DE GRUYTER

Mathematica Slovaca

DOI: 10.1515/ms-2022-0001
Math. Slovaca **72** (2022), No. 1, 1–10

EXPANDING LATTICE ORDERED ABELIAN GROUPS TO RIESZ SPACES

Antonio Di Nola — Giacomo Lenzi — Gaetano Vitale [c]

(*Communicated by Roberto Giuntini*)

ABSTRACT. First we give a necessary and sufficient condition for an abelian lattice ordered group to admit an expansion to a Riesz space (or vector lattice). Then we construct a totally ordered abelian group with two non-isomorphic Riesz space structures, thus improving a previous paper where the example was a non-totally ordered lattice ordered abelian group. This answers a question raised by Conrad in 1975. We give also a partial solution to another problem considered in the same paper. Finally, we apply our results to MV-algebras and Riesz MV-algebras.

1. Introduction

Our objects of interest are two kinds of algebraic structures: Riesz spaces (or vector lattices) and Riesz MV-algebras. See the preliminary section for the definitions.

Riesz spaces find applications in several fields like functional analysis, economy, etc. whereas Riesz MV-algebras find applications in many-valued logic, fuzzy logic, quantum mechanics, etc. For Riesz spaces, an example of monography is [10], whereas Riesz MV-algebras are more recent and we know no extensive treatment of the subject. These structures are enrichments of simpler structures: abelian l-groups and MV-algebras. For abelian l-groups see [1] or [2], whereas for MV-algebras see [4].

The relations between all these kinds of structures are very interesting and are studied, for instance, in [5, 7] and [11] . This paper is devoted to the study of these relations.

This paper is in a sense a continuation of [9], whereas [5] is our source of inspiration. Our main theme is the relation between abelian lattice ordered groups (abelian l-groups) and Riesz spaces. Each Riesz space is also, by definition, an abelian l-group. The problem now is: given an abelian l-group, can it be expanded to a Riesz space? and in how many ways?

First of all, one would like to have simple necessary and sufficient conditions for an l-group to admit a Riesz space structure. A necessary and sufficient condition is given in [5]. We propose another one. More generally we attempt to make a theory of l-groups, or MV-algebras, which admit at least one Riesz structure (we call these structures extendible).

For instance, a countable l-group cannot become a Riesz space, and Archimedean l-groups can have at most one structure. On the other hand, [5] gives examples of l-groups G with at least two Riesz space structures: for instance, $G = R\ lex\ R$, where R is the ordered group of the real numbers and *lex* denotes lexicographic product. More generally [5] proves that every totally

2020 Mathematics Subject Classification: Primary 06D35, 46A40.
Keywords: Abelian lattice ordered group, Riesz space, MV-algebra, Riesz MV-algebra.
[c] Corresponding author.

ordered, non-Archimedean l-group has either no structure, or at least two. Another example of double structure is given by [12: Example 11.54].

In [5], several open problems are left. One of them (question II thereof) is whether all Riesz spaces over a given abelian l-group G are isomorphic. This problem is solved in the negative in [9] with an explicit counterexample of G. However, G is not totally ordered, whereas [5] asks for a totally ordered example. In this paper, we solve in the negative this problem, by exhibiting a totally ordered abelian group G with two non-isomorphic Riesz space structures. The construction of the example is similar to that of [9], but somewhat simpler. Another question of [5] is whether every l-group with exactly one Riesz space structure is Archimedean. We give a partial positive solution concerning the l-groups embedded in a product of totally ordered abelian groups which are closed under finite variant.

Finally, we turn to MV-algebras. As it often happens, we can use the Mundici functor of [11] to transfer information from l-groups to MV-algebras. This paper is no exception: the previous results on l-groups can be transferred to MV-algebras. In particular, via the Mundici functor, we can prove that there is a totally ordered MV-algebra with two non-isomorphic Riesz MV-algebra structures.

