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On vertex PI index of certain triangular tessellation networks

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Abstract: The Wiener index, due to its many applications is considered to be one of very important distance-based index. But the Padmaker-Ivan (PI) index is kind of the only distance related index linked to parallelism of edges. The PI index like other distance related indices has great disseminating power. The index was firstly investigated by Khadikar et al. (2001), they have probed the chemical applications of the PI index. They proved that the proposed PI index correlates highly with the physicochemical properties and biological activities of a large number of diverse and complex chemical compounds and the Wiener and Szeged indices. Recently, the vertex Padmarkar-Ivan (PI_v) index of a chemical graph G was introduced as the sum over all edges uv of a molecular graph G of the vertices of the graph that are not equidistant to the vertices u and v. In this paper, the vertex PI_v index of certain triangular tessellation are computed by using graph-theoretic analysis, combinatorial computing, and edge-dividing technology.

Keywords: PI index, triangular tessellation, hexagonal network

2010 Mathematics Subject Classification: 05C90

1 Introduction

The first molecular topological index that used in chemistry was Wiener index. It was in 1947, when Harold Wiener introduced a topological descriptor, known as the Wiener index that later become one of most useful and popular molecular descriptor. Wiener applied this index to determine the physical properties of certain types of alkanes known as paraffins. The Wiener index, due to its many applications is considered to be one of very important distance based index. But, the Padmaker-Ivan (PI) index is kind of the only distance related index linked to parallelism of edges.

There are many degree based and distance based topological indices that are defined by the mathematician and a lots of work has been done in this regard. Topological properties of molecular graphs in this regard are explored. The details about the work done in this direction can be found in Ali et al. (2017), Ashrafi and Loghman (2008), Baig et al. (2015), Caporossi et al. (2003), Estarda et al. (1998), Gao et al. (2016), Ghorbani and Hosseinzadeh (2010), Graovoc et al. (2011), Gutman and Das (2004), Gutman and Trinajstc (1972), and Kartica et al. (2012).

By introducing these topological indices, the graph theory has provided the chemist with a variety of very useful tool to investigate the chemical properties of certain chemical networks. Formally, a topological index is a numeric quantity from the structural graph of a molecule. There are many applications of these indices which are found in chemistry, pharmaceutics, and biology. For a thorough survey on this topic, consult the work of Mansour and Schork (2009).

In Amić et al. (1998), the authors defined a new distance related molecular topological index and named it as Padmarker-Ivan index. The newly defined index was abbreviated as PI_v. This newly defined index was a distances related index and does not coincide with Wiener index in general and in particular, for acyclic (trees) molecules. One important property of the purposed index PI_v is that it simple in calculation and has similar impact as that of the Wiener index, for detail see Ashrafi and Loghman (2006).

2 Preliminaries

We consider throughout the text that G is a simple undirected connected graph with vertex and edge sets, $V(G)$ and $E(G)$, respectively. $d(x,y)$ denote the distance

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between two vertices x and y and it is defined to be the length of the path that contain least edges between the vertices x and y. The distance of an edge $e=uv\in E(G)$ to the vertex $w\in V(G)$ is defines as the minimum of the distances of its ends vertices to w, that is:

$$d(w,e)=\min\{d(w,v),d(w,u)\}.$$

The vertices which have shorter distance from the u than to v of the edge e are denoted by $n_u(e\mid G)$ and similarly, the vertices which have shorter distance to the vertex v than to the vertex u are denoted by $n_v(e\mid G)$. Thus:

$$n_u(e\mid G):=|\{a\in V(G)\mid d(u,a)<d(v,a\}|$$

Similarly, we can define $n_v(e\mid G)$. The vertex Padmarker-Ivan (PI_v) index (Mansour and Schork, 2009) of a graph G is defined as:

$$PI_v(G):=\sum_{e\in E(G)} n_u(e\mid G)+n_v(e\mid G),$$

From definition, it is clear that the vertices lying at equal distance from u and v are not counted. These vertices are said to be parallel to e. Thus, we can write:

$$PI_v(G)=\sum_{e\in E(G)} n_e(G).$$

3 The vertex PI index of graph derived from hexagonal networks

The graph derived from hexagonal networks are finite subgraphs of the triangular grid. In this section, the PI index of graph of hexagonal network is computed.

