + e_2e_3^2 + e_1^2e_2 + e_1^2e_3 + e_1e_2^2 + e_1e_3^2 + e_1e_2e_3) + e_1e_2e_3S_1(-A_3)\Big)z^3 \\
&\quad - S_1(-A_3)S_3(-A_3)\left(e_1^2e_2e_3 + e_1e_2^2e_3 + e_1e_2e_3^2\right)z^4 + e_1^2e_2^2e_3^2S_3^2(-A_3)z^6\Big] \\
&\quad \times \left[\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\Big)\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\Big)\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\Big)\right]^{-1}
\end{aligned}$$

we will have

$$\sum_{n=0}^{+\infty} S_n(A_3)S_n\left(E_3\right)z^n \quad = \quad \frac{P_1(z)}{D_1(z)}.$$

Thus, this completes the proof. $\square$

Based on the relation (3.1), we have

$$\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-1}(E_3)z^n = \frac{P_2(z)}{D_1(z)}, \tag{3.2}$$

with

$$\begin{aligned} P_2(z) = z &- S_2(-A_3)S_2(-E_3)\,z^3 \\ &+ [S_3(-A_3)\,(S_2(-E_3)\,S_1(-E_3) - 2S_3(-E_3)) + S_1(-A_3)\,S_3(-E_3)]\,z^4 \\ &- S_1(-A_3)S_1(-E_3)\,S_3(-A_3)\,S_3(-E_3)\,z^5 + S_3^2(-A_3)S_3^2(-E_3)\,z^7. \end{aligned}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-2}(E_3)z^n = \frac{P_3(z)}{D_1(z)}, \tag{3.3}$$

with

$$\begin{aligned} P_3(z) = z^2 &- S_2(-A_3)S_2(-E_3)\,z^4 \\ &+ [S_3(-A_3)\,(S_2(-E_3)\,S_1(-E_3) - 2S_3(-E_3)) + S_1(-A_3)\,S_3(-E_3)]\,z^5 \\ &- S_1(-A_3)S_1(-E_3)\,S_3(-A_3)\,S_3(-E_3)\,z^6 + S_3^2(-A_3)S_3^2(-E_3)\,z^8. \end{aligned}$$

Theorem 2. *Let $A_3 = \{a_1, a_2, a_3\}$ and $E_3 = \{e_1, e_2, e_3\}$ be two alphabets. Then we have*

$$\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n = \frac{P_4(z)}{D_1(z)}, \tag{3.4}$$

with

$$\begin{aligned} P_4(z) = &-S_1(-A_3)z + S_2(-A_3)S_1(-E_3)z^2 - S_3(-A_3)\left[S_1^2(-E_3) - S_2(-E_3)\right]z^3 \\ &+ \left[S_1(-A_3)S_3(-A_3)S_3(-E_3) - S_2^2(-A_3)S_3(-E_3)\right]z^4 \\ &+ S_2(-A_3)S_3(-A_3)S_1(-E_3)S_3(-E_3)z^5 - S_3^2(-A_3)S_2(-E_3)S_3(-E_3)z^6. \end{aligned}$$

P r o o f. We have (see[16])

$$\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(e_1+e_2)z^n = \frac{-S_1(-A_3)z - (e_1+e_2)S_2(-A_3)z^2 - ((e_1+e_2)^2 - e_1e_2)S_3(-A_3)z^3}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)}.$$

Suffices to apply the operator $\pi_{e_2e_3}$to the identity above, we obtain

$$\begin{aligned} &\pi_{e_2e_3}\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(e_1+e_2)z^n \\ &= \frac{\pi_{e_2e_3}-S_1(-A_3)z - (e_1+e_2)S_2(-A_3)z^2 - ((e_1+e_2)^2 - e_1e_2)S_3(-A_3)z^3}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)}, \end{aligned}$$

$$\begin{aligned} &\sum_{n=0}^{+\infty} S_n(A_3)\pi_{e_2e_3}S_{n-1}(e_1+e_2)z^n \\ &= e_2\frac{-S_1(-A_3)z - (e_1+e_2)S_2(-A_3)z^2 - \left((e_1+e_2)^2 - e_1e_2\right)S_3(-A_3)z^3}{(e_2-e_3)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)} \end{aligned}$$

$$- e_3 \frac{-S_1(-A_3)z - (e_1+e_3)S_2(-A_3)z^2 - ((e_1+e_3)^2 - e_1e_3)S_3(-A_3)z^3}{(e_2-e_3)\Big(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\Big)\Big(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\Big)},$$

$$\begin{aligned}
&\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(e_1+e_2+e_3)z^n \\
&= \Big[-(e_2-e_3)S_1(-A_3)z - (e_2-e_3)(e_1+e_2+e_3)S_2(-A_3)z^2 \\
&\quad - (e_2-e_3)S_3(-A_3)\left((e_1+e_2+e_3)^2 - (e_1e_2+e_1e_3+e_2e_3)\right)z^3 \\
&\quad - (e_2-e_3)e_1e_2e_3\left(S_1(-A_3)S_3(-A_3) - S_2^2(-A_3)\right)z^4 \\
&\quad + (e_2-e_3)e_1e_2e_3(e_1+e_2+e_3)S_2(-A_3)S_3(-A_3)z^5 \\
&\quad + (e_2-e_3)e_1e_2e_3(e_1e_3+e_1e_2+e_2e_3)S_3^2(-A_3)z^6\Big] \\
&\quad \times \left[(e_2-e_3)\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\Big)\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\Big)\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\Big)\right]^{-1},
\end{aligned}$$

$$\begin{aligned}
&\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n \\
&= \Big[-S_1(-A_3)z + S_2(-A_3)S_1(-E_3)z^2 - S_3(-A_3)\left(S_1^2(-E_3) - S_2(-E_3)\right)z^3 \\
&+ S_3(-E_3)\left(S_1(-A_3)S_3(-A_3) - S_2^2(-A_3)\right)z^4 \\
&+ S_2(-A_3)S_3(-A_3)S_1(-E_3)S_3(-E_3)z^5 \\
&- S_3^2(-A_3)S_2(-E_3)S_3(-E_3)z^6\Big] \\
&\quad \times \left[\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\Big)\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\Big)\Big(\sum_{n=0}^{+\infty} S_n(-A_3)e_3^n z^n\Big)\right]^{-1}.
\end{aligned}$$