2. Preliminaries

In this preliminary section, we mostly follow [9]. We denote by N, Z, Q, R the sets of natural numbers (starting from 0), the integers, the rationals and the reals respectively.

2.1. l-groups

A *lattice ordered abelian group* (l-group) is a structure $(G, +, \leq)$ such that:

- $(G, +)$ is an abelian group;
- $(G, \leq)$ is a lattice;
- $x \leq y$ implies $x + z \leq y + z$.

The infimum and supremum of two elements $x, y \in G$ will be denoted by $x \wedge y$ and $x \vee y$. A particular case is when the lattice order is total, in which case we say that $(G, +, \leq)$ is a *totally ordered abelian group.*

A *strong unit* of an l-group G is an element $u \in G$ such that for every $x \in G$ there is $n \in N$ such that $x \leq nu$.

The *absolute value* of an element $x \in G$ is $|x| = x \vee -x$.

When G is totally ordered, we simply have $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.

Given $x, y \in G$, we say that x *dominates* y if there is $n \in N$ such that $|y| \leq n|x|$. We say that x, y are equidominant if they dominate each other. Note that equidominance is an equivalence relation (in literature the equidominance relation is often called Archimedean equivalence). We say that x *strictly dominates* y if x dominates y but y does not dominate x.

An l-group G is called *Archimedean* if any two non-zero elements of G are equidominant (id est Archimedean equivalent).

Note that l-groups are an equational class, so there is a natural notion of homomorphism of l-groups and a natural category of l-groups.

2.2. Riesz spaces

A *Riesz space* is a structure $(G, +, \leq, \rho)$ which is an l-group with a structure of vector space over R, formally a map $+\colon G \times G \to G$ and a map $\rho\colon R \times G \to G$, satisfying the usual vector space axioms, that is (letting $rv = \rho(r, v)$):

- $r(v + w) = rv + rw$
- $(r + s)v = rv + sv$
- $r(sv) = (rs)v$
- $1v = v$

and such that if $v \geq 0$ and r is a positive real, then $rv \geq 0$.

Like l-groups, Riesz spaces form an equational class, so there is a natural notion of homomorphism of Riesz spaces and of the category of Riesz spaces.

2.3. MV-algebras

An *MV-algebra* is a structure $(A, \oplus, 0, 1, \neg)$ where:

- $(A, \oplus, 0)$ is a commutative monoid where 0 is the neutral element;
- $1 = \neg 0$;
- $x \oplus 1 = 1$;
- $\neg\neg x = x$;
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Intuitively, $\oplus$ is a kind of sum, and $\neg$ is a kind of negation.

The most important, and motivating, example of an MV-algebra is based on the unit real interval, where $A = [0, 1]$, $x \oplus y = \min(x + y, 1)$ and $\neg x = 1 - x$.

Other derived connectives in MV-algebras are $x \odot y = \neg(\neg x \oplus \neg y)$ (a kind of a product, dual to the sum) and $x \ominus y = x \odot \neg y$ (a kind of difference).

Once again we have an equational class, a natural notion of homomorphism and category.

2.4. Riesz MV-algebras

A *Riesz MV-algebra* is a structure $(A, \oplus, 0, \neg, \rho)$, where $(A, \oplus, 0, \neg)$ is an MV-algebra and $\rho \colon [0, 1] \times A \to A$ verifies the following axioms (where $rx = \rho(r, x)$):

- $r(x \ominus y) = (rx) \ominus (ry)$;
- $(r \ominus q)x = (rx) \ominus (qx)$;
- $r(qx) = (rq)x$;
- $1x = x$.

Once again we have an equational class, a natural notion of homomorphism and category.

3. A categorial equivalence

In [5], a necessary and sufficient condition is given for an abelian l-group to admit an expansion to a Riesz space structure. In this paper, we give another one, and we build a category of "expanded l-groups" equivalent to the category of Riesz spaces.