The graph of hexagonal network of dimension n is denoted by $HX(n)$. The graph contain $3n^2-3n+1$ vertices and $9n^2-15n+6$ edges, where n is the number of vertices on one side of the hexagon (Ashrafi and Loghman, 2006). There is only one vertex v which has distance $n-1$ from every other corner vertices. This vertex is said to be the center of $HX(n)$ and is represented by O. In this section, The vertex PI index of hexagonal network $HX(n)$ will be computed. We assume that $V=V(HX(n))$ is the set of vertices of $HX(x)$ and for every $e=uv$; $R_e=\{x\in V\mid d_H(u,x)=d_H(v,x)\}$:

$$PI_v(HX(n))=\sum_{e\in E(HX(n))} n_e(HX(n))=\sum_{e\in E(HX(n))} |E||V|-|N_e|,$$

where $|V|=3n^2-3n+1$. Manuel et al. (2008) proposed a co-ordinate system in which direction of each

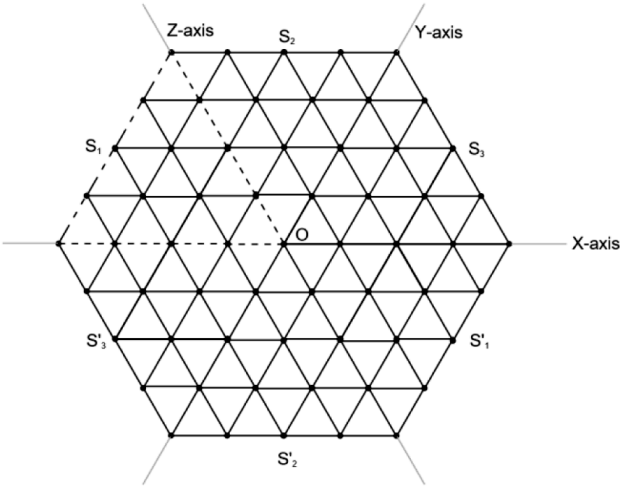

Figure 1: Hexagon networks.

correspond to an axis denoted by X, Y, and Z and the angle between any two axes is 120. It is therefore enough to calculate $|N_e|$ for every $e\in E$. To calculate N_e, we divide the hexagon into six equilateral triangles $S_i;i=1$, 2, 3 and $S'_i;i=1$, 2, 3. Let $N_e(x_k)$ denote the number of vertices in the k^{th} row of $HX(n)$ which are equidistance to edge e. Consider top left triangle S_1. Now, we see that there are three types of edges, that are, horizontal edges e^x parallel to X-axis, edges e^y parallel to Y-axis and edges e^z parallel to Z-axis. It is easy to see that N_e for edges parallel to X-axis and Z-axis in S_1 and S'_1 are same. Similarly, N_e for edges parallel to Y-axis and Z-axis in S_2 and S'_2 are same. N_e for edges parallel to X-axis and Y-axis in S_3 and S'_3 are same.

Lemma 3.1
Let e is an horizontal edge in S_1 then $|N_e|=\frac{1}{12}\left(5n^4-10n^3+7n^2-2n\right)$.

Proof. Let $V(S_1)=\{x^1_{lm}:1\le l\le n,1\le m\le l\}$ be the set of vertices of S_1 and $E^x(S_1)=\{e^x_{lm}:2\le l\le n,1\le m\le l-1\}$ be the set of edges parallel to X-axis in S_1. Let x_p denotes the number of vertices in the p^{th} row of the graph $HX(n)$. Since the graph $HX(n)$ has $2n-1$ rows, therefore the number of vertices in the p^{th} row of the graph $HX(n)$ which are equidistance to end vertices of edge e^x_{lm} are:

$$N_{e^x_{lm}}=\begin{cases} m & for\ 1\le p\le l-m,\\ l-p & for\ l-m+1\le p\le l,\\ p-l & for\ l+1\le p\le n+m,\\ n-(l-m) & for\ n+m+1\le p\le 2n-1.\end{cases}$$

Now:

$$
\begin{aligned}
|N_{e^x_{lm}}| &= \left|\sum_{p=1}^{2n-1} x_p\right| \\
&= \left|\sum_{p=1}^{l-m} x_p\right| + \left|\sum_{p=(l-m)+1}^{l} x_p\right| + \left|\sum_{p=l+1}^{n+m} x_p\right| + \left|\sum_{p=n+m+1}^{2n-1} x_p\right| \\
&= \sum_{p=1}^{l-m} m + \sum_{p=(l-m)+1}^{l} l-m+\sum_{p=l+1}^{n+m} p-l+\sum_{p=n+m+1}^{2n-1} n-(l-m) \\
&= \frac{1}{2}\begin{pmatrix} 3n^2-n(l+m+3l-3m+1) \\ +2lm-2m^2-2m+l^2+l \end{pmatrix}.
\end{aligned} \tag{1}
$$