Therefore

$$\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n = \frac{P_4(z)}{D_1(z)}.$$

This completes the proof. □

From (3.4) we conclude the following relationships

$$\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-2}(E_3)z^n = \frac{P_5(z)}{D_1(z)}, \tag{3.5}$$

with

$$\begin{aligned}
P_5(z) = \; & -S_1(-A_3)z^2 + S_2(-A_3)S_1(-E_3)z^3 - S_3(-A_3)\left[S_1^2(-E_3) - S_2(-E_3)\right]z^4 \\
& + \left[S_1(-A_3)S_3(-A_3)S_3(-E_3) - S_2^2(-A_3)S_3(-E_3)\right]z^5 \\
& + S_2(-A_3)S_3(-A_3)S_1(-E_3)S_3(-E_3)z^6 - S_3^2(-A_3)S_2(-E_3)S_3(-E_3)z^7.
\end{aligned}$$

Theorem 3. *Let A_3 and E_3 be two alphabets, defined by $\{a_1, a_2, a_3\}$, $\{e_1, e_2, e_3\}$, respectively. Then we have*

$$\sum_{n=0}^{+\infty} S_{n-1}(A_3) S_n(E_3) z^n = \frac{P_6(z)}{D_1(z)}, \tag{3.6}$$

with

$$\begin{aligned} P_6(z) = & -S_1(-E_3)z + S_1(-A_3)S_2(-E_3)z^2 - S_3(-E_3)\left(S_1^2(-A_3) - S_2(-A_3)\right)z^3 \\ & - S_3(-A_3)\left(S_2^2(-E_3) - S_1(-E_3)S_3(-E_3)\right)z^4 \\ & + S_1(-A_3)S_3(-A_3)S_2(-E_3)S_3(-E_3)z^5 - S_3^2(-E_3)S_2(-A_3)S_3(-A_3)z^6. \end{aligned}$$

Proof. By the same method given in Theorem 2, the proof can be easily made. □

Based to the relation (3.6), we obtain

$$\sum_{n=0}^{+\infty} S_{n-2}(A_3) S_{n-1}(E_3) z^n = \frac{P_7(z)}{D_1(z)}, \tag{3.7}$$

with

$$\begin{aligned} P_7(z) = & -S_1(-E_3)z^2 + S_1(-A_3)S_2(-E_3)z^3 - S_3(-E_3)\left(S_1^2(-A_3) - S_2(-A_3)\right)z^4 \\ & - S_3(-A_3)\left(S_2^2(-E_3) - S_1(-E_3)S_3(-E_3)\right)z^5 \\ & + S_1(-A_3)S_3(-A_3)S_2(-E_3)S_3(-E_3)z^6 - S_3^2(-E_3)S_2(-A_3)S_3(-A_3)z^7. \end{aligned}$$

Theorem 4. *Let $A_3 = \{a_1, a_2, a_3\}$ and $E_3 = \{e_1, e_2, e_3\}$ be two alphabets. Then we have*

$$\sum_{n=0}^{+\infty} S_{n-2}(A_3) S_n(E_3) z^n = \frac{P_8(z)}{D_1(z)}, \tag{3.8}$$

with

$$\begin{aligned} P_8(z) = & \left(S_1^2(-E_3) - S_2(-E_3)\right)z^2 - S_1(-A_3)\left(S_1(-E_3)S_2(-E_3) - S_3(-E_3)\right)z^3 \\ & + \left[S_1^2(-A_3)S_1(-E_3)S_3(-E_3) + S_2(-A_3)\left(S_2^2(-E_3) - 2S_1(-E_3)S_3(-E_3)\right)\right]z^4 \\ & - S_2(-E_3)S_3(-E_3)\left[S_1(-A_3)S_2(-A_3) - S_3(-A_3)\right]z^5 \\ & + \left[S_2^2(-A) - S_1(-A_3)S_3(-A_3)\right]S_3^2(-E_3)z^6. \end{aligned}$$

Proof. We have (see [16])

$$\sum_{n=0}^{+\infty} S_{n-2}(A_3) S_n(e_1+e_2) z^n = \frac{\left((e_1+e_2)^2 - e_1e_2\right)z^2 + e_1e_2\left(e_1+e_2\right)S_1(-A_3)z^3 + e_1^2e_2^2S_2(-A_3)z^4}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_1^n z^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A_3)e_2^n z^n\right)}.$$

By following the same demonstration steps as those of identity (3.4), we get (3.8). □

From (3.8), we have

$$\sum_{n=0}^{+\infty} S_n(A_3) S_{n-2}(E_3) z^n = \frac{P_9(z)}{D_1(z)}, \tag{3.9}$$

with

$$\begin{aligned} P_9(z) = & \left(S_1^2(-A_3) - S_2(-A_3)\right)z^2 - S_1(-E_3)\left(S_1(-A_3)S_2(-A_3) - S_3(-A_3)\right)z^3 \\ & + \left[S_1^2(-E_3)S_1(-A_3)S_3(-A_3) + S_2(-E_3)\left(S_2^2(-A_3) - 2S_1(-A_3)S_3(-A_3)\right)\right]z^4 \\ & - S_2(-A_3)S_3(-A_3)\left[S_1(-E_3)S_2(-E_3) - S_3(-E_3)\right]z^5 \end{aligned}$$