We will say that an *expanding family* of a divisible group G is a family $G(b)_{b \in G^+}$ of subgroups of G such that:

- $b \in G(b)$;
- $G(b)$ is isomorphic to the reals as an ordered group;
- if $c \in G(b)$ and $c \in G^+$ then $G(c) = G(b)$;
- if $x \in G(b), y \in G(c)$ and for every rational q, $x < qb$ if and only if $y < qc$, then $x + y \in G(b + c)$ and $x \wedge y \in G(b \wedge c)$.

We will say that an abelian l-group G is *Riesz expandable* if G is divisible and admits an expanding family.

THEOREM 3.1. *An abelian l-group G is the reduct of a Riesz space if and only if G is Riesz extendable.*

P r o o f. All conditions are clearly necessary since $G(b)$ is the set of all real multiples of b.

Conversely, suppose all conditions are met. If $b \in G^+$, and $r \in R$, we let rb be the image of r in the unique ordered group isomorphism between R and $G(b)$ sending 1 to b. Note that if $b, c \in G^+$ then $r(b+c) = rb + rc$.

If g is any element of G, then $g = b_1 - b_2$ for some $b_1, b_2 \in G^+$, so we let $rg = rb_1 - rb_2$. Note that the decomposition of g as a difference of two positive elements is not unique, but the definition of rg is independent of the decomposition: if $g = a - b = c - d$ then $a + d = b + c$, so $ra + rd = r(a+d) = r(b+c) = rb + rc$ and $ra - rb = rc - rd$.

This gives a Riesz space structure on G. In fact, from the last item, it follows for every $r \in R$ and for every $b, c > 0$ that $rb + rc = r(b+c)$ and $rb \wedge rc = r(b \wedge c)$. The properties extend from G^+ to G. □

We note that expanding families are not unique.

PROPOSITION 3.1. *There is an l-group G with at least two different expanding families.*

P r o o f. By the examples given first (to our knowledge) in [5], there is an l-group G with at least two Riesz space structures $\rho \neq \rho'$. But if $G(b), G'(b)$ are the corresponding expanding families, then $G(b) \neq G'(b)$ for some b. In fact, suppose $G(b) = G'(b)$ for every $b \in G^+$. Then $\rho(q, b) = \rho'(q, b)$ for every $q \in Q$, and for every real r, $\rho(r, b)$ is the unique element of $G(b)$ such that $\rho(q, b) < \rho(r, b) < \rho(q', b)$ for every pair of rationals $q < r < q'$, and $\rho'(r, b)$ is the unique element of $G'(b)$ such that $\rho(q, b) < \rho(r, b) < \rho(q', b)$ for every pair of rationals $q < r < q'$. So $\rho(r, b) = \rho'(r, b)$ for every real r and $b \in G^+$, so $\rho = \rho'$. □

Let us call expanded l-group a structure $(G, G(b)_{b \in G^+})$ where G is a divisible l-group and $G(b)$ is an expanding family of subgroups of G.

We can make a category of expanded l-groups by taking as morphisms between $(G, G(b))$ and $(H, H(c))$ the homomorphism of groups $f\colon G \to H$ such that $f(G(b)) = H(f(b))$.

PROPOSITION 3.2. *The categories of Riesz spaces and expanded l-groups are equivalent.*

P r o o f. Given a Riesz space G we associate the expanded l-group $(G, G(b))$ where $G(b)$ is the group of the real multiples of G. This association is functorial and is an equivalence by Theorem 3.1. □

4. On convex expanded families

PROPOSITION 4.1. *Let G be an l-group such that $G(b)$ is convex for every b. Then G is Archimedean.*

P r o o f. Suppose for a contradiction that G is not Archimedean. Then there are $b, \epsilon \in G^+$ such that $n\epsilon \leq b$ for every $n \in N$. Now $b \leq b + \epsilon \leq 2b$, so $b + \epsilon \in G(b)$ and $\epsilon \in G(b)$. But every nonzero element of $G(b)$ dominates b, whereas ϵ dominates b. This contradiction concludes the proof. □

Note that if G is Archimedean then G has at most one Riesz space structure, hence at most one expanding family.