Now:

$$
\begin{aligned}
|N_e| &= \left|\sum_{l=2}^{n}\left(\sum_{m=1}^{l-1} N_{e^x_{lm}}\right)\right| \\
&= \frac{1}{2}\sum_{l=2}^{n}\begin{pmatrix} \sum_{m=1}^{l-1}(3n^2-l(4n-1)+m(2n-2) \\ -n+2lm-2m^2+l^2) \end{pmatrix} \\
&= \frac{1}{12}\left(5n^4-10n^3+7n^2-2n\right).
\end{aligned}
$$

Corollary 3.1

Let e_{nm} be the edge on central line then $N_e = \frac{1}{6}[4n^3-6n^2+2n]$.

Proof. Let e^x_{nm} be the edge on the central line. By putting $l = n$ in Eq. 1 we get:

$$|N_{e^x_{nm}}| = 2nm-m^2-m.$$

Now:

$$
\begin{aligned}
|N_e| &= \sum_{m=1}^{n-1} 2nm-m^2-m \\
&= \frac{1}{6}\left[4n^3-6n^2+2n\right].
\end{aligned}
$$

Lemma 3.2

Let e^y_{lm} is an edge of S_1 parallel to Y-axis then $|N_e| = \frac{2}{3}n^4-\frac{4}{3}n^3+\frac{5}{6}n^2-\frac{1}{6}n$.

Proof. Let $V(S_1) = \{x_{lm} : 0 \le m \le l, 0 \le l \le n-1\}$ be the set of vertices of S_1 and $E^y(S_1) = \{e^y_{lm} : 1 \le m \le l, 1 \le l \le n-1\}$ be the set of edges parallel to Y-axis in S_1. Let, the number of vertices in p-th row is denoted by x_p of the graph $HX(n)$. Since the graph $HX(n)$ has $2n-1$ rows, therefore the number of vertices in the graph $HX(n)$ which are equidistance to end vertices of edge e^y_{lm} are:

For $2 \le l \le n-1$ and $1 \le m \left\lfloor \frac{l}{2} \right\rfloor$

$$
N_{e^y_{lm}} = \begin{cases} n+p-1 & for\ 1 \le p \le m, \\ n+m-1 & for\ m+1 \le p \le l-m+1, \\ n-p+l & for\ l-m+2 \le p \le n+l, \\ p-n-l & for\ n+l+1 \le p \le 2n-1. \end{cases}
$$

For $2 \le l \le n-1$ and $\left\lfloor \frac{l}{2} \right\rfloor + 1 \le m \le l$

$$
N_{e^y_{lm}} = \begin{cases} n+p-1 & for\ 1 \le p \le l-m+1, \\ n+m-1 & for\ l-m+2 \le p \le m, \\ n-p+l & for\ m+1 \le p \le n+l, \\ p-n-l & for\ n+l+1 \le p \le 2n-1. \end{cases}
$$

For $l = 1,3,5,\ldots$ and $m = \frac{l+1}{2}$

$$
N_{e^y_{lm}} = \begin{cases} n-1+p & for\ 1 \le p \le m, \\ n+l-p & for\ m+1 \le p \le n+l, \\ p-l-n & for\ n+l+1 \le p \le 2n-1. \end{cases}
$$

Therefore:

$$
\begin{aligned}
|N_{e^y_{lm}}| &= \left|\sum_{p=1}^{2n-1} x_p\right| \\
&= \left|\sum_{p=1}^{m} x_p\right| + \left|\sum_{p=m+1}^{(l-m)+1} x_p\right| + \left|\sum_{p=(l-m)+2}^{n+l} x_p\right| + \left|\sum_{p=(n+l)+1}^{2n-1} x_p\right| \\
&= \left|\sum_{p=1}^{m}(n+p)-1\right| + \left|\sum_{p=m+1}^{(l-m)+1}(n+m)-1\right| \\
&\quad + \left|\sum_{p=(l-m)+2}^{n+l}(n-p)+l\right| + \left|\sum_{p=(n+l)+1}^{2n-1}(p-n)-l\right| \\
&= (nm-m)+\frac{(m^2+m)}{2}+((n+m)-1)(l-2m+1) \\
&\quad +(n+l)(n+m-1)-[(l-m)+2+\ldots(l-m) \\
&\quad +m+n]-(n+l)(n-l-1)+[(n+l)+1+\ldots+(n+l) \\
&\quad +n-l-1] \\
&= \frac{1}{2}[l^2-l-2m^2+2m+2lm+2n^2-2n].
\end{aligned}
$$