$$+\left[S_2^2(-E)-S_1(-E_3)S_3(-E_3)\right]S_3^2(-A_3)z^6,$$

and

$$\begin{aligned}D_1(z)=1&-\left[S_1(-E_3)S_1(-A_3)+S_2(-E_3)S_1^2(-A_3)\right]z\\&+\left[S_2(-A_3)S_1^2(-E_3)-2S_2(-E_3)S_2(-A_3)\right]z^2\\&-\Big[S_1^3(-E_3)S_3(-A_3)+S_3(-E_3)\left(S_1^3(-A_3)-6S_3(-A_3)\right)\\&\quad+\left(S_1(-A_3)S_2(-A_3)-3S_3(-A_3)\right)\left(S_1(-E_3)S_2(-E_3)-3S_3(-E_3)\right)\Big]z^3\\&+\Big[S_3(-E_3)S_1(-E_3)S_2(-A_3)S_1^2(-A_3)\\&\quad+\left(S_2^2(-A_3)-2S_3(-A_3)S_1(-A_3)\right)\left(S_2^2(-E_3)-2S_3(-E_3)S_1(-E_3)\right)\\&\quad+S_3(-A_3)S_1(-A_3)\left(S_1^2(-E_3)S_2(-E_3)-5S_3(-E_3)S_1(-E_3)\right)\Big]z^4\\&-\Big[S_3(-E_3)S_2(-E_3)\left(S_1(-A_3)S_2^2(-A_3)+2S_3(-A_3)S_1(-A_3)-S_3(-A_3)S_2(-A_3)\right)\\&\quad+S_3(-E_3)S_1^2(-E_3)\left(S_3(-A_3)S_1^2(-A_3)-2S_3(-A_3)S_2(-A_3)\right)\\&\quad+S_1(-E_3)S_2(-E_3)S_2(-A_3)S_3(-A_3)\Big]z^5\\&+\Big[S_3^2(-E_3)S_2^3(-A_3)+S_3(-E_3)S_3(-A_3)\left(S_1(-A_3)S_2(-A_3)-3S_3(-A_3)\right)\\&\quad\times\left(S_1(-E_3)S_2(-E_3)-3S_3(-E_3)\right)+S_3^2(-A_3)\left(S_2^3(-E_3)-6S_3^2(-E_3)\right)\Big]z^6\\&-\Big[S_3^2(-E_3)S_1(-E_3)S_3(-A_3)\left(S_2^2(-A_3)-2S_3(-A_3)S_1(-A_3)\right)\\&\quad+S_3^2(-A_3)S_1(-A_3)S_3(-E_3)S_2(-E_3)\Big]z^7\\&+S_3^2(-E_3)S_2(-E_3)S_3^2(-A_3)S_2(-A_3)z^8-S_3^3(-E_3)S_3^3(-A_3)z^9.\end{aligned}$$

4. Applications

All the identities appearing in the following section are corollaries of (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) and (3.9).

4.1. Generating functions of square of some known numbers

In this part, we now derive the new generating functions of the square of some known numbers. This case consists of four related parts. **Firstly**, the substitutions

$$\begin{cases}S_1(-A_3)=-1\\S_2(-A_3)=-1\\S_3(-A_3)=-1\end{cases}\quad\text{and}\quad\begin{cases}S_1(-E_3)=-1\\S_2(-E_3)=-1\\S_3(-E_3)=-1\end{cases}$$

in (3.1)–(3.5) and (3.9), we have

$$\sum_{n=0}^{+\infty}S_n(A_3)S_n(E_3)z^n=\frac{1-z^2-2z^3-z^4+z^6}{1-z-4z^2-12z^3-6z^4-4z^5+8z^6+2z^7+z^8-z^9},\tag{4.1}$$

$$\sum_{n=0}^{+\infty}S_{n-1}(A_3)S_{n-1}(E_3)z^n=\frac{z-z^3-2z^4-z^5+z^7}{1-z-4z^2-12z^3-6z^4-4z^5+8z^6+2z^7+z^8-z^9},\tag{4.2}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-2}(E_3)z^n = \frac{z^2 - z^4 - 2z^5 - z^6 + z^8}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9}, \tag{4.3}$$

$$\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n = \frac{z + z^2 + 2z^3 + z^5 - z^6}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9}, \tag{4.4}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-2}(E_3)z^n = \frac{z^2 + z^3 + 2z^4 + z^6 - z^7}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9}, \tag{4.5}$$

$$\sum_{n=0}^{+\infty} S_n(A_3)S_{n-2}(E_3)z^n = \frac{2z^2 + 2z^3 + 2z^4 - 2z^5}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9}. \tag{4.6}$$

Then, we have the following proposition and theorem.

Proposition 3. *For $n \in \mathbb{N}$, the new generating function of the square of Tribonacci numbers is given by*

$$\sum_{n=0}^{+\infty} T_n^2 z^n = \frac{1 - z^2 - 2z^3 - z^4 + z^6}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9}.$$

We can state the following corollary.

Corollary 2. *The following identity holds true:*

$$T_n^2 = S_n(A_3)S_n(E_3).$$

Theorem 5. *For $n \in \mathbb{N}$, the new generating function of the square of Tribonacci Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} K_n^2 z^n = \frac{9 - 8z - 28z^2 - 54z^3 - 22z^4 - 6z^5 + 24z^6 + z^8}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9}.$$

Proof. In [10], we have $K_n = 3S_n(A_3) - 2S_{n-1}(A_3) - S_{n-2}(A_3)$, then we see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} K_n^2 z^n &= \sum_{n=0}^{+\infty} [3S_n(A_3) - 2S_{n-1}(A_3) - S_{n-2}(A_3)]\,[3S_n(E_3) - 2S_{n-1}(E_3) - S_{n-2}(E_3)]\, z^n \\
&= 9\sum_{n=0}^{+\infty} S_n(A_3)S_n(E_3)z^n + 4\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-1}(E_3)z^n + \sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-2}(E_3)z^n \\
&\quad - 12\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n - 6\sum_{n=0}^{+\infty} S_n(A_3)S_{n-2}(E_3)z^n + 4\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-2}(E_3)z^n \\
&= 9\sum_{n=0}^{+\infty} T_n^2 z^n + 4\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-1}(E_3)z^n + \sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-2}(E_3)z^n \\
&\quad - 12\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n - 6\sum_{n=0}^{+\infty} S_n(A_3)S_{n-2}(E_3)z^n + 4\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-2}(E_3)z^n.
\end{aligned}$$