Conversely, suppose G is an Archimedean l-group with an expanding family $G(b)$. Then $G(b)$ is not necessarily convex for every b, for example: $G = R \times R$, $b = (1,1)$, we have $(2,2) < (2,3) < (3,3)$ but $(2,3) \notin G(b)$.

5. On l-groups closed under finite variant

An open question of [5] is whether a non-Archimedean Riesz space can have only one Riesz space structure (compatible with its l-group structure). The following is a partial answer (recall that every l-group is embeddable in a product of totally ordered l-groups).

Theorem 5.1. *Let G be a non-Archimedean l-group embedded in a product of totally ordered l-groups and closed under finite variant. Then G admits either zero or more than one Riesz space structure.*

Proof. Let $G \subseteq \Pi_{i \in I} G_i$, where each G_i is a totally ordered abelian group. Since G is closed under finite variant, for every $i \in$I, G contains the vector u_i consisting of 1 in position i and 0 elsewhere. Let ρ be a Riesz space structure on G. Let $r \in R^+$ and $q, q' \in Q$ such that $0 < q < r < q'$. Then $\rho(r, u_i)$ is between $\rho(q, u_i)$ and $\rho(q', u_i)$, hence $\rho(r, u_i)$ must have zero in all components different from i. Moreover, suppose any $v \in G$ has $v_i = 0$ for some $i \in I$. Then v is orthogonal to u_i (i.e. $v \wedge u_i = 0$), and by definition of Riesz structure, $\rho(v) \wedge \rho(u_i) = 0$. That is, $\rho(v)$ has i-th coordinate equal to zero. By additivity, if v and w have the same i-th component, then $\rho(r, v)$ and $\rho(r, w)$ have the same i-th component. So, for every $i \in I$, there is a map $\rho_i \colon G_i \times R \to G_i$ such that $\rho_i(r, v_i) = \rho(r, v)_i$, and ρ_i is a Riesz space structure on G_i. In other words, all the Riesz space structures on G are products of Riesz space structures on G_i.

Now suppose G has a Riesz space structure ρ. By the argument above, every G_i has a Riesz space structure ρ_i. Since G is non-Archimedean, some G_i must be non-Archimedean. Any such G_i must have another Riesz space structure ρ'_i. Now, let $\rho' \colon R \times G \to G$ be the map such that $\rho'(r, v) = w$ if and only if $w_i = \rho'_i(r, v_i)$ and $w_j = \rho_i(r, v_j)$ for every $j \neq i$. ρ' is well defined because $\rho'(r, v)$ is a finite variant of $\rho(r, v)$, and is a Riesz space structure on G. Since $\rho_i \neq \rho'_i$, we conclude $\rho \neq \rho'$. □

Like in [9], we call atom of an l-group G an element $a \in G^+$ such that for every $b, c \in G^+$ with $b, c \leq a$ we have $b \wedge c \neq 0$.

Note that every l-group closed under finite variant is atomic (i.e. below every positive element there is a positive atom). In fact, every positive real multiple of u_i is an atom, and every positive element is above some positive real multiple of u_i for some $i \in I$. We conjecture that the previous theorem generalizes to atomic l-groups.

6. A totally ordered example

We have said that [9] gives a construction of a unital l-group with two non-isomorphic Riesz space structures. In this section we adapt the construction of [9] to the case of totally ordered abelian groups, and we obtain:

Theorem 6.1. *There is a totally ordered abelian group G with a strong unit u, such that G has two non-isomorphic Riesz space structures ρ_1 and ρ_2.*

P r o o f. Like in [9], the idea is to build a group with two "asymmetric" Riesz structures.

Let R^a be the field of the real algebraic numbers. R and R^a are real closed fields, so they are elementarily equivalent. By Frayne's Theorem there is an embedding $j_1 \colon R \to *R^a$, where $*R^a = (R^a)^I/U$ is an ultrapower of R^a (so I is a set and U is an ultrafilter over I).