For $l=1,3,5,\ldots$ and $m=\frac{l+1}{2}$. Therefore:

$$\begin{aligned}|N_{e^y_{lm}}| &= \left|\sum_{p=1}^{2n-1} x_p\right| \\ &= \left|\sum_{p=1}^{m} x_p\right| + \left|\sum_{p=m+1}^{n+l} x_p\right| + \left|\sum_{p=(n+l)+1}^{2n-1} x_p\right| \\ &= \left|\sum_{p=1}^{m}(n+p)-1\right| + \left|\sum_{p=m+1}^{n+l}(n-p)+l\right| + \left|\sum_{p=n+l+1}^{2n-1} p-n-l\right| \\ &= \frac{1}{2}[2l^2+2m^2-2lm+2n^2-2n].\end{aligned}$$

Now,

$$\begin{aligned}|N_e| &= \left|\sum_{l=1}^{n-1}\sum_{m=1}^{l} N_{e^y_{lm}}\right| \\ &= \sum_{l=1\text{lisodd}}^{n-1}\left(\sum_{m=1}^{\frac{l-1}{2}} N_{e^y_{lm}} + \sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}} N_{e^y_{lm}} + \sum_{m=\frac{l+3}{2}}^{l} N_{e^y_{lm}}\right) \\ &\quad + \sum_{l=2\text{lseven}}^{n-1}\left(\sum_{m=1}^{\frac{l}{2}} N_{e^y_{lm}} + \sum_{m=\frac{1}{2}+1}^{l} N_{e^y_{lm}}\right) \\ &= 2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}} N_{e^y_{lm}} + \sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}} N_{e^y_{lm}} + 2\sum_{l=2\text{liseven}}^{n-1}\sum_{m=1}^{\frac{l}{2}} N_{e^y_{lm}}.\end{aligned}$$

Case 1. When n is even:

$$\begin{aligned}2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}} N_{e^y_{lm}} &= 2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}}\left(\frac{1}{2}[l^2-l-2m^2+2m \right. \\ &\quad \left. +2lm+2n^2-2n]\right) \\ &= \sum_{l=3\text{lisodd}}^{n-1}\left[\frac{2}{3}l^3-\frac{3}{4}l^2+\left(\frac{1}{3}+n^2-n\right)l-n^2+n-\frac{1}{4}\right] \\ &= \frac{2}{3}(L^2(2L^2-1)-1)-\frac{3}{4}\left(\frac{(L(2L+1)(2L-1)}{1}-1\right) \\ &\quad +\left(\frac{1}{3}+n^2-n\right)(L^2-1)-\left(n^2-n+\frac{1}{4}\right)(L-1)\end{aligned}$$

where the number of terms are denoted by L. Putting $L=\frac{n}{2}$ gives:

$$2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}} N_{e^y_{lm}} = \frac{1}{3}n^4-\frac{7}{8}n^3+\frac{5}{12}n^2.$$

Now:

$$\begin{aligned}\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}} N_{e^y_{lm}} &= \sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}(n^2-n+l^2+m^2-lam) \\ &= \sum_{l=1\text{lisodd}}^{n-1}\left[(n^2-n)(1)+l^2(1)+\left(\frac{l+1}{2}\right)^2-l\left(\frac{l+1}{2}\right)\right] \\ &= \sum_{l=1\text{lisodd}}^{n-1}(n^2-n)+\frac{3}{4}l^2+\frac{1}{4} \\ &= \left(n^2-n+\frac{1}{4}\right)L+\frac{3}{4}\left(\frac{L(2L+1)(2L-1)}{3}\right).\end{aligned}$$