By using (4.1)–(4.6), we obtain

$$\sum_{n=0}^{+\infty} K_n^2 z^n = \frac{9 - 8z - 28z^2 - 54z^3 - 22z^4 - 6z^5 + 24z^6 + z^8}{1 - z - 4z^2 - 12z^3 - 6z^4 - 4z^5 + 8z^6 + 2z^7 + z^8 - z^9},$$

which completes the proof of this theorem. □

Secondly, let us now consider the following conditions for Eqs. (3.1)–(3.5) and (3.9):

$$\begin{cases} S_1(-A_3) = -1 \\ S_2(-A_3) = -1 \\ S_3(-A_3) = -2 \end{cases} \quad \text{and} \quad \begin{cases} S_1\left(-E_3\right) = -1 \\ S_2\left(-E_3\right) = -1 \\ S_3\left(-E_3\right) = -2 \end{cases}.$$

Then it yields

$$\sum_{n=0}^{+\infty} S_n(A_3)S_n(E_3)z^n = \frac{1 - z^2 - 8z^3 - 4z^4 + 16z^6}{1 - z - 4z^2 - 29z^3 - 15z^4 - 16z^5 + 92z^6 + 32z^7 + 16z^8 - 64z^9}, \tag{4.7}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-1}(E_3)z^n = \frac{z - z^3 - 8z^4 - 4z^5 + 16z^7}{1 - z - 4z^2 - 29z^3 - 15z^4 - 16z^5 + 92z^6 + 32z^7 + 16z^8 - 64z^9}, \tag{4.8}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-2}(E_3)z^n = \frac{z^2 - z^4 - 8z^5 - 4z^6 + 16z^8}{1 - z - 4z^2 - 29z^3 - 15z^4 - 16z^5 + 92z^6 + 32z^7 + 16z^8 - 64z^9}, \tag{4.9}$$

$$\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n = \frac{z + z^2 + 4z^3 - 2z^4 + 4z^5 - 8z^6}{1 - z - 4z^2 - 29z^3 - 15z^4 - 16z^5 + 92z^6 + 32z^7 + 16z^8 - 64z^9}, \tag{4.10}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-2}(E_3)z^n = \frac{z^2 + z^3 + 4z^4 - 2z^5 + 4z^6 - 8z^7}{1 - z - 4z^2 - 29z^3 - 15z^4 - 16z^5 + 92z^6 + 32z^7 + 16z^8 - 64z^9}, \tag{4.11}$$

$$\sum_{n=0}^{+\infty} S_n(A_3)S_{n-2}(E_3)z^n = \frac{2z^2 + 3z^3 + 5z^4 - 6z^5 - 4z^6}{1 - z - 4z^2 - 29z^3 - 15z^4 - 16z^5 + 92z^6 + 32z^7 + 16z^8 - 64z^9}. \tag{4.12}$$

Therefore, we state the following proposition and theorem.

Proposition 4. *For $n \in \mathbb{N}$, the new generating function of square of third order of Jacobsthal numbers is given by*

$$\sum_{n=0}^{+\infty} \left(J_n^{(3)}\right)^2 z^n = \frac{z - z^3 - 8z^4 - 4z^5 + 16z^7}{1 - z - 4z^2 - 29z^3 - 15z^4 - 16z^5 + 92z^6 + 32z^7 + 16z^8 - 64z^9}.$$

We can state the following corollary.

Corollary 3. *The following identity holds true:*

$$\left(J_n^{(3)}\right)^2 = S_{n-1}(A_3)S_{n-1}\left(E_3\right).$$

Theorem 6. *For $n \in \mathbb{N}$, the new generating function of square of third order of Jacobsthal Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} \left(j_n^{(3)}\right)^2 z^n = \frac{4 - 3z + 8z^2 - 29z^3 + 4z^4 - 92z^5 + 32z^6 + 48z^7 + 64z^8}{1 - z - 4z^2 - 29z^3 - 15z^4 - 16z^5 + 92z^6 + 32z^7 + 16z^8 - 64z^9}.$$

P r o o f. By referred to [10], we have $j_n^{(3)} = 2S_n(A_3) - S_{n-1}(A_3) + 2S_{n-2}(A_3)$, then

$$\begin{aligned}
\sum_{n=0}^{+\infty} \left(j_n^{(3)}\right)^2 z^n &= \sum_{n=0}^{+\infty} \left[2S_n(A_3) - S_{n-1}(A_3) + 2S_{n-2}(A_3)\right]\left[2S_n(E_3) - S_{n-1}(E_3) + 2S_{n-2}(E_3)\right] z^n \\
&= 4\sum_{n=0}^{+\infty} S_n(A_3)S_n(E_3)z^n - 4\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n + 8\sum_{n=0}^{+\infty} S_n(A_3)S_{n-2}(E_3)z^n \\
&\quad - 4\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-2}(E_3)z^n + \sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-1}(E_3)z^n \\
&\quad + 4\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-2}(E_3)z^n \\
&= 4\sum_{n=0}^{+\infty} S_n(A_3)S_n(E_3)z^n - 4\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n + 8\sum_{n=0}^{+\infty} S_n(A_3)S_{n-2}(E_3)z^n \\
&\quad - 4\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-2}(E_3)z^n + \sum_{n=0}^{+\infty} \left(J_n^{(3)}\right)^2 z^n \\
&\quad + 4\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-2}(E_3)z^n.
\end{aligned}$$

By using (4.7), (4.8), (4.9), (4.10), (4.11), and (4.12), we obtain

$$\sum_{n=0}^{+\infty} \left(j_n^{(3)}\right)^2 z^n = \frac{4 - 3z + 8z^2 - 29z^3 + 4z^4 - 100z^5 + 32z^6 + 48z^7 + 64z^8}{1 - z - 4z^2 - 29z^3 - 15z^4 - 16z^5 + 92z^6 + 32z^7 + 16z^8 - 64z^9},$$

which completes the proof of this theorem. □

Thirdly, the substitutions

$$\begin{cases} S_1(-A_3) = -1 \\ S_2(-A_3) = 0 \\ S_3(-A_3) = -1 \end{cases} \quad \text{and} \quad \begin{cases} S_1(-E_3) = -1 \\ S_2(-E_3) = 0 \\ S_3(-E_3) = -1 \end{cases}$$

in (3.2), we obtain the following proposition.

