Research Article

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Structures devised by the generalizations of two graph operations and their topological descriptors

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Abstract: Graph theory served in different fields of sciences, especially in chemistry in which creating complex structures and studying their enormous properties. Graph operation is a tool to construct complex chemical structures using basic graphs. While studying their properties, topological descriptors are a well-known methodology introduced by chemists, and even after half of a century past, it is still serving. Formally, a topological descriptor or index is a numerical value corresponding to a chemical structure. This numerical value can be easily accessed by a particular equation, for example, the second Zagreb index, the first reformulated Zagreb, and also from the forgotten topological descriptor. In this particular work, we generalized two existing graph operations, and by using these newly developed graph operations, we created two complex structures by using two graph operations, namely, the corona product and double graph operation. Furthermore, to evaluate the chemical properties of these newly generated structures, we used the methodology of topological descriptors, particularly the first and second Zagreb, the first reformulated Zagreb, and forgotten topological descriptors. Moreover, we also presented the closed formulas of the first and second Zagreb co-indices for these newly generated structures.

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1 Introduction

In mathematics, graph theory is the study of graphs that are mathematical structures used to model the pairwise relationships between objects (Bender and Gill Williamson, 2010). It is being actively used in fields as varied as biochemistry (genomics), electrical engineering (communication networks and coding theory), computer science (algorithms and computation), and operations research (scheduling) (Pirzada, 2007).

Graph operations are extremely vital in chemical graph theory. By performing graph operations on some general or specific graphs, different chemically important graphs can be obtained. A particular example is given in the study by Kwun et al. (2018), and the linear polynomial chain is actually a devised chemical structure made by the Cartesian product of two path graphs. Another particular example of a chemical structure made by the Cartesian product of two different path graphs is a C_4 nanotube or named as $TUC4(m, n)$ and C_4 nanotorus named as $TC4(m, n)$ is constructed by two different cycle graphs using the Cartesian product, given by De et al. (2016). Motivated by these particular examples, we construct some generalized chemical structures by generalizing the two graphs operations, namely, the corona product and double graph product.

Let G be a simple, finite, and connected graph with vertex set $V(G)$ and edge set $E(G)$. In graph G, the number of vertices and the numbers of edges are called the order and the size, respectively. For any vertex $v \in V(G)$, the degree of v is the number of edges incident with the vertex v, and it is written as $d_G(v)$. For two vertices u and v, the distance, $d_G(u, v)$, is the length of the shortest path between these vertices. In graph G, $N_G^k(v)$ is the set of k-distance neighborhood of $v \in V(G)$ and contains all the vertices that are at distance k from vertex v, i.e., $N_G^k(v) = \{u \mid u \in V(G), d(v,u) = k\}$.

A topological index of a graph is a number that is invariant under graph automorphism. This number also called a molecular structure descriptor or graph theoretical descriptor (Ilić et al., 2011). The first and the second Zagreb indices for a graph G are defined by Gutman and Das (2004) and Gutman and Trinajstic (1972) as follows:

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]$$

$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

In 1972, these topological indices applied for the first time to find the total π energy of molecular graphs (Gutman and Trinajstic, 1972). Later, the Zagreb indices developed important applications in QSPR/QSAR studies, and a lot of research papers have been published on these. For details, see the studies by Bollobas and Erdos (1998), Das et al. (2013), De (2017), Gao et al. (2018), Gutman and Das. (2004), and Gutman et al. (2015).

In 2004, Miličević et al. (2004) reformulated the Zagreb indices in terms of edge degree, which are defined as follows:

$$EM_1(G) = \sum_{e \in E(G)} d(e)^2,$$

$$EM_2(G) = \sum_{e \sim f} d(e)d(f),$$

where $d(e)$ denotes the degree of the edge e in G, which is defined as $d(e) = d(u) + d(v) - 2$ for edge $e = uv$ and $e \sim f$ shows that the edges e and f are adjacent.

For a graph G, the forgotten topological index or F-index is defined by Furtula and Gutman (2015) and Furtula et al. (2015) as follows:

$$F(G) = \sum_{v \in V(G)} d_G(v)^3.$$

In 2013, Shirdel et al. (2013) proposed a new version of Zagreb indices called the first hyper Zagreb index, and it is defined as follows:

$$HM_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^2.$$

The first and the second Zagreb co-indices are defined by Ramane et al. (2017) and Rasi et al. (2017) as follows:

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)],$$

$$\overline{M_2}(G) = \sum_{uv \notin E(G)} d_G(u) \cdot d_G(v).$$

Many of the topological indices have found applications as means to model chemical, pharmaceutical, and other properties of molecules.

Graph operations play a very important role in chemical graph theory, as some chemically interesting graphs can be obtained by different graphs operations on some general or particular graphs (Bondy and Murty, 2008; Harary, 1994).

Let $G_i = (V(G_i), E(G_i))$, $i = 1, 2$ be two graphs of order n_i and size m_i. For graphs G_1 and G_2, suppose $X \subseteq V(G_2)$, then the generalized corona product is denoted by $G_1 \circ G_2(X)$, and it is obtained by taking one copy of G_1 and n_1 copies of G_2 and joining each vertex of X of ith*ith* copy of G_2 with ith*ith* vertex of G_1. We obtain classical corona product for $X = V(G_2)$. The illustration of this operations is shown in Figure 1 by letting $X = \{u_2, u_3\}$.

The idea of this definition is taken from Jamil (2017), where the author used this operation to find the t-generalized quasi extremal trees. For a graph G, the generalized double graph $gD_k[G]$, is obtained by taking G with one copy of G, say G', and joining a vertex $u \in V(G)$ with $v \in V(G')$ if $d_G(u,v) = k$, where $1 \leqslant k \leqslant$ diameter. An illustration is shown in Figure 2.

From the definition, one can notice that $gD_k[P_n] = C_{2n}$ for $k = n-1$ and $gD_k[G] = D[G]$ for $k = 1$, $D[G]$ is a double

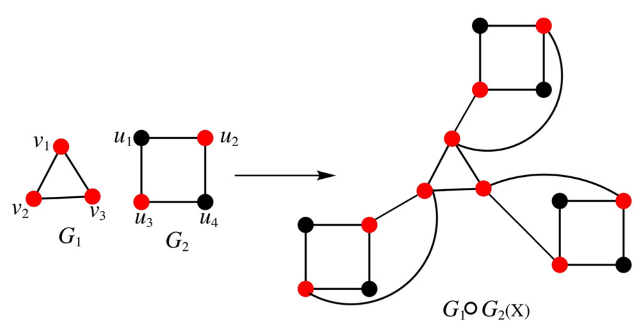

Figure 1: Generalized corona product of graphs G_1 and G_2 with $X = \{u_2, u_3\}$.