Putting $L=\frac{n}{2}$ gives:

$$\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}} N_{e^y_{lm}} = \frac{5}{8}n^3-\frac{1}{2}n^2.$$

$$\begin{aligned}2\sum_{l=1\text{liseven}}^{n-1}\sum_{m=2}^{\frac{l}{2}} N_{e^y_{lm}} &= 2\sum_{l=2\text{lseven}}^{n-1}\sum_{m=1}^{\frac{l}{2}}\frac{1}{2}\left[l^2-l-2m^2+2m+2lm \right. \\ &\quad \left. +2n^2-2n\right] \\ &= \sum_{l=1\text{lseven}}^{n-1} l^2\left(\frac{l}{2}\right)-l\left(\frac{l}{2}\right)-2\frac{\frac{l}{2}\frac{l+2}{2}(l+1)}{6} \\ &\quad +2\frac{2\frac{l}{2}\frac{l+2}{2}}{2}+\frac{2l\frac{l}{2}\frac{l+2}{2}}{2}+\frac{(2n^2-2n)l}{2} \\ &= \sum_{l=1\text{lseven}}^{n-1}\frac{8}{12}l^3+(\frac{1}{3}+n^2-n)l \\ &= \frac{8}{12}2(L(L+1)^2+\left(\frac{1}{3}+n^2-n\right)L(L+1)\end{aligned}$$

where L is number of terms. By putting $L=\frac{n-2}{2}$ we get:

$$2\sum_{l=1\text{liseven}}^{n-1}\sum_{m=2}^{\frac{l}{2}} N_{e^y_{lm}} = \frac{1}{3}n^4-\frac{13}{12}n^3+\frac{11}{12}n^2-\frac{1}{6}n.$$

By adding above equations, we get: $|N_e|=\frac{2}{3}n^4-\frac{4}{3}n^3+\frac{5}{6}n^2-\frac{1}{6}n.$

Case 2. When n is odd:

$$2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}}N_{e^y_{lm}}=2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}}\left(\frac{1}{2}[l^2-l-2m^2+2m+2lm+2n^2-2n]\right)$$

$$=\sum_{l=3\text{lisodd}}^{n-1}l^2\frac{l-1}{2}-l\frac{l-1}{2}2\left[\frac{\frac{l-1}{2}\left(\frac{l-1}{2}+1\right)(l-1+1)}{6}\right]$$

$$+2\left(\frac{l-1}{2}\right)\left(\frac{l+1}{2}\right)+l\left(\frac{l-1}{2}\right)\left(\frac{l+1}{2}\right)$$

$$+(2n^2-2n)\left(\frac{l-1}{2}\right)$$

$$=\sum_{l=3\text{lisodd}}^{n-1}\left[\frac{2}{3}l^3-\frac{3}{4}l^2+\left(\frac{1}{3}=n^2-n\right)l-n^2+n-\frac{1}{4}\right]$$

$$=\frac{2}{3}\left(L^2(2L^2-1)-1\right)-\frac{3}{4}\left(\frac{(L(2L+1)(2L-1)}{1}-1\right)$$

$$+\left(\frac{1}{3}+n^2-n\right)(L^2-1)-\left(n^2-n+\frac{1}{4}\right)(L-1)$$

where L is number of terms. Putting $L=\frac{n-1}{2}$ gives:

$$2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}}N_{e^y_{lm}}=\frac{1}{3}n^4-\frac{41}{24}n^3+\frac{61}{24}n^2-\frac{31}{24}n+\frac{1}{8}.$$

Now:

$$\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}N_{e^y_{lm}}=\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}(n^2-n+l^2+m^2-lam)$$

$$=\sum_{l=1\text{lisodd}}^{n-1}\left[(n^2-n)(1)+l^2(1)+\left(\frac{l+1}{2}\right)^2\right]$$

$$-l\left(\frac{l+1}{2}\right)$$

$$=\sum_{l=1\text{lisodd}}^{n-1}(n^2-n)+\frac{3}{4}l^2+\frac{1}{4}$$

$$=\left(n^2-n+\frac{1}{4}\right)L+\frac{3}{4}\left(\frac{L(2L+1)(2L-1)}{3}\right).$$

Putting $L=\frac{n-1}{2}$ gives:

$$\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}N_{e^y_{lm}}=\frac{5}{8}n^3-\frac{11}{8}n^2+\frac{7}{8}n-\frac{1}{8}$$