Proposition 5. *For $n \in \mathbb{N}$, the new generating function of square of Narayana numbers is given by*

$$\sum_{n=0}^{+\infty} N_n^2 z^n = \frac{z - z^4 - z^5 + z^7}{1 - z - 5z^3 - z^4 - z^5 + 3z^6 + 2z^7 - z^9}.$$

We can state the following corollary.

Corollary 4. *The following identity holds true:*

$$N_n^2 = S_{n-1}(A_3)S_{n-1}(E_3).$$

Fourthly, the substitutions

$$\begin{cases} S_1(-A_3) = 0 \\ S_2(-A_3) = -1 \\ S_3(-A_3) = -1 \end{cases} \quad \text{and} \quad \begin{cases} S_1(-E_3) = 0 \\ S_2(-E_3) = -1 \\ S_3(-E_3) = -1 \end{cases}$$

in (3.3) we deduce the following proposition.

Proposition 6. *For $n \in \mathbb{N}$, the new generating function of square of Padovan Perin numbers is given by*

$$\sum_{n=0}^{+\infty} S_n^2 z^n = \frac{z^2 - z^4 - 2z^5 + z^8}{1 - 2z^2 - 3z^3 + z^4 + z^5 + z^6 + z^8 - z^9}.$$

Corollary 5. *The following identity holds true:*

$$S_n^2 = S_{n-2}(A_3) S_{n-2}(E_3).$$

4.2. Generating functions of product of some known polynomials

In this part, we now derive the new generating functions of the product of some known polynomials. This case consists of three related parts. **Firstly,** let us now consider the following conditions for equations (3.1)–(3.9):

$$\begin{cases} S_1(-A_3) = -x^2 \\ S_2(-A_3) = -x \\ S_3(-A_3) = -1 \end{cases} \quad \text{and} \quad \begin{cases} S_1(-E_3) = -y^2 \\ S_2(-E_3) = -y \\ S_3(-E_3) = -1 \end{cases}.$$

Then it yields

$$\sum_{n=0}^{+\infty} S_n(A_3) S_n(E_3) z^n = \frac{1 - xyz^2 + \left(x^2 - y^3 - 2\right) z^3 - x^2y^2z^4 + z^6}{D_2(z)}, \tag{4.13}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(A_3) S_{n-1}(E_3) z^n = \frac{z - xyz^3 + \left(x^2 - y^3 - 2\right) z^4 - x^2y^2z^5 + z^7}{D_2(z)}, \tag{4.14}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(A_3) S_{n-2}(E_3) z^n = \frac{z^2 - xyz^4 + \left(x^2 - y^3 - 2\right) z^5 - x^2y^2z^6 + z^8}{D_2(z)}, \tag{4.15}$$

$$\sum_{n=0}^{+\infty} S_n(A_3) S_{n-1}(E_3) z^n = \frac{x^2z + xy^2z^2 + y\left(1 + y^3\right) z^3 + xy^2z^5 - yz^6}{D_2(z)}, \tag{4.16}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(A_3) S_{n-2}(E_3) z^n = \frac{x^2z^2 + xy^2z^3 + \left(1 + y^3\right) yz^4 + xy^2z^6 - yz^7}{D_2(z)}, \tag{4.17}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(A_3) S_n(E_3) z^n = \frac{y^2z + x^2yz^2 + \left(1 + x^3\right) xz^3 + x^2yz^5 - xz^6}{D_2(z)}, \tag{4.18}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(A_3) S_{n-1}(E_3) z^n = \frac{y^2z^2 + x^2yz^3 + \left(1 + x^3\right) xz^4 + x^2yz^6 - xz^7}{D_2(z)}, \tag{4.19}$$

$$\sum_{n=0}^{+\infty} S_{n-2}(A_3) S_n(E_3) z^n = \frac{\left(1 + y^3\right) yz^2 + \left(1 + y^3\right) x^2z^3 + xy^2\left(1 + x^3\right) z^4 - \left(1 + x^3\right) yz^5}{D_2(z)}, \tag{4.20}$$

$$\sum_{n=0}^{+\infty} S_n(A_3) S_{n-2}(E_3) z^n = \frac{\left(1 + x^3\right) xz^2 + \left(1 + x^3\right) y^2z^3 + \left(1 + y^3\right) x^2yz^4 - \left(1 + y^3\right) xz^5}{D_2(z)}, \tag{4.21}$$

with

$$\begin{aligned} D_2(z) = 1 &- x^2y^2z - xy\left(x^3 + y^3 + 2\right) z^2 - \left(x^6 + y^6 + x^3y^3 + 3x^3 + 3y^3 + 3\right) z^3 \\ &- x^2y^2\left(x^3 + y^3 + 4\right) z^4 + xy\left(x^3 - 2x + 1 - y^2\left(\left(x^3 + 2\right) y + 1\right)\right) z^5 \\ &+ \left(2y^3 + 2x^3 + x^3y^3 + 3\right) z^6 + x^2y\left(y + 1\right) z^7 + xyz^8 - z^9. \end{aligned}$$

Therefore, we state the following results.