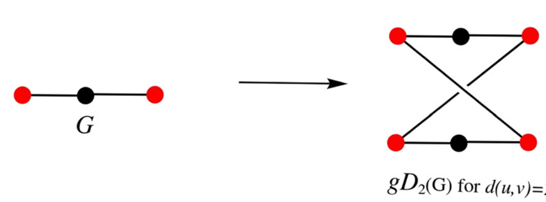

Figure 2: Generalized double graph $gD_k[G]$ of G for $k = 2$.

graph of (Jamil and Tomescu, 2015; Munarini et al., 2008). Arezoomand and Taeri (2013) presented the exact expressions of Zagreb indices of the generalized hierarchical product of graphs. In the study by Khalifeh et al. (2009), the authors investigated the first and second Zagreb indices of some graph operations. Devillers and Balaban (2000) computed the forgotten topological index of different corona products of graphs. A lot of researches have been done on the topics of graph operation (Arezoomand and Taeri, 2013; Ashrafi et al., 2010; Azari and Iranmanesh, 2013; Das and Gutman, 2004; Eliasi and Taeri; 2009, Khalifeh et al., 2008, 2009).

2 Auxiliary lemmas

From the definition of the generalized corona product and generalized double graph, we generated the following given lemmas, and these are easy to prove by the basic definitions of generalized corona product and double graph operations. We will use these two lemmas in our main results in the next section.

Lemma 2.1. *Let G_1 and G_2 be two vertex disjoint graphs and $X \subseteq V(G_2)$. Then, the degree behavior of vertices in the generalized corona product $G_1 \circ G_2(X)$ is given as follows*:

$$d_{G_1 \circ G_2(X)}(v) = \begin{cases} d_{G_1}(v) + |X|, & \text{if } v \in V(G_1), \\ d_{G_2}(v) + 1, & \text{if } v \in X, \\ d_{G_2}(v), & \text{otherwise}. \end{cases}$$

Lemma 2.2. *Let G be a graph and $gD_k[G]$ be its generalized double graphs. Then, the degree of vertices in $gD_k[G]$ is given as follows:*

$$d_{gD_k[G]}(u) = d_G(u) + |N_G^k(u)|.$$

3 Main results

In this section, we investigate the degree-based topological indices of the defined general graph operations, and some of the existing results are represented as the corollaries of the results.

Theorem 3.1. *Let G_1 and G_2 be two graphs and $X \subseteq V(G_2)$. Then, the first Zagreb index of $G_1 \circ G_2(X)$ is given as follows:*

$$\begin{aligned} M_1(G_1 \circ G_2(X)) &= M_1(G_1) + n_1|X|^2 + 4m_1|X| \\ &\quad + n_1M_1(G_2) + n_1|X| + 2n_1\sum_{v \in X} d_{G_2}(v). \end{aligned}$$

Proof. Lemma 2.1 gives the information of degrees of each vertex in $G_1 \circ G_2(X)$. From the definition of the first Zagreb index and Lemma 2.1, we have:

$$\begin{aligned} &M_1(G_1 \circ G_2(X)) \\ &= \sum_{v \in V(G_1 \circ G_2(X))} d_{G_1 \circ G_2(X)}(v)^2 \\ &= \sum_{v \in V(G_1)} \big(d_{G_1}(v) + |X|\big)^2 + n_1 \sum_{v \in X} \big(d_{G_2}(v) + 1\big)^2 + n_1 \sum_{v \notin X} d_{G_2}(v)^2 \\ &= \sum_{v \in V(G_1)} (d_{G_1}(v)^2 + |X|^2 + 2|X| d_{G_1}(v)) + n_1 \sum_{v \in X} \big(d_{G_2}(v)^2 + 1 + 2d_{G_2}(v)\big) + n_1 \sum_{v \notin X} d_{G_2}(v)^2 \\ &= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + |X|^2 \sum_{v \in V(G_1)} 1 + 2|X| \sum_{v \in V(G_1)} d_{G_1}(v) + n_1 \left[\sum_{v \in X} d_{G_2}(v)^2 + \sum_{v \notin X} d_{G_2}(v)^2\right] + n_1 \sum_{v \in X} 1 + 2n_1 \sum_{v \in X} d_{G_2}(v) \\ &= \sum_{v \in V(G_1)} d_{G_1}(v)^2 + |X|^2 \sum_{v \in V(G_1)} 1 + 2|X| \sum_{v \in V(G_1)} d_{G_1}(v) + n_1 \sum_{v \in V(G_2)} d_{G_2}(v)^2 \\ &\quad + n_1 \sum_{v \in X} 1 + 2n_1 \sum_{v \in X} d_{G_2}(v) \\ &= M_1(G_1) + n_1|X|^2 + 4m_1|X| + n_1M_1(G_2) + n_1|X| + 2n_1 \sum_{v \in X} d_{G_2}(v) \end{aligned}$$

which is the required result.

As mentioned earlier for $X = V(G_2)$, the generalized corona product becomes the classical corona product, and we have the following corollary.

Corollary 3.2. *Let G_1 and G_2 be two graphs and $X = V(G_2)$* (Yarahmadi and Ashrafi, 2012). *Then:*

$$G_1M_1(G_1 \circ G_2) = M_1(G_1) + n_1M_1(G_2) + 4(n_2m_1 + n_1m_2) + n_1n_2(n_2 + 1).$$

Theorem 3.3. *Let G_1 and G_2 be two graphs and $G_1 \circ G_2(X)$ is their generalized corona product for $X \subseteq V(G_2)$. Then, the second Zagreb index of $G_1 \circ G_2(X)$ is expressed as follows*:

$$M_2(G_1 \circ G_2(X)) = M_2(G_1) + |X|M_1(G_1) + m_1|X|^2 + n_1M_2(G_2) + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\in X}} [d_{G_2}(u) + d_{G_2}(v)] + n_1|E(G_2)_{\substack{e=uv\\ u,v\in X}}| + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X, v\in X}} d_{G_2}(u) + \sum_{u\in X} d_{G_2}(u)(2m_1 + n_1|X|) + 2m_1|X| + n_1|X|^2,$$

where $|E(G_2)_{\substack{e=uv\\ u,v\in X}}|$ is the number of edges of graph G_2 whose both vertices from set X.