Now:

$$2\sum_{l=1\text{liseven}}^{n-1}\sum_{m=2}^{\frac{l}{2}}N_{e^y_{lm}}=\sum_{l=1\text{llseven}}^{n-1}\frac{8}{12}l^3+\left(\frac{1}{3}+n^2-n\right)l$$

$$=\frac{8}{12}2\left(L(L+1)\right)^2+\left(\frac{1}{3}+n^2-n\right)L(L+1)$$

where the number of terms are denoted by L. By putting $L=\frac{n-1}{2}$ we get:

$$2\sum_{l=1\text{liseven}}^{n-1}\sum_{m=2}^{\frac{l}{2}}N_{e^y_{lm}}=\frac{1}{3}n^4-\frac{1}{4}n^3-\frac{1}{3}n^2+\frac{1}{4}n.$$

By addling above equations, we get $|N_e|=\frac{2}{3}n^4-\frac{4}{3}n^3+\frac{5}{6}n^2-\frac{1}{6}n.$

Lemma 3.3

For $n\geq 2$ then $|N_e|=9n^4-22n^3+18n^2-5n$.

Proof. It is easy to see that:

$$|N_{e^x\ln S_1}|=|N_{e^z\ln S_1}|=|N_{e^x\ln S_1'}|=|N_{e^z\ln S_1'}|=|N_{e^y\ln S_2}|$$

$$=|N_{e^z\ln S_2}|=|N_{e^y\ln S_2'}|=|N_{e^z\ln S_2'}|=|N_{e^x\ln S_3}|$$

$$=|N_{e^y\ln S_3}|=|N_{e^x\ln S_3'}|=|N_{e^y\ln S_3'}|.$$

Moreover, it is easy to see that:

$$|N_{e^y\ln S_1}|=|N_{e^y\ln S_1'}|=|N_{e^x\ln S_2}|=|N_{e^x\ln S_2'}|=|N_{e^z\ln S_3}|$$

$$=|N_{e^z\ln S_3'}|.$$

This implies:

$$|N_e|=12|N_{e^x\ln S_1}|+6|N_{e^y\ln S_1}|-6|N_{e^c}|.$$

We conclude from Lemma 3.1, Lemma 3.2, and Corollary 3.1 that:

$$\begin{aligned}|N_e| &= \left[5n^4-10n^3+7n^2-2n\right]+6\left[\frac{2}{3}n^4-\frac{4}{3}n^3+\frac{5}{6}n^2-\frac{1}{6}n\right]\\ &\quad -6\left[4n^3-6n^2+2n\right]\\ &= 9n^4-22n^3+18n^2-5n.\end{aligned}$$

Theorem 3.1
If G is the graph of hexagonal networks $HX(n)$ then $PI_v(G)=8n^4-50n^3+54n^2-28n$ for $n\geq 2$.

Proof. Since:

$$PI_v(G)=|V(G)||E(G)|-|N_e|$$

Using $|V(G)|=3n^2-3n+1$, $|E(G)|=9n^2-15n+6$ and $|N_e|=9n^4-22n^3+18n^2-5n$, we get

$$PI_v(G)=18n^4-50n^3+54n^2-28n+6.$$

4 The vertex PI index of the graph derived from triangular mesh

This section will start with the definition and properties of triangular mesh. The radix-n triangular mesh network, denoted by Tn, has the set of vertices $V(Tn)=\{x_{lm}: 0\leq l\leq n-1, 0\leq m\leq l\}$ and the set of horizontal edges $E^x(Tn)=\{e^x_{lm}: 1\leq l\leq n-1, 1\leq m\leq l\}$.

The number of vertices of the graph Tn is $n(n+1)/2$. The degree of vertices are two, four, or six. The vertices which have degree two are three and we call them as corner vertices. In Figure 2, the corner vertices are labeled by a, b, and c. The number of edges of Tn is $3n(n-1)/2$.

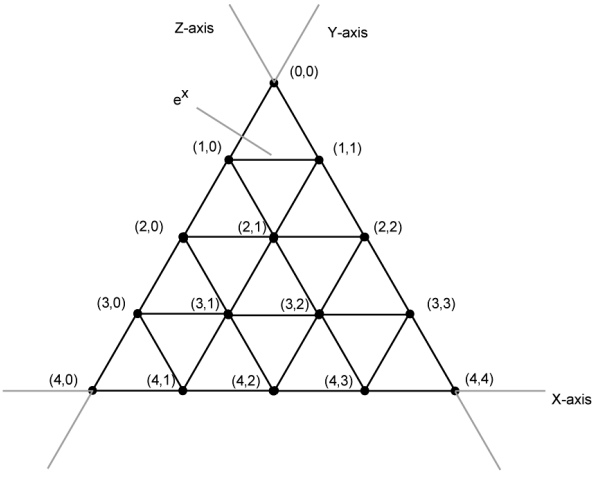

Figure 2: Triangular mesh.