Proposition 7. *Let n be a natural number. Then we have the new generating function for the product of Tribonacci polynomials:*

$$\begin{aligned}\sum_{n=0}^{+\infty} T_n(x)T_n(y)\,z^n = {} & \left[1 - xyz^2 + \left(x^2 - y^3 - 2\right)z^3 - x^2y^2z^4 + z^6\right] \\ & \times \Big[1 - x^2y^2z - xy\left(x^3 + y^3 + 2\right)z^2 - \left(x^6 + y^6 + x^3y^3 + 3x^3 + 3y^3 + 3\right)z^3 \\ & - x^2y^2\left(x^3 + y^3 + 4\right)z^4 + xy\left(x^3 - 2x + 1 - y^2\left(\left(x^3 + 2\right)y + 1\right)\right)z^5 \\ & + \left(2y^3 + 2x^3 + x^3y^3 + 3\right)z^6 + x^2y\left(y + 1\right)z^7 + xyz^8 - z^9\Big]^{-1}.\end{aligned}$$

We can state the following corollary.

Corollary 6. *The following identity holds true:*

$$T_n(x)T_n(y) = S_n(A_3)S_n(E_3).$$

Theorem 7. *Let n be a natural number. Then we have the new generating function for the product of Tribonacci Lucas polynomials:*

$$\sum_{n=0}^{+\infty} K_n(x)K_n(y)\,z^n = \frac{P_{10}(z)}{D_2(z)},$$

and

$$\begin{aligned}P_{10}(z) = {} & 9 - 8x^2y^2z - 7xy\left(2 + x^3 + y^3\right)z^2 \\ & + 3\left(3x^2 - 3x^3 - 2x^6 - 6y^3 - 2y^6 - 2x^3y^3 - 6\right)z^3 \\ & - x^2y^2\left(20 + 5y^3 + x^3 - 4x^2\right)z^4 - xy\left(4x^3y^3 - x^2 + 4y^3 + 3x^3 - 4\right)z^5 \\ & + 3\left(3 + x^3y^3 + 2x^3 + 2y^3\right)z^6 + xyz^8.\end{aligned}$$

P r o o f. Recall that, we have (see [16]) $K_n(x) = 3S_n(A_3) - 2x^2S_{n-1}(A_3) - xS_{n-2}(A_3)$.

$$\begin{aligned}\sum_{n=0}^{+\infty} K_n(x)K_n(y)z^n = {} & \sum_{n=0}^{+\infty}\left[3S_n(A_3) - 2x^2S_{n-1}(A_3) - xS_{n-2}(A_3)\right] \\ & \quad \times \left[3S_n(E_3) - 2y^2S_{n-1}(E_3) - yS_{n-2}(E_3)\right]z^n \\ = {} & 9\sum_{n=0}^{+\infty} S_n(A_3)S_n(E_3)z^n + 4x^2y^2\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-1}(E_3)z^n \\ & \quad + xy\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-2}(E_3)z^n - 6y^2\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n \\ & \quad - 3y\sum_{n=0}^{+\infty} S_n(A_3)S_{n-2}(E_3)z^n - 6x^2\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_n(E_3)z^n \\ & \quad + 2x^2y\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-2}(E_3)z^n - 3x\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_n(E_3)z^n \\ & \quad + 2xy^2\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-1}(E_3)z^n\end{aligned}$$

$$\begin{aligned}
&= 9\sum_{n=0}^{+\infty} T_n(x)T_n(y)z^n + 4x^2y^2\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-1}(E_3)z^n \\
&\quad + xy\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-2}(E_3)z^n - 6y^2\sum_{n=0}^{+\infty} S_n(A_3)S_{n-1}(E_3)z^n \\
&\quad - 3y\sum_{n=0}^{+\infty} S_n(A_3)S_{n-2}(E_3)z^n - 6x^2\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_n(E_3)z^n \\
&\quad + 2x^2y\sum_{n=0}^{+\infty} S_{n-1}(A_3)S_{n-2}(E_3)z^n - 3x\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_n(E_3)z^n \\
&\quad + 2xy^2\sum_{n=0}^{+\infty} S_{n-2}(A_3)S_{n-1}(E_3)z^n.
\end{aligned}$$

By using (4.13)–(4.21), we obtain

$$\sum_{n=0}^{+\infty} K_n(x)K_n(y)z^n = \frac{P_{10}(z)}{D_2(z)},$$

which completes the proof of this theorem. □

Secondly, the substitutions

$$\begin{cases} S_1(-A_3) = -1 \\ S_2(-A_3) = -x \\ S_3(-A_3) = -x^2 \end{cases} \quad \text{and} \quad \begin{cases} S_1(-E_3) = -1 \\ S_2(-E_3) = -y \\ S_3(-E_3) = -y^2 \end{cases}$$

in (3.1), we have the following proposition.

Proposition 8. *We have the following new generating function of the product of Tricobsthal polynomials:*

$$\sum_{n=0}^{+\infty} J_n^{(3)}(x)J_n^{(3)}(y)z^n = \frac{1 - xyz^2 + y\left(y - x^2(1+2y)\right)z^3 - x^2y^2z^4 + x^4y^4z^6}{D_3(z)},$$

with

$$\begin{aligned}
D_3(z) = 1 &- z - (x + y + 2xy)z^2 - \left(y^2 + x^2 + 3x^2y^2 + xy + 3x^2y + 3xy^2\right)z^3 \\
&- xy(x + 4xy + y)z^4 - x^2y\left(x + y + 2xy - xy^2 + y^2\right)z^5 \\
&+ x^3y^3(2x + 2y + 3xy + 1)z^6 + x^4y^3(1+y)z^7 + x^5y^5z^8 - x^6y^6z^9.
\end{aligned}$$

Corollary 7. *The following identity holds true:*

$$J_n^{(3)}(x)J_n^{(3)}(y) = S_n(A_3)S_n(E_3).$$

Thirdly, the substitutions

$$\begin{cases} S_1(-A_3) = -x \\ S_2(-A_3) = \alpha \\ S_3(-A_3) = -\gamma \end{cases} \quad \text{and} \quad \begin{cases} S_1(-E_3) = -y \\ S_2(-E_3) = \alpha \\ S_3(-E_3) = -\gamma \end{cases}$$

in (3.1), we get the following proposition.