Let $j_2 \colon R \to R^I/U$ be the diagonal embedding. So the field $K = R^I/U$ has two natural Riesz space structures ρ_1, ρ_2, where $\rho_1(r,x) = j_1(r)x$ and $\rho_2(r,x) = j_2(r)x$.

Let K_0 be the set of finite sums $\Sigma_i j_1(r_i) j_2(s_i)$ where $r_i, s_i \in R$. K_0 is a Riesz subspace of K in both Riesz structures ρ_1 and ρ_2, and has a strong unit $j_1(1)$ (note that $j_1(1) = j_2(1)$).

Note that K_0 has the cardinality of the continuum, so K_0 is included in at most $2^{\aleph_0}$ Archimedean classes (to our knowledge the exact number of Archimedean classes of K_0 is not known, note that K_0 is defined in an indirect way by an ultraproduct construction).

The idea is to consider certain sequences of elements of K_0 indexed by a regular cardinal Λ sufficiently large. More precisely, we fix two regular cardinals η, Λ such that $2^{\aleph_0} < \eta < \Lambda$.

Note that any two elements of K_0 have Archimedean distance less than η.

Let us equip the group K_0^Λ with the lexicographic ordering. That is, we let $g < h$ if and only if the first nonzero component of $h - g$ is positive. In this way K_0^Λ is a totally ordered abelian group.

Let $G \subseteq K_0^\Lambda$ be the set of all sequences $g \in K_0^\Lambda$ such that for every $\alpha < \Lambda$, $g(\alpha)$ can be written, in the vector space (K_0, ρ_1), as a real linear combination of some finite set $F \subseteq K_0$ independent of α. In other words, the range of g has finite dimension in (K_0, ρ_1). In symbols,

$$g(\alpha) = \Sigma_{i \in F} j_1(r_{i,\alpha}) k_i$$

where $r_{i,\alpha} \in R$, F is finite, and $k_i \in K_0$.

Note that G inherits from K_0^Λ (and from K) the two vector space structures above, which we will still call ρ_1 and ρ_2.

An example of strong unit of G is simply $u = (j_1(1), 0, 0, \ldots)$. Note that in order to have a strong unit, we do not need the condition (present in [9]) that the components of the elements of G are bounded.

Since G is totally ordered, the absolute value of an element g of G, written $|g|$, is simply g if $g \geq 0$, and $-g$ if $g < 0$; and the Archimedean classes (id est equidominance classes) of G are also totally ordered in the natural way.

Let us call *Archimedean distance* between $g, h \in G$ the number, possiby infinite, of Archimedean classes between g and h.

More simply than [9], we let $\Gamma = j_1(R)^\Lambda$. Similarly to [9] we have:

Lemma 6.1. *$\Gamma \subseteq G$ and Γ generates the vector space (G, ρ_2).*

P r o o f. $\Gamma \subseteq G$ because we can take $F = \{1\}$ and $k_1 = 1$.

For the second point, consider $g \in G$. Then $g(\alpha) = \Sigma_{i \in F} j_1(r_{i,\alpha}) k_i$, where F is finite and $k_i \in K_0$.

Since $k_i \in K_0$ we have $k_i = \Sigma_{j \in J_i} j_1(r_{ij}) j_2(s_{ij})$, where J_i is finite. So

$$g(\alpha) = \Sigma_{i \in F} \Sigma_{j \in J_i} j_1(r_{i,\alpha}) j_1(r_{ij}) j_2(s_{ij}).$$

Let γ_{ij} the Λ-sequence such that $\gamma_{ij}(\alpha) = j_1(r_{i,\alpha}) j_1(r_{ij})$. Then $\gamma_{ij} \in \Gamma$. Moreover

$$g(\alpha) = \Sigma_{i \in F} \Sigma_{j \in J_i} \gamma_{ij}(\alpha) j_2(s_{ij})$$

and, letting α range over Λ, we have

$$g = \Sigma_{i \in F} \Sigma_{j \in J_i} \gamma_{ij} j_2(s_{ij})$$

that is, g is a linear combination of Γ in the vector space (G, ρ_2). □

The following corollary, instead, is new.