Lemma 4.1
If G is a graph of triangular mesh Tn and e be the horizontal edge then $|N_e|=\frac{1}{12}n^4-\frac{1}{12}n^2$.

Proof. Let e^x_{lm} be the horizontal edge of Tn. Let x_p denotes the number of vertices in the p^{th} row of Tn which are equidistance to end vertices of edge e^x_{lm}. Then:

For $2\leq l\leq n-1$ and $1\leq m\leq \left\lfloor \frac{l}{2}\right\rfloor$

$$N_{e^x_{lm}}=\begin{cases}p & for\ 1\leq p\leq m,\\ m & for\ m+1\leq p\leq l-m+1,\\ l-p-1 & for\ l-m21\leq p\leq l,\\ p-l-1 & for\ l+1\leq p\leq n.\end{cases}$$

For $2\leq l\leq n-1$ and $\left\lfloor \frac{l}{2}\right\rfloor+1\leq m\leq l$

$$N_{e^x_{lm}}=\begin{cases}p & for\ 1\leq p\leq l-m+1,\\ l-m+1 & for\ l-m+2\leq p\leq l,\\ p-l-1 & for\ l+1\leq p\leq n.\end{cases}$$

For $l=1,3,5,\ldots$ and $m=\frac{l+1}{2}$

$$N_{e^x_{lm}}=\begin{cases}p & for\ 1\leq p\leq m,\\ l-p-1 & for\ m+1\leq p\leq l,\\ p-l-1 & for\ l+1\leq p\leq n.\end{cases}$$

$$\begin{aligned}|N_{e^x_{lm}}| &= \left|\sum_{p=1}^{n}x_p\right|\\ &= \left|\sum_{p=1}^{m}x_p\right|+\left|\sum_{p=m+1}^{l-m+1}x_p\right|+\left|\sum_{p=l-m+2}^{l}x_p\right|+\left|\sum_{p=l+1}^{n}x_p\right|\\ &= \left|\sum_{p=1}^{m}p\right|+\left|\sum_{p=m+1}^{l-m+1}m\right|+\left|\sum_{p=l-m+2}^{l}l-p-1\right|+\left|\sum_{p=l+1}^{n}p-l-1\right|\\ &= \frac{m(m+1)}{2}+(l+1)(m-2+l)+[(l-m+2)\\ &\quad +\ldots+(l-m+m)]\\ &\quad +[(l+1)+\ldots+(l+n-l)]-(l+1)(n-l)\\ &= lm-m^2+m+\frac{1}{2}\{n^2-2ln+l^2+l-n\}.\end{aligned}$$

For $l=1,3,5,\ldots$ and $m=\frac{l+1}{2}$

$$|N_{e^x_{lm}}|=\left|\sum_{p=1}^{n}x_p\right|$$

$$=\left|\sum_{p=1}^{m}x_p\right|+\left|\sum_{p=m+1}^{l}x_p\right|+\left|\sum_{p=l+1}^{n}x_p\right|$$

$$=\left|\sum_{p=1}^{m}p\right|+\left|\sum_{p=m+1}^{l}l-p-1\right|+\left|\sum_{p=l+1}^{n}p-l-1\right|$$

$$=\frac{1}{2}(m(m+1))+(l+1)(l-m)-[(m+1)+\ldots+m+(l-m)]-(l+1)(n-l)+[(l+1)+\ldots+l+(n-l)]$$

$$=\frac{1}{2}[2l^2+2m^2-2lm+2(1-n)l+n^2-n].$$

Now:

$$|N_e|=\left|\sum_{l=1}^{n-1}\sum_{m=1}^{l}N_{e^x_{lm}}\right|$$

$$=\sum_{l=1\text{lisodd}}^{n-1}\left(\sum_{m=1}^{\frac{l-1}{2}}N_{e^x_{lm}}+\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}N_{e^x_{lm}}+\sum_{m=\frac{l+3}{2}}^{l}N_{e^x_{lm}}\right)+\sum_{l=2\text{lseven}}^{n-1}\left(\sum_{m=1}^{\frac{l}{2}}N_{e^x_{lm}}+\sum_{m=\frac{1}{2}+1}^{l}N_{e^x_{lm}}\right)$$

$$=2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}}N_{e^x_{lm}}+\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}N_{e^x_{lm}}+2\sum_{l=2\text{liseven}}^{n-1}\sum_{m=1}^{\frac{l}{2}}N_{e^x_{lm}}.$$