Proposition 9. *Let n be a natural number. Then we have the new generating function for the product of 2-Chebyshev polynomials of first kind:*

$$\sum_{n=0}^{+\infty} \widehat{T}_n(x)\widehat{T}_n(y) z^n = \frac{1-\alpha^2 z^2 + \gamma(x-2\gamma+\alpha y) z^3 - \gamma^2 xyz^4 + \gamma^4 z^6}{D_4(z)},$$

with

$$\begin{aligned} D_4(z) = 1 - xyz &+ \alpha\left(x^2+y^2-2\alpha\right) z^2 - \left(\gamma y^3 + \gamma x^3 + 3\gamma^2 + \alpha^2 xy - 3\alpha\gamma y - 3\alpha\gamma x\right) z^3 \\ &+ \left(\alpha\gamma x^2 y + \alpha\gamma x y^2 - \gamma^2 xy - 2\alpha^2\gamma x - 2\alpha^2\gamma y + \alpha^4\right) z^4 \\ &- \left(\alpha\gamma x\left(\alpha^2-2\gamma\right) + \gamma^2 y^2 x^2 - \alpha^2\gamma^2 + \alpha\gamma^2 y(\alpha-2\gamma)\right) z^5 \\ &+ \left(2\alpha^3\gamma^2 + \gamma^2(\alpha y - 3\gamma)(\alpha x - 3\gamma) - 6\gamma^4\right) z^6 \\ &- \gamma^3\left(y\left(\alpha^2-2\gamma x\right)+\alpha x\right) z^7 + \alpha^2\gamma^4 z^8 - \gamma^6 z^9. \end{aligned}$$

We can state the following corollary.

Corollary 8. *The following identity holds true:*

$$\widehat{T}_n(x)\widehat{T}_n(y) = S_n(A_3) S_n(E_3).$$

4.3. The case $A_3 = \{1,2,3\}$ and $E_3 = \{1,2,3\}$

Let us now consider the following conditions for equations (3.1)–(3.5) and (3.9):

$$\begin{cases} S_1(-A_3) = S_1(-E_3) = -6 \\ S_2(-A_3) = S_2(-E_3) = 11 \\ S_3(-A_3) = S_3(-E_3) = -6 \end{cases}.$$

We have the following results.

Proposition 10. *For $n \in \mathbb{N}$, the new generating functions of Stirling numbers of the first kind is given by*

$$\begin{aligned} &\sum_{n=0}^{+\infty}\left(\frac{3^{n+2}+1}{2} - 2^{n+2}\right)^2 z^n \\ &= \frac{1-121z^2+360z^3-1296z^4+1296z^6}{(1-z)(1-2z)^2(1-3z)^2(1-4z)(1-6z)^2(1-9z)}, \end{aligned}$$

$$\begin{aligned} &\sum_{n=0}^{+\infty}\left(\frac{3^{n+1}+1}{2} - 2^{n+1}\right)^2 z^n \\ &= \frac{z-121z^3+360z^4-1296z^5+1296z^7}{(1-z)(1-2z)^2(1-3z)^2(1-4z)(1-6z)^2(1-9z)}. \end{aligned}$$

Proposition 11. *Let n be a natural number. Then we have the new generating functions of Stirling numbers of the first type:*

$$\begin{aligned} &\sum_{n=0}^{+\infty}\left(\frac{3^{n}+1}{2} - 2^{n}\right)^2 z^n \\ &= \frac{z^2-121z^4+360z^5-1296z^6+1296z^8}{(1-z)(1-4z)(1-9z)(1-2z)^2(1-3z)^2(1-6z)^2}, \end{aligned}$$

$$\sum_{n=0}^{+\infty}\Big(\frac{3^{n+2}+1}{2}-2^{n+2}\Big)\Big(\frac{3^{n+1}+1}{2}-2^{n+1}\Big)z^n$$
$$=\frac{6z-66z^2+150z^3+510z^4-2376z^5+2376z^6}{(1-z)(1-4z)(1-9z)(1-2z)^2(1-3z)^2(1-6z)^2}.$$

Proposition 12. *For $n\in\mathbb{N}$, the new generating functions of Stirling numbers of the first type is given by*

$$\sum_{n=0}^{+\infty}\Big(\frac{3^{n+1}+1}{2}-2^{n+1}\Big)\Big(\frac{3^{n}+1}{2}-2^{n}\Big)z^n$$
$$=\frac{6z^2-66z^3+150z^4+510z^5-2376z^6+2376z^7}{(1-z)(1-4z)(1-9z)(1-2z)^2(1-3z)^2(1-6z)^2},$$
$$\sum_{n=0}^{+\infty}\Big(\frac{3^{n+2}+1}{2}-2^{n+2}\Big)\Big(\frac{3^{n}+1}{2}-2^{n}\Big)z^n$$
$$=\frac{25z^2-360z^3+1835z^4-3960z^5+3060z^6}{(1-z)(1-4z)(1-9z)(1-2z)^2(1-3z)^2(1-6z)^2}.$$

4.4. The case $A_3=\{1,1,1\}$ and $E_3=\{1,1,1\}$

Let us now consider the following conditions for equations (3.1)–(3.5) and (3.9):

$$\begin{cases} S_1(-A_3)=S_1(-E_3)=-3\\ S_2(-A_3)=S_2(-E_3)=3\\ S_3(-A_3)=S_3(-E_3)=-1\end{cases}.$$