Proof. From the definition of second Zagreb index and Lemma 2.1, we have:

$$\begin{aligned}
M_2&(G_1 \circ G_2(X))\\
&= \sum_{uv\in E(G_1\circ G_2(X))} d_{G_1\circ G_2(X)}(u)d_{G_1\circ G_2(X)}(v)\\
&= \sum_{e=uv\in E(G_1)} (d_{G_1}(u) + |X|)(d_{G_1}(v) + |X|) + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\notin X}} (d_{G_2}(u).\ d_{G_2}(v))\\
&\quad + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\in X}} (d_{G_2}(u) + 1)(d_{G_2}(v) + 1) + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X, v\in X}} (d_{G_2}(u))(d_{G_2}(v) + 1)\\
&\quad + \sum_{\substack{e=uv\\ u\in X\\ v\in V(G_1)}} (d_{G_2}(u) + 1)(d_{G_1}(v) + |X|)\\
&= \sum_{e=uv\in E(G_1)} \left[d_{G_1}(u)d_{G_1}(v) + |X|d_{G_1}(u) + |X|d_{G_1}(v) + |X|^2\right]\\
&\quad + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\notin X}} \left[d_{G_2}(u)\cdot d_{G_2}(v)\right] + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\in X}} \left[d_{G_2}(u)\cdot d_{G_2}(v) + d_{G_2}(u) + d_{G_2}(v) + 1\right]\\
&\quad + \sum_{\substack{e=uv\\ u,v\in V(G_1)}} \left[d_{G_2}(u).\ d_{G_1}(v) + |X|d_{G_2}(u) + d_{G_1}(v) + |X| \right]
\end{aligned}$$

which is our required result.

Corollary 3.4. *Let G_1 and G_2 be two graphs and $X = V(G_2)$* (Munarini et al., 2008). *Then:*

$$M_2(G_1 \circ G_2) = n_1[M_1(G_2) + M_2(G_2) + m_2] + (2m_2 + n_2)(2m_1 + n_2n_1) + n_2M_1(G_1) + M_2(G_1) + m_1n_2^2.$$

$$= \sum_{e=uv\in E(G_1)} d_{G_1}(u)d_{G_1}(v) + |X| \sum_{e=uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v)) + |X|^2 \sum_{e=uv\in E(G_1)} 1$$
$$+ n_1\left[\sum_{\substack{e=uv\in E(G_2)\\ u,v\notin X}} (d_{G_2}(u)\cdot d_{G_2}(v)) + \sum_{\substack{e=uv\in E(G_2)\\ u,v\in X}} (d_{G_2}(u)\cdot d_{G_2}(v)) \right.$$
$$\left. + \sum_{\substack{e=uv\in E(G_2)\\ u\notin X, v\in X}} (d_{G_2}(u).\ d_{G_2}(v)) \right] + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\in X}} (d_{G_2}(u) + d_{G_2}(v)) + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\in X}} 1$$
$$+ n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X, v\in X}} d_{G_2}(u) + \sum_{u\in X} d_{G_2}(u) \sum_{v\in V(G_1)} d_{G_1}(v) + |X| \sum_{u\in X} d_{G_2}(u) \sum_{v\in V(G_1)} 1$$
$$+ \sum_{u\in X} 1 \sum_{v\in V(G_1)} d_{G_1}(v) + |X| \sum_{u\in X} 1 \sum_{v\in V(G_1)}$$
$$= M_2(G_1) + |X|M_1(G_1) + m_1|X|^2 + n_1M_2(G_2) + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\in X}} \left[d_{G_2}(u) + d_{G_2}(v) \right]$$
$$+ n_1|E(G_2)_{\substack{e=uv\\ u,v\in X}}| + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X\\ v\in X}} d_{G_2}(u) + \sum_{u\in X} d_{G_2}(u)(2m_1 + n_1|X|) + 2m_1|X| + n_1|X|^2.$$

Theorem 3.5. *Let G_1 and G_2 be two vertex disjoint graphs. Then, first reformulated Zagreb index of $G_1 \circ G_2(X)$ is equal to*:

$$EM_1(G_1 \circ G_2(X))$$
$$= HM_1(G_1) + 4m_1(|X| - 1)^2 + 4(|X| - 1)M_1(G_1) + n_1HM_1(G_2)$$
$$+ 4n_1|E(G_2)_{\substack{e=uv\\ u,v\notin X}}| - 4n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\notin X}} [d_{G_2}(u) + d_{G_2}(v)] + n_1|E(G_2)_{\substack{e=uv\\ u\notin X\\ v\in X}}|$$
$$- 2n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X\\ v\in X}} [d_{G_2}(u) + d_{G_2}(v)] + n_1 \sum_{u\in X} d_{G_2}(u)^2 + |X| \sum_{v\in G_1} d_{G_1}(v)^2$$
$$+ 4m_1 \sum_{u\in X} d_{G_2}(u) + n_1|X|(|X| - 1)^2 + 2n_1(|X| - 1) \sum_{u\in X} d_{G_2}(u) + 4m_1|X|(|X| - 1)$$

where $|E(G_2)_{\substack{e=uv\\ u,v\notin X}}|$ is the number of edges of graph G_2 whose both vertices not from set X, similarly $|E(G_2)_{\substack{e=uv\\ u\notin X\\ v\in X}}|$ is the number of edges in G_2 whose exactly one end belongs to X.