Corollary 6.1. *Every element $g \in G$ is generated in (G, ρ_2) by positive elements of Γ which have Archimedean distance less than η from g.*

P r o o f. We can suppose $g \neq 0$. Let α be the first nonzero component of g. Given $h_1, \dots, h_n \in \Gamma$ positive elements which generate g, define $h'_1, \dots, h'_n$ such that $h'_i(\beta) = 0$ if $\beta < \alpha$, and $h'_i(\beta) = |h_i(\beta)|$ otherwise.

Then $h'_i \in \Gamma$, h'_i still generate g and either $h'_i(\alpha) = 0$, or h'_i has Archimedean distance less than η from g. Since the h'_i generate g, there must be some index i_1 such that $h'_{i_1}(\alpha) \neq 0$. So, for every i such that $h'_i(\alpha) = 0$, we replace h'_i with $h'_{i_1} + h'_i$. □

Let g_n be a sequence of elements of G. A *weak sum* of g_n, if it exists, is an element $s \in G$ such that, for every n, $s - g_1 - g_2 - \cdots - g_n$ is dominated by g_n at a distance at least η. Note that a weak sum is not necessarily unique, because the components $\alpha < \Lambda$ of s beyond the first nonzero components of all g_n are not specified (and such components exist because Λ is an uncountable regular cardinal).

We have the following key lemma.

Lemma 6.2. *Every positive decreasing sequence g_n of elements of Γ with distances at least η admits a weak sum s.*

P r o o f. Let α_n be the first nonzero component of g_n for $n \in N$ and let α be the supremum of the α_n (note $\alpha < \Lambda$). Then for $\beta < \alpha_0$ we let $s(\beta) = 0$, for $\alpha_n \leq \beta < \alpha_{n+1}$ we let $s(\beta) = g_1(\beta) + \cdots + g_n(\beta)$, and for $\beta \geq \alpha$ we let $s(\beta) = 0$. □

Like in [9] we say that an *enriched Riesz space* is a triple (G, ρ, B), where (G, ρ) is a Riesz space and B is a subset of G. An *isomorphism* of enriched Riesz spaces (G, ρ, B) and (G', ρ', B') is an isomorphism between the Riesz spaces (G, ρ) and (G', ρ') which sends B bijectively onto B'.

Now suppose by contradiction that (G, ρ_1) is isomorphic to (G, ρ_2). Then for some subset Δ of G, the enriched Ries space (G, ρ_2, Γ) is isomorphic to (G, ρ_1, Δ). So Δ must satisfy Corollary 6.1 and Lemma 6.2 (up to replacing ρ_2 with ρ_1). Let us choose a sequence (t_n) of real transcendental numbers linearly independent over the subfield R^a of R. Then already in [9] it was observed:

Lemma 6.3 ([9]). *The sequence $(j_2(t_n))$ is linearly independent in the vector space (K_0, ρ_1).*

P r o o f. In fact, let us suppose that $\Sigma_{n \in F} j_1(r_n) j2(t_n) = 0$, where $F \neq \emptyset$ is finite and $r_n \neq 0$ for every $n \in F$.

Note that $j_1(r_n) \in {}^*R^a$ and that ${}^*R^a$ is the ultrapower $(R^a)^I/\mathcal{U}$.

Instead, $j_2 \colon R \to R^I/\mathcal{U}$ is the diagonal embedding, so $j_2(t_n)$ is the $\mathcal{U}$-class of the constant sequence t_n.