Using the same procedure, we obtain the new generating functions:

$$\sum_{n=0}^{+\infty}\binom{2n+1}{n}^2 z^n=\frac{1-9z^2+10z^3-9z^4+z^6}{1-9z+36z^2-84z^3+126z^4-126z^5+84z^6-36z^7+9z^8-z^9},$$
$$\sum_{n=0}^{+\infty}\binom{2n}{n-1}^2 z^n=\frac{z-9z^3+10z^4-9z^5+z^7}{1-9z+36z^2-84z^3+126z^4-126z^5+84z^6-36z^7+9z^8-z^9},$$
$$\sum_{n=0}^{+\infty}\binom{2n-1}{n-2}^2 z^n=\frac{z^2-9z^4+10z^5-9z^6+z^8}{1-9z+36z^2-84z^3+126z^4-126z^5+84z^6-36z^7+9z^8-z^9},$$
$$\sum_{n=0}^{+\infty}\binom{2n+1}{n}\binom{2n}{n-1} z^n=\frac{3z-9z^2+6z^3+6z^4-9z^5+3z^6}{1-9z+36z^2-84z^3+126z^4-126z^5+84z^6-36z^7+9z^8-z^9},$$
$$\sum_{n=0}^{+\infty}\binom{2n}{n-1}\binom{2n-1}{n-2} z^n=\frac{3z^2-9z^3+6z^4+6z^5-9z^6+3z^7}{1-9z+36z^2-84z^3+126z^4-126z^5+84z^6-36z^7+9z^8-z^9},$$
$$\sum_{n=0}^{+\infty}\binom{2n+1}{n}\binom{2n-1}{n-2} z^n=\frac{6z^2-24z^3+36z^4-24z^5+6z^6}{1-9z+36z^2-84z^3+126z^4-126z^5+84z^6-36z^7+9z^8-z^9}.$$

4.5. The case $A_3=\{1,1,1\}$ and $E_3=\{1,2,3\}$

Let us now consider the following conditions for equations (3.1)–(3.9) :

$$\begin{cases} S_1(-A_3)=-3\\ S_2(-A_3)=3\\ S_3(-A_3)=-1\end{cases}\quad\text{and}\quad\begin{cases} S_1(-E_3)=-6\\ S_2(-E_3)=11\\ S_3(-E_3)=-6\end{cases}.$$

We have the following propositions.

Proposition 13. *For $n \in \mathbb{N}$, the new generating functions is given by*

$$\sum_{n=0}^{+\infty} \binom{2n+1}{n} \left(\frac{3^{n+2}+1}{2} - 2^{n+2} \right) z^n$$
$$= \frac{1 - 33z^2 + 72z^3 - 108z^4 + 36z^6}{1 - 18z + 141z^2 - 630z^3 + 1767z^4 - 2034z^5 + 3815z^6 - 846z^7 + 1188z^8 - 216z^9},$$

$$\sum_{n=0}^{+\infty} \binom{2n}{n-1} \left(\frac{3^{n+1}+1}{2} - 2^{n+1} \right) z^n$$
$$= \frac{z - 33z^3 + 72z^4 - 108z^5 + 36z^7}{1 - 18z + 141z^2 - 630z^3 + 1767z^4 - 2034z^5 + 3815z^6 - 846z^7 + 1188z^8 - 216z^9}.$$

Proposition 14. *Let n be a natural number. Then we have the new generating functions*

$$\sum_{n=0}^{+\infty} \binom{2n-1}{n-2} \left(\frac{3^{n}+1}{2} - 2^{n} \right) z^n$$
$$= \frac{z^2 - 33z^4 + 72z^5 - 108z^6 + 36z^8}{1 - 18z + 141z^2 - 630z^3 + 1767z^4 - 2034z^5 + 3815z^6 - 846z^7 + 1188z^8 - 216z^9},$$

$$\sum_{n=0}^{+\infty} \binom{2n+1}{n} \left(\frac{3^{n+1}+1}{2} - 2^{n+1} \right) z^n$$
$$= \frac{3z - 18z^2 + 25z^3 + 36z^4 - 108z^5 + 66z^6}{1 - 18z + 141z^2 - 630z^3 + 1767z^4 - 2034z^5 + 3815z^6 - 846z^7 + 1188z^8 - 216z^9}.$$

Proposition 15. *For $n \in \mathbb{N}$, it is given the new generating functions by*

$$\sum_{n=0}^{+\infty} \binom{2n}{n-1} \left(\frac{3^{n+2}+1}{2} - 2^{n+2} \right) z^n$$
$$= \frac{6z - 33z^2 + 36z^3 + 85z^4 - 198z^5 + 108z^6}{1 - 18z + 141z^2 - 630z^3 + 1767z^4 - 2034z^5 + 3815z^6 - 846z^7 + 1188z^8 - 216z^9},$$

$$\sum_{n=0}^{+\infty} \binom{2n-1}{n-2} \left(\frac{3^{n+1}+1}{2} - 2^{n+1} \right) z^n$$
$$= \frac{6z^2 - 33z^3 + 36z^4 + 85z^5 - 198z^6 + 108z^7}{1 - 18z + 141z^2 - 630z^3 + 1767z^4 - 2034z^5 + 3815z^6 - 846z^7 + 1188z^8 - 216z^9}.$$

Proposition 16. *Let n be a natural number. Then we have the new generating functions*

$$\sum_{n=0}^{+\infty} \binom{2n-1}{n-2} \left(\frac{3^{n+2}+1}{2} - 2^{n+2} \right) z^n$$
$$= \frac{25z^2 - 180z^3 + 471z^4 - 528z^5 + 216z^6}{1 - 18z + 141z^2 - 630z^3 + 1767z^4 - 2034z^5 + 3815z^6 - 846z^7 + 1188z^8 - 216z^9},$$

$$\sum_{n=0}^{+\infty} \binom{2n+1}{n} \left(\frac{3^n+1}{2} - 2^n \right) z^n$$
$$= \frac{6z^2 - 48z^3 + 141z^4 - 180z^5 + 85z^6}{1 - 18z + 141z^2 - 630z^3 + 1767z^4 - 2034z^5 + 3815z^6 - 846z^7 + 1188z^8 - 216z^9}.$$

5. Conclusion

In this paper, we have derived new theorems in order to determine new generating functions of some numbers and 2-orthogonal polynomials. The derived theorems and corollaries are based on symmetric functions.

Acknowledgement. Authors are grateful to the Editor-In-Chief of the Journal and the anonymous reviewers for their constructive comments which improved the quality and the presentation of the paper.

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Received 29. 12. 2020
Accepted 11. 4. 2021

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