Proof. Applying Lemma 2.1 on the definition of first reformulated Zagreb index gives:

$$
\begin{aligned}
&EM_1(G_1 \circ G_2(X))\\
&= \sum_{uv\in E(G_1\circ G_2(X))} \left[d_{G_1\circ G_2(X)}(u) + d_{G_1\circ G_2(X)}(v) - 2\right]^2\\
&= \sum_{e=uv\in E(G_1)} \left[d_{G_1}(u) + |X| + d_{G_1}(v) + |X| - 2\right]^2 + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\notin X}} \left[d_{G_2}(u) + d_{G_2}(v) - 2\right]^2\\
&\quad + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\in X}} \left[d_{G_2}(u) + 1 + d_{G_2}(v) + 1 - 2\right]^2 + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X, v\in X}} \left[d_{G_2}(u) + d_{G_2}(v) + 1 - 2\right]^2\\
&\quad + \sum_{\substack{e=uv\\ u,v\in V(G_1)}} \left[d_{G_2}(u) + 1 + d_{G_1}(v) + |X| - 2\right]^2\\
&= \sum_{e=uv\in E(G_1)} [(d_{G_1}(u) + d_{G_1}(v))^2 + (2(|X|-1))^2 + 2(d_{G_1}(u) + d_{G_1}(v)).2(|X|-1)]\\
&\quad + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\notin X}} \left[(d_{G_2}(u) + d_{G_2}(v))^2 + 4 - 4(d_{G_2}(u) + d_{G_2}(v))\right]\\
&\quad + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\in X}} \left[d_{G_2}(u) + d_{G_2}(v)\right]^2 + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X, v\in X}} \Big[(d_{G_2}(u) + d_{G_2}(v))^2 + 1\\
&\quad - 2(d_{G_2}(u) + d_{G_2}(v))\Big] + \sum_{\substack{e=uv\\ u\in X, v\in V(G_1)}} \Big[(d_{G_2}(u) + d_{G_1}(v))^2 + (|X|-1)^2\\
&\quad + 2(|X|-1)(d_{G_2}(u) + d_{G_1}(v))\Big]\\
&= \sum_{e=uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v))^2 + (2(|X|-1))^2 \sum_{e=uv\in E(G_1)} 1 + 4(|X|-1)\\
&\quad \times \sum_{e=uv\in E(G_1)} (d_{G_1}(u) + d_{G_1}(v)) + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X, v\notin X}} (d_{G_2}(u) + d_{G_2}(v))^2 + 4n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\notin X}} 1\\
&\quad - 4n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\notin X}} (d_{G_2}(u) + d_{G_2}(v)) + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u,v\in X}} (d_{G_2}(u) + d_{G_2}(v))^2\\
&\quad + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X, v\in X}} (d_{G_2}(u) + d_{G_2}(v))^2 + n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X, v\in X}} 1 - 2n_1 \sum_{\substack{e=uv\in E(G_2)\\ u\notin X, v\in X}} (d_{G_2}(u) + d_{G_2}(v))\\
&\quad + \sum_{\substack{e=uv\\ u\in X, v\in V(G_1)}} (d_{G_2}(u) + d_{G_1}(v))^2 + (|X|-1)^2 \cdot \sum_{\substack{e=uv\\ u\in X, v\in V(G_1)}} 1 + 2(|X|-1) \cdot \sum_{\substack{e=uv\\ u\in X, v\in V(G_1)}} (d_{G_2}(u)\\
&+ d_{G_1}(v))
\end{aligned}
$$

and hence the result.

The following two corollaries are the direct consequence of the aforementioned result.

Corollary 3.6. *Let G_1 and G_2 be two graphs and choose $X = V(G_2)$. Then*:

$$\begin{aligned} EM_1(G_1 \circ G_2) &= HM_1(G_1) + n_1[HM_1(G_2) \\ &+ M_1(G_2) - 4m_2] + n_2[5M_1(G_1) - 12m_1] - 4M_1(G_1) \\ &+ 4m_1(2n_2{}^2 + 1) + n_1n_2(n_2 - 1)^2 + 4m_2(n_1n_2 + 2m_1). \end{aligned}$$

Corollary 3.7. *Let G_1 and G_2 be two graphs and choose $X = V(G_2)$. Then*:

$$\begin{aligned} EM_1(G_1 \circ G_2) &= EM_1(G_1) + n_1EM_1(G_2) + 8m_1n_2{}^2 \\ &+ n_1[5M_1(G_2) - 8m_2] + n_2[5M_1(G_1) - 12m_1] \\ &+ n_1n_2(n_2 - 1)^2 + 4m_2(n_1n_2 + 2m_1). \end{aligned}$$

Moreover, the aforementioned corollary is also discussed in the study by Jamil et al. (2017) particularly, and it is a subcase of our main result.

Theorem 3.8. *Let G_1 and G_2 be two graphs and $G_1 \circ G_2(X)$ is their generalized corona product. Then, the forgotten topological index of $G_1 \circ G_2(X)$ is expressed as follows*:

$$\begin{aligned} F(G_1 \circ G_2(X)) &= F(G_1) + n_1|X|^3 + 3|X|M_1(G_1) + 6m_1|X|^2 \\ &+ n_1F(G_2) + n_1|X| \\ &+ 3n_1 \sum_{v\in X} d_{G_2}(v)^2 + 3n_1 \sum_{v\in X} d_{G_2}(v). \end{aligned}$$

Proof. From the definition of forgotten topological index, we have:

$$F(G_1 \circ G_2(X)) = \sum_{v\in V(G_1 \circ G_2(X))} d_{(G_1 \circ G_2(X))}(v)^3.$$

Now apply the Lemma 2.1 in the aforementioned equation:

$$\begin{aligned} &= \sum_{v\in V(G_1)} \big(d_{G_1}(v) + |X|\big)^3 + n_1 \sum_{v\in X} \big(d_{G_2}(v) + 1\big)^3 + n_1 \sum_{v\notin X} d_{G_2}(v)^3 \\ &= \sum_{v\in V(G_1)} \big[d_{G_1}(v)^3 + |X|^3 + 3|X|d_{G_1}(v)^2 + 3|X|^2 d_{G_1}(v)\big] \\ &= \sum_{v\in V(G_1)} \big[d_{G_1}(v)^3 + |X|^3 + 3|X|d_{G_1}(v)^2 + 3|X|^2 d_{G_1}(v)\big] \\ &+ n_1 \sum_{v\in X} [d_{G_2}(v)^3 + 1 + 3d_{G_2}(v)^2 + 3d_{G_2}(v)] + n_1 \sum_{v\notin X} d_{G_2}(v)^3 \\ &= \sum_{v\in V(G_1)} d_{G_1}(v)^3 + |X|^3 \sum_{v\in V(G_1)} 1 + 3|X| \sum_{v\in V(G_1)} d_{G_1}(v)^2 + 3|X|^2 \sum_{v\in V(G_1)} d_{G_1}(v) \\ &+ n_1 \sum_{v\in X} d_{G_2}(v)^3 + n_1 \sum_{v\in X} 1 + 3n_1 \sum_{v\in X} d_{G_2}(v)^2 + 3n_1 \sum_{v\in X} d_{G_2}(v) + n_1 \sum_{v\notin X} d_{G_2}(v)^3 \\ &= \sum_{v\in V(G_1)} d_{G_1}(v)^3 + |X|^3 \sum_{v\in V(G_1)} 1 + 3|X| \sum_{v\in V(G_1)} d_{G_1}(v)^2 + 3|X|^2 \sum_{v\in V(G_1)} d_{G_1}(v) \\ &+ n_1 \left[\sum_{v\in X} d_{G_2}(v)^3 + \sum_{v\notin X} d_{G_2}(v)^3\right] + n_1 \sum_{v\in X} 1 + 3n_1 \sum_{v\in X} d_{G_2}(v)^2 + 3n_1 \sum_{v\in X} d_{G_2}(v) \\ &= \sum_{v\in V(G_1)} d_{G_1}(v)^3 + |X|^3 \sum_{v\in V(G_1)} 1 + 3|X| \sum_{v\in V(G_1)} d_{G_1}(v)^2 + 3|X|^2 \sum_{v\in V(G_1)} d_{G_1}(v) \\ &+ n_1 \sum_{v\in V(G_2)} d_{G_2}(v)^3 + n_1 \sum_{v\in X} 1 + 3n_1 \sum_{v\in X} d_{G_2}(v)^2 + 3n_1 \sum_{v\in X} d_{G_2}(v) \\ &= F(G_1) + n_1|X|^3 + 3|X|M_1(G_1) + 6m_1|X|^2 + n_1F(G_2) + n_1|X| + 3n_1 \sum_{v\in X} d_{G_2}(v)^2 \\ &+ 3n_1 \sum_{v\in X} d_{G_2}(v) \end{aligned}$$

which is our required result.

Corollary 3.9. *Let G_1 and G_2 be two graphs and take $X = V(G_2)$. Then*:

$$F(G_1 \circ G_2) = F(G_1) + n_1F(G_2) + 3n_2M_1(G_1) + 3n_1M_1(G_2) + 6m_1n_2^2 + 6n_1m_2 + n_1n_2(n_2^2 + 1).$$

Theorem 3.10. *Let G be a graph and $gD_k[G]$ is its generalized double graph. Then, the first Zagreb index of $gD_k[G]$ is given as follows*:

$$M_1(gD_k[G]) = 2M_1(G) + 2\sum_{u\in(G)}|N_G^k(u)|^2 + 4\sum_{u\in(G)}d_G(u)|N_G^k(u)|.$$

Proof. Apply the definition of the first Zagreb index on $gD_k[G]$ and using Lemma 2.2:

$$\begin{aligned}M_1(gD_k[G]) &= \sum_{u\in V(gD_k[G])} d_{gD_k[G]}(u)^2\\ &= \sum_{u\in V(gD_k[G])}[d_G(u) + |N_G^k(u)|]^2\\ &= \sum_{u\in V(gD_k[G])}[d_G(u)^2 + |N_G^k(u)|^2 + 2d_G(u)|N_G^k(u)|]\\ &= 2\sum_{u\in V(G)}d_G(u)^2 + 2\sum_{u\in V(G)}|N_G^k(u)|^2 + 4\sum_{u\in V(G)}d_G(u)|N_G^k(u)|\\ &= 2M_1(G) + 2\sum_{u\in V(G)}|N_G^k(u)|^2 + 4\sum_{u\in V(G)}d_G(u)|N_G^k(u)|\end{aligned}$$

and hence, the required result.

Theorem 3.11. *Let G_1 be a graph and G_2 is copy of G_1. Then, second Zagreb index of $gD_k(G_1)$ is expressed as follows*:

$$\begin{aligned}M_2(gD_k(G_1)) = 2M_2(G_1) &+ 2\sum_{uv\in E(G_1)}[d_{G_1}(u)|N_{G_1}^k(v)| + d_{G_1}(v)|N_{G_1}^k(u)| + |N_{G_1}^k(u)||N_{G_1}^k(v)|]\\ &+ \sum_{\substack{uv\in E(gD_k(G_1)),\\ u\in G_1, v\in G_2}}[d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u)|N_{G_2}^k(v)| + d_{G_2}(v)|N_{G_1}^k(u)| + |N_{G_1}^k(u)||N_{G_2}^k(v)|]\end{aligned}$$

Proof. Using Lemma 2.2 and the definition of the second Zagreb index for $gD_k[G]$, we have:

$$\begin{aligned}M_2(gD_k(G_1)) &= \sum_{uv\in E(gD_k(G_1))} d_{gD_k(G_1)}(u)d_{gD_k(G_1)}(v)\\ &= \sum_{uv\in E(gD_k(G_1))}\left[d_{G_1}(u) + |N_G^k(u)|\right]\left[d_{G_2}(v) + |N_G^k(v)|\right]\end{aligned}$$

There are three types of edges in $gD_k(G_1)$:

1. $uv \in E(G_1)$,
2. $uv \in E(G_2)$,
3. $uv \in E(gD_k(G_1))$, where $u \in G_1$ and $v \in G_2$.