Suppose the $\mathcal{U}$-class $j_1(r_n)$ contains a tuple of real algebraic numbers $(r_{n,i})_{i \in I}$. By Łoś's Theorem on ultraproducts, we obtain $\Sigma_{n \in F} r_{n,i} t_n = 0$ and $r_{n,i} \neq 0$ for every $n \in F$ and for almost all $i \in I$ with respect to $\mathcal{U}$. So, for some $i \in \mathcal{U}$, we have $\Sigma_{n \in F} r_{n,i} t_n = 0$ and $r_{n,i} \in R^a \smallsetminus \{0\}$. But this is not possible since the sequence (t_n) is linearly independent over R^a. □

The idea of the following lemma (which gives the main construction) is to use the sequence $j_2(t_n)$ and define a sequence δ_n of positive elements of Δ such that Lemma 6.2 may be applied to δ_n.

We denote by $n\eta$ the ordinal $\eta + \eta + \cdots + \eta$ (a sum with n occurrences of η).

Lemma 6.4. *There is a sequence of "quasiconstant" elements $f_n \in G$, a sequence of integers $k_n \in N$, a sequence of finite sets $\Delta_n \subseteq \Delta$ and positive elements $\delta_n \in \Delta_n$ such that:*

- *for $\beta < n\eta$, $f_n(\beta) = 0$;*
- *for $n\eta \leq \beta < \Lambda$, $f_n(\beta) = j_2(t_{k_n})$;*
- *the elements of Δ_n generate f_n in (G, ρ_1) and all of them have distance less than η from f_n;*
- *$\delta_n(n\eta)$ is linearly independent in (K_0, ρ_1) from the components of all elements of $\Delta_1 \cup \cdots \cup \Delta_{n-1}$.*

P r o o f. The proof goes by complete induction.

As a base step, we let $k_1 = 1$. Let f_1 be the corresponding quasiconstant element of G. By Corollary 6.1, f_1 is generated in (G, ρ_1) by a finite set Δ_1 of positive elements of Δ, which cannot be empty. Let δ_1 be any element of Δ_1.

The inductive step $n+1$ is as follows. We have Δ_i, δ_i, k_i for $1 \leq i \leq n$. By definition of G, the components of every element of $\Delta_1 \cup \cdots \cup \Delta_n$ have finite dimension in (K_0, ρ_1), whereas the sequence $j_2(t_n)$ has infinite dimension. So we can find a number $k_{n+1} \in N$ so high that $j_2(t_{k_{n+1}})$ is not generated by $\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_n$.

Let f_{n+1} be the quasiconstant element of G associated to k_{n+1}. By Corollary 6.1, f_{n+1} is generated in (G, ρ_1) by a finite set Δ_{n+1} of positive elements of Δ with distance less than η from f_{n+1}. In particular, $f_{n+1}((n+1)\eta) = j_2(t_{k_{n+1}})$ is generated in (K_0, ρ_1) by the elements $\delta((n+1)\eta)$ with $\delta \in \Delta_{n+1}$. So, by definition of k_{n+1}, there must be $\delta_{n+1} \in \Delta_{n+1}$ such that $\delta_{n+1}((n+1)\eta)$ is also linearly independent in (K_0, ρ_1) from the components of $\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_n$.

The inductive construction is thus completed. □

Now by Lemma 6.2 the positive sequence $\delta_n \in \Delta$ constructed in the previous lemma admits a weak sum $s \in G$. By definition of weak sum, for every $n \in N$, we have

$$s(n\eta) = \delta_1(n\eta) + \cdots + \delta_n(n\eta).$$

By construction, for every $n \in N$, $\delta_n(n\eta)$ is linearly independent in (K_0, ρ_1) from all components of the sequences $\delta_1, \ldots, \delta_{n-1}$. So also $s(n\eta)$ is linearly independent in (K_0, ρ_1) from $s(\eta), \ldots, s(n-1\eta)$. Summing up, we conclude that the range of s has infinite dimension in (K_0, ρ_1), so $s \notin G$, a contradiction. □

7. Applications to MV-algebras

We note that a condition for the MV-algebra reducts of Riesz MV-algebras can be inferred from Theorem 3.1 by applying the results of [7], where the Mundici equivalence (Γ, Ξ) of [11] between MV-algebras and abelian unital l-groups is specialized to an equivalence (Γ', Ξ') between Riesz MV-algebras and unital Riesz spaces. In fact, we have:

Lemma 7.1. *An MV-algebra A is the reduct of a Riesz MV-algebra if and only if the abelian l-group image $\Xi(A)$ of A in the Mundici equivalence (Γ, Ξ) is the reduct of a Riesz space.*

P r o o f. Suppose A is the reduct of a Riesz MV-algebra R. Then by [7], $\Xi(A)$ is the l-group reduct of the Riesz space $\Xi'(R)$.

Conversely, if the l-group $\Xi(A)$ is the reduct of a Riesz space $S = \Xi'(R)$, then A is the reduct of the Riesz MV-algebra R. □

However we have also a direct characterization in terms of MV-algebras. For this aim we call *difference structure* a structure $(A, \ominus)$ where $\ominus$ is a binary operation on A. For instance, every MV-algebra is a difference structure with respect to its usual truncated difference operation.

What we call difference structures are related to the D-posets of [3].

We will say that an MV-algebra A is Riesz-extendable if A is divisible and there is a family of difference substructures of A, $R(a)_{a \in A}$ such that:

- $R(0) = 0$;
- if $a > 0$ then $R(a)$ is isomorphic to $[0, 1]$ as a difference structure, and the (unique) isomorphism sends a to 1;
- if $a \in R(a')$ then $R(a) \subseteq R(a')$;
- if $x \in R(a), x' \in R(a')$ and for every rational $q \in [0, 1]$, $x < qa$ if and only if $x' < qa'$, then $x \ominus x' \in R(a \ominus a')$.

Theorem 7.1. *An MV-algebra A is the reduct of a Riesz MV-algebra if and only if A is Riesz-extendable.*

P r o o f. The conditions are necessary since $R(a)$ is the set of ra for $r \in [0, 1]$.

Conversely, suppose all conditions are met. Let $r \in [0, 1]$ and $a \in A$. If $a = 0$ then we let $ra = 0$. If $a > 0$ then we let ra be the image of r in the unique difference isomorphism from $[0, 1]$ to $R(a)$. This is a Riesz MV-algebra structure on A. □

We note that, by [9], the Riesz MV-algebra structure on an MV-algebra, when it exists, is not necessarily unique, not even up to isomorphism. This means that the family $R(a)$ is not uniquely determined by A.

As a corollary of Theorem 6.1, we obtain:

Corollary 7.1. *There is a totally ordered MV-algebra with two non-isomorphic Riesz MV-algebra structures.*

P r o o f. Let (G, u) be a totally ordered abelian group with a strong unit u such that G has two non-isomorphic Riesz space structures (such a group exists by Theorem 6.1). Consider the MV-algebra $A = \Gamma_M(G, u)$, where Γ_M is the Mundici functor of [11]. So the universe of A is the set $\{x \in G | 0 \leq x \leq u\}$ and the MV-algebra operations are $x \oplus y = \min(x + y, u)$ and $\neg x = u - x$.

We note (see [7: Theorem 3]) that every Riesz space structure on G gives a Riesz MV-algebra structure on A. Actually, in [7] there is an equivalence Γ_{DL} between the category of Riesz spaces with strong unit and Riesz MV-algebras, which coincides with Γ_M when restricted to the MV-algebra reducts of Riesz MV-algebras and the abelian l-group reducts of Riesz spaces.

So, the structures ρ_1 and ρ_2 on A cannot be isomorphic, otherwise by the functor Γ_{DL} we should have an isomorphism between (G, ρ_1) and (G, ρ_2), contrary to Theorem 6.1. □

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Received 30. 7. 2020
Accepted 12. 1. 2021

Department of Mathematics
University of Salerno
Via Giovanni Paolo II 132
84084 Fisciano
ITALY
E-mail: adinola@unisa.it
gilenzi@unisa.it
gvitale@unisa.it