So, we have the following:

$$\begin{aligned}&= \sum_{uv\in E(G_1)}[d_{G_1}(u) + |N_{G_1}^k(u)|][d_{G_1}(v) + |N_{G_1}^k(v)|]\\ &\quad + \sum_{uv\in E(G_2)}[d_{G_2}(u) + |N_{G_2}^k(u)|][d_{G_2}(v) + |N_{G_2}^k(v)|]\\ &\quad + \sum_{\substack{uv\in E(gD_k(G_1))\\ u\in G_1, v\in G_2}}[d_{G_1}(u) + |N_{G_1}^k(u)|][d_{G_2}(v) + |N_{G_2}^k(v)|]\\ &= \sum_{uv\in E(G_1)}[d_{G_1}(u)d_{G_1}(v) + d_{G_1}(u)|N_{G_1}^k(v)|\\ &\quad + d_{G_1}(v)|N_{G_1}^k(u)| + |N_{G_1}^k(u)||N_{G_1}^k(v)|] + \sum_{uv\in E(G_2)}[d_{G_2}(u)d_{G_2}(v) + d_{G_2}(u)|N_{G_2}^k(v)|\\ &\quad + d_{G_2}(v)|N_{G_2}^k(u)| + |N_{G_2}^k(u)||N_{G_2}^k(v)|] + \sum_{\substack{uv\in E(gD_k(G_1))\\ u\in G_1, v\in G_2}}[d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u)|N_{G_2}^k(v)| + d_{G_2}(v)|N_{G_1}^k(u)| + |N_{G_1}^k(u)||N_{G_2}^k(v)|].\end{aligned}$$

Notice that in $gD_k(G_1)$, we have: $d_{G_1}(u) = d_{G_2}(u)$, $d_{G_1}(v) = d_{G_2}(v)$, $|N^k_{G_1}(u)| = |N^k_{G_2}(u)|$, and $|N^k_{G_1}(v)| = |N^k_{G_2}(v)|$. Therefore, we can write:

$$\begin{aligned}
&= 2\sum_{uv\in E(G_1)} [d_{G_1}(u)d_{G_1}(v) + d_{G_1}(u)|N^k_{G_1}(v)| + d_{G_1}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_1}(v)|] \\
&\quad + \sum_{\substack{uv\in E(gD_k(G_1))\\ u\in G_1, v\in G_2}} \left[d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u)|N^k_{G_2}(v)| + d_{G_2}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_2}(v)|\right] \\
&= 2\sum_{uv\in E(G_1)} d_{G_1}(u)d_{G_1}(v) + 2\sum_{uv\in E(G_1)} [d_{G_1}(u)|N^k_{G_1}(v)| + d_{G_1}(v)|N^k_{G_1}(u)| \\
&\quad + |N^k_{G_1}(u)||N^k_{G_1}(v)|] + \sum_{\substack{uv\in E(gD_k(G_1))\\ u\in G_1, v\in G_2}} [d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u)|N^k_{G_2}(v)| + d_{G_2}(v)|N^k_{G_1}(u)| \\
&\quad + |N^k_{G_1}(u)||N^k_{G_2}(v)|] \\
&= 2M_2(G_1) + 2\sum_{uv\in E(G_1)} [d_{G_1}(u)|N^k_{G_1}(v)| + d_{G_1}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_1}(v)|] \\
&\quad + \sum_{\substack{uv\in E(gD_k(G_1))\\ u\in G_1, v\in G_2}} [d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u)|N^k_{G_2}(v)| + d_{G_2}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_2}(v)|],
\end{aligned}$$

and hence, the required result.

Theorem 3.12. *Let G_1 be a graph and G_2 is copy of G_1. Then, the formula of the first reformulated Zagreb index for $gD_k(G_1)$ is given as follows*:

$$\begin{aligned}
EM_1(gD_k(G_1)) &= 2EM_1(G_1) + 2\sum_{uv\in E(G_1)} \left[\left|N^k_{G_1}(u)\right| + \left|N^k_{G_1}(v)\right|\right]^2 \\
&\quad + 4\sum_{uv\in E(G_1)} \left[d_{G_1}(u) + d_{G_1}(v) - 2\right]\left[\left|N^k_{G_1}(u)\right| + \left|N^k_{G_1}(v)\right|\right] \\
&\quad + \sum_{\substack{uv\in E(gD_k(G_1)),\\ u\in G_1, v\in G_2}} \left[d_{G_1}(u) + d_{G_2}(v) + |N^k_{G_1}(u)| + |N^k_{G_2}(v)| - 2\right]^2
\end{aligned}$$

Proof. By using the definition of the first reformulated Zagreb index and Lemma 2.2:

$$\begin{aligned}
EM_1(gD_k(G_1)) &= \sum_{e\in E(gD_k(G_1))} d_{gD_k(G_1)}(e)^2 \\
&= \sum_{uv\in E(gD_k(G_1))} \left[d_{gD_k(G_1)}(u) + d_{gD_k(G_1)}(v) - 2\right]^2 \\
&= \sum_{uv\in E(gD_k(G_1))} \left[d_{G_1}(u) + |N^k_{G_1}(u)| + d_{G_1}(v) + |N^k_{G_1}(v)| - 2\right]^2 \\
&= \sum_{uv\in E(G_1)} \left[d_{G_1}(u) + d_{G_1}(v) + |N^k_{G_1}(u)| + |N^k_{G_1}(v)| - 2\right]^2 \\
&\quad + \sum_{uv\in E(G_2)} \left[d_{G_2}(u) + d_{G_2}(v) + |N^k_{G_2}(u)| + |N^k_{G_2}(v)| - 2\right]^2 \\
&\quad + \sum_{uv\in E(gD_k(G_1))} \left[d_{G_1}(u) + d_{G_2}(v) + |N^k_{G_1}(u)| + |N^k_{G_2}(v)| - 2\right]^2
\end{aligned}$$

$$= 2\sum_{uv\in E(G_1)}\left[d_{G_1}(u)+d_{G_1}(v)+|N_{G_1}^k(u)|+|N_{G_1}^k(v)|-2\right]^2$$
$$+\sum_{uv\in E(gD_k(G_1))}\left[d_{G_1}(u)+d_{G_2}(v)+|N_{G_1}^k(u)|+|N_{G_2}^k(v)|-2\right]^2$$
$$= 2\sum_{uv\in E(G_1)}\left[d_{G_1}(u)+d_{G_1}(v)-2\right]^2+2\sum_{e\in E(G_1)}\left[\left|N_{G_1}^k(u)\right|+|N_{G_1}^k(v)|\right]^2$$
$$+4\sum_{e\in E(G_1)}[d_{G_1}(u)+d_{G_1}(v)-2][|N_{G_1}^k(u)|+|N_{G_1}^k(v)|]$$
$$+\sum_{uv\in E(gD_k(G_1))}\left[d_{G_1}(u)+d_{G_2}(v)+|N_{G_1}^k(u)|+|N_{G_2}^k(v)|-2\right]^2$$
$$= 2\sum_{uv\in E(G_1)}d(e)^2+2\sum_{e\in E(G_1)}\left[\left|N_{G_1}^k(u)\right|+|N_{G_1}^k(v)|\right]^2$$
$$+4\sum_{e\in E(G_1)}[d_{G_1}(u)+d_{G_1}(v)-2][|N_{G_1}^k(u)|+|N_{G_1}^k(v)|]$$
$$+\sum_{uv\in E(gD_k(G_1))}\left[d_{G_1}(u)+d_{G_2}(v)+|N_{G_1}^k(u)|+|N_{G_2}^k(v)|-2\right]^2$$
$$= 2EM_1(G_1)+2\sum_{uv\in E(G_1)}\left[\left|N_{G_1}^k(u)\right|+|N_{G_1}^k(v)|\right]^2$$
$$+4\sum_{e\in E(G_1)}[d_{G_1}(u)+d_{G_1}(v)-2][|N_{G_1}^k(u)|+|N_{G_1}^k(v)|]$$
$$+\sum_{uv\in E(gD_k(G_1))}\left[d_{G_1}(u)+d_{G_2}(v)+|N_{G_1}^k(u)|+|N_{G_2}^k(v)|-2\right]^2$$

which is the required result.

Theorem 3.13. *For a graph G, the forgotten topological index of $gD_k[G]$ is given as follows*:

$$F(gD_k[G]) = 2F(G)+2\sum_{u\in V(G)}|N_G^k(u)|[|N_G^k(u)|^2+3d_G(u)^2$$
$$+3d_G(u)|N_G^k(u)|]$$

Proof. By definition of the forgotten index and the Lemma 2.2:

$$F(gD_k[G])$$
$$=\sum_{u\in V(gD_k[G])}d_{gD_k[G]}(u)^3=\sum_{u\in V(gD_k[G])}[d_G(u)+|N_G^k(u)|]^3$$
$$=\sum_{u\in V(gD_k[G])}[d_G(u)^3+|N_G^k(u)|^3+3d_G(u)|N_G^k(u)|(d_G(u)$$
$$+|N_G^k(u)|)]$$
$$=\sum_{u\in V(gD_k[G])}[d_G(u)^3+|N_G^k(u)|^3+3d_G(u)^2|N_G^k(u)|$$
$$+3d_G(u)|N_G^k(u)|^2]$$
$$=2\sum_{u\in V(G)}[d_G(u)^3+|N_G^k(u)|^3+3d_G(u)^2|N_G^k(u)|$$
$$+3d_G(u)|N_G^k(u)|^2]$$
$$=2F(G)+2\sum_{u\in V(G)}|N_G^k(u)|[|N_G^k(u)|^2+3d_G(u)^2$$
$$+3d_G(u)|N_G^k(u)|],$$

and hence the required formula.

Theorem 3.14. *Let G_1 be a graph and G_2 is a copy of G_1. Then, first Zagreb co-index of $gD_k(G_1)$ is expressed as follows*:

$$\overline{M_1}(gD_k(G_1)) = 2\overline{M_1}(G_1)+2\sum_{uv\notin E(G_1)}[|N_{G_1}^k(u)|+|N_{G_1}^k(v)|]$$
$$+\sum_{uv\notin E(gD_k(G_1))}[d_{G_1}(u)+d_{G_2}(v)+|N_{G_1}^k(u)|+|N_{G_2}^k(v)|].$$

Proof. By using the definition of first Zagreb co-index and Lemma 2.2, we have:

$$\overline{M_1}(gD_k(G_1))$$
$$=\sum_{uv\notin E(gD_k(G_1))}[d_{gD_k(G_1)}(u)+d_{gD_k(G_1)}(v)]$$
$$=\sum_{uv\notin E(gD_k(G_1))}[d_{G_1}(u)+|N_{G_1}^k(u)|+d_{G_1}(v)+|N_{G_1}^k(v)|]$$
$$=\sum_{uv\notin E(gD_k(G_1))}[d_{G_1}(u)+d_{G_1}(v)+|N_{G_1}^k(u)|+|N_{G_1}^k(v)|]$$

There are three types of nonadjacent vertices in $gD_k(G_1)$:
(1) $uv \notin E(gD_k(G_1))$ and $u, v \in V(G_1)$,
(2) $uv \notin E(gD_k(G_1))$ and $u, v \in V(G_2)$,
(3) $uv \notin E(gD_k(G_1))$ and $u \in V(G_1), v \in V(G_2)$.

$$\begin{aligned}
&\overline{M_1}(gD_k(G_1))\\
&= \sum_{\substack{uv\notin E(G_1),\\ u,v\in G_1}} [d_{G_1}(u) + d_{G_1}(v) + |N^k_{G_1}(u)| + |N^k_{G_1}(v)|] + \sum_{\substack{uv\notin E(G_2),\\ u,v\in G_2}} [d_{G_2}(u) + d_{G_2}(v) + |N^k_{G_2}(u)| + |N^k_{G_2}(v)|]\\
&\quad + \sum_{\substack{uv\notin E(gD_k[G]),\\ u\in G_1, v\in G_2}} [d_{G_1}(u) + d_{G_2}(v) + |N^k_{G_1}(u)| + |N^k_{G_2}(v)|]\\
&= 2\sum_{\substack{uv\notin E(G_1),\\ u,v\in G_1}} [d_{G_1}(u) + d_{G_1}(v) + |N^k_{G_1}(u)| + |N^k_{G_1}(v)|] + \sum_{\substack{uv\notin E(gD_k[G]),\\ u\in G_1, v\in G_2}} [d_{G_1}(u) + d_{G_2}(v) + |N^k_{G_1}(u)| + |N^k_{G_2}(v)|]\\
&= 2\sum_{\substack{uv\notin E(G_1),\\ u,v\in G_1}} (d_{G_1}(u) + d_{G_1}(v)) + 2\sum_{\substack{uv\notin E(G_1),\\ u,v\in G_1}} [|N^k_{G_1}(u)| + |N^k_{G_1}(v)|] + \sum_{\substack{uv\notin E(gD_k[G]),\\ u\in G_1, v\in G_2}} [d_{G_1}(u) + d_{G_2}(v) + |N^k_{G_1}(u)| + |N^k_{G_2}(v)|]\\
&= 2\overline{M_1}(G_1) + 2\sum_{uv\notin E(G_1)} [|N^k_{G_1}(u)| + |N^k_{G_1}(v)|] + \sum_{uv\notin E(gD_k(G_1))} [d_{G_1}(u) + d_{G_2}(v) + |N^k_{G_1}(u)| + |N^k_{G_2}(v)|]
\end{aligned}$$

which is the required expression.

Theorem 3.15. *Let G_1 be a graph and G_2 is a copy of G_1. Then second Zagreb co-index of $gD_k(G_1)$ is expressed as follows*:

$$\begin{aligned}
&\overline{M_2}(gD_k(G_1))\\
&= 2\overline{M_2}(G_1) + 2\sum_{uv\notin E(G_1)} [d_{G_1}(u)|N^k_{G_1}(v)| + d_{G_1}(v)|N^k_{G_1}(u)|\\
&\quad + |N^k_{G_1}(u)||N^k_{G_1}(v)|] + \sum_{uv\notin E(gD_k(G_1))} [d_{G_1}(u)d_{G_2}(v)\\
&\quad + d_{G_1}(u)|N^k_{G_2}(v)| + d_{G_2}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_2}(v)|]
\end{aligned}$$

Proof. By using the definition of second Zagreb co-index and Lemma 2.2:

$$\begin{aligned}
EM_1(gD_k(G_1)) &= \sum_{e\in E(gD_k(G_1))} d_{gD_k(G_1)}(e)^2\\
&= \sum_{uv\in E(gD_k(G_1))} \left[d_{gD_k(G_1)}(u) + d_{gD_k(G_1)}(v) - 2\right]^2\\
&= \sum_{uv\in E(gD_k(G_1))} \left[d_{G_1}(u) + |N^k_{G_1}(u)| + d_{G_1}(v) + |N^k_{G_1}(v)| - 2\right]^2\\
&= \sum_{uv\in E(G_1)} \left[d_{G_1}(u) + d_{G_1}(v) + |N^k_{G_1}(u)| + |N^k_{G_1}(v)| - 2\right]^2\\
&\quad + \sum_{uv\in E(G_2)} \left[d_{G_2}(u) + d_{G_2}(v) + |N^k_{G_2}(u)| + |N^k_{G_2}(v)| - 2\right]^2\\
&\quad + \sum_{uv\in E(gD_k(G_1))} \left[d_{G_1}(u) + d_{G_2}(v) + |N^k_{G_1}(u)| + |N^k_{G_2}(v)| - 2\right]^2
\end{aligned}$$

$$= \sum_{\substack{uv\notin E(G_1),\\ u,v\in G_1}} [d_{G_1}(u)d_{G_1}(v) + d_{G_1}(u)|N^k_{G_1}(v)| + d_{G_1}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_1}(v)|]$$
$$+ \sum_{\substack{uv\notin E(G_2),\\ u,v\in G_2}} [d_{G_2}(u)d_{G_2}(v) + d_{G_2}(u)|N^k_{G_2}(v)| + d_{G_2}(v)|N^k_{G_2}(u)| + |N^k_{G_2}(u)||N^k_{G_2}(v)|]$$
$$+ \sum_{\substack{uv\notin E(gD_k(G_1)),\\ u\in G_1, v\in G_2}} [d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u)|N^k_{G_2}(v)| + d_{G_2}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_2}(v)|]$$
$$= 2\sum_{uv\notin E(G_1)} [d_{G_1}(u)d_{G_1}(v) + d_{G_1}(u)|N^k_{G_1}(v)| + d_{G_1}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_1}(v)|]$$
$$+ \sum_{uv\notin E(gD_k(G_1))} [d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u)|N^k_{G_2}(v)| + d_{G_2}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_2}(v)|]$$
$$= 2\sum_{uv\notin E(G_1)} d_{G_1}(u)d_{G_1}(v)$$
$$+ 2\sum_{uv\notin E(G_1)} [d_{G_1}(u)|N^k_{G_1}(v)| + d_{G_1}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_1}(v)|]$$
$$+ \sum_{uv\notin E(gD_k(G_1))} [d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u)|N^k_{G_2}(v)| + d_{G_2}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^K_{G_2}(v)|]$$
$$= 2\overline{M}_2(G_1) + 2\sum_{uv\notin E(G_1)} [d_{G_1}(u)|N^k_{G_1}(v)| + d_{G_1}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_1}(v)|]$$
$$+ \sum_{uv\notin E(gD_k(G_1))} [d_{G_1}(u)d_{G_2}(v) + d_{G_1}(u)|N^k_{G_2}(v)| + d_{G_2}(v)|N^k_{G_1}(u)| + |N^k_{G_1}(u)||N^k_{G_2}(v)|],$$

and hence, the required formula.

4 Discussion and conclusion

The graph theory is used to create complex structures and study their enormous properties by graph operations, it creates complex chemical structures from simple graphs. While studying their properties, topological descriptors are a well-known methodology introduced by chemists that is still in use after half a century. In this work, we used two graph operations, the corona product and the double graph operation, to create two complex structures. Furthermore, we used topological descriptor methodology to evaluate the chemical properties of these newly generated structures, specifically the first and second Zagreb, the first reformulated Zagreb, and forgotten topological descriptors. Furthermore, for these newly generated structures, we presented the closed formulas of the first and second Zagreb co-indices.

In the main section of our results, there are ten theorems in total, in which we discussed the first, second Zagreb and their co-indices, forgotten index and first reformulated Zagreb indices are computed, for two generalized graph operations, namely, the generalized corona product and double graph operation. In particular, Theorems 3.1, 3.3, 3.10, and 3.11 discussed the first and second Zagreb indices, respectively, in terms of the two generalized graph operations which are named the generalized corona product and double graph operation. namely, the generalized corona product and double graph operation. While Theorems 3.5 and 3.12 are the results for the first reformulated Zagreb indices for same graph operations defined earlier. Forgotten index for the two generalized graph operations, namely, the generalized corona product and double graph operation, is computed in Theorems 3.8 and 3.13. Finally, we computed the first and second co-indices generalized graph operation, namely, the generalized double graph operation, in Theorems 3.14 and 3.15.

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