Case 1. When n is even:

$$2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}}N_{e^x_{lm}}=2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}}lam-m^2+m+\frac{1}{2}\{n^2-2ln+l^2+l-n\}$$

$$=\sum_{l=3\text{lisodd}}^{n-1}\frac{1}{3}l^3+\left(\frac{1}{8}-\frac{n}{2}\right)l^2+\left(\frac{1}{4}n^2+\frac{1}{4}n-\frac{1}{3}\right)l+\left(-\frac{1}{4}n^2+\frac{n}{4}-\frac{1}{8}\right)$$

$$=2\left[\frac{1}{3}(L^2(2L^2-1)-1)+\left(\frac{1}{8}-\frac{n}{2}\right)\left[\frac{L(2L+1)(2L-1)}{3}-1\right]+\left(\frac{1}{4}n^2+\frac{1}{4}n-\frac{1}{3}\right)(L^2-1)+\left(-\frac{1}{4}n^2+\frac{n}{4}-\frac{1}{8}(L-1)\right)\right]$$

where the number of terms are denoted by L. Putting $L=\frac{n}{2}$, we get:

$$2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}}N_{e^x_{lm}}=\frac{1}{24}n^4-\frac{1}{24}n^3+\frac{1}{24}n^2-\frac{1}{12}n.$$

$$\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}N_{e^x_{lm}}=\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}\frac{1}{2}[2l^2+2m^2-2lm+2(1-n)l+n^2-n]$$

$$=\sum_{l=1\text{lisodd}}^{n-1}\frac{1}{2}\left[\frac{3}{2}l^2+(2-2n)l+\frac{1}{2}+n^2-n\right]$$

$$=\frac{1}{2}\left[\frac{2}{3}\left(\frac{L(2L+1)(2L-1)}{3}\right)\right]+(2-2n)L^2+\left(\frac{1}{2}+n^2-n\right)L$$

where the number of terms are denoted by L, putting $L=\frac{n}{2}$ we get:

$$\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}N_{e^x_{lm}}=\frac{1}{8}n^3.$$

$$2\sum_{l=2\text{liseven}}^{n-1}\sum_{m=1}^{\frac{l}{2}}N_{e^x_{lm}}=2\sum_{l=2\text{lseven}}^{n-1}\sum_{m=1}^{\frac{l}{2}}lam-m^2+m+\frac{1}{2}\{n^2-2ln+l^2+l-n\}$$

$$=2\sum_{l=2\text{lseven}}^{n-1}\frac{1}{3}l^3+\left(\frac{1}{2}-\frac{n}{2}\right)l^2+\left(\frac{1}{6}+\frac{n^2}{4}-\frac{n}{4}\right)l$$

$$=2\left[\frac{1}{3}(2L^2(L+1)^2)+\left(\frac{1}{2}-\frac{n}{2}\right)\frac{2L(L+1)(2L+1)}{3}l^2+\left(\frac{1}{6}+\frac{n^2}{4}-\frac{n}{4}\right)(L+1)\right]$$

where the number of terms are denoted by L. Putting $L=\frac{n-2}{2}$ we get:

$$2\sum_{l=2\text{liseven}}^{n-1}\sum_{m=1}^{\frac{l}{2}}N_{e^x_{lm}}=\frac{1}{24}n^4-\frac{1}{24}n^3+\frac{1}{6}n^2+\frac{1}{6}n.$$

Addling above equations, we get:

$$|N_e|=\frac{1}{12}n^4-\frac{1}{12}n^2.$$

Case 2. When n is odd:

$$2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}}lam-m^2+m+\frac{1}{2}\{n^2-2ln+l^2+l-n\}$$

$$=\sum_{l=3\text{lisodd}}^{n-1}\frac{1}{3}l^3+\left(\frac{1}{8}-\frac{n}{2}\right)l^2+\left(\frac{1}{4}n^2+\frac{1}{4}n-\frac{1}{3}\right)l$$

$$+\left(-\frac{1}{4}n^2+\frac{n}{4}-\frac{1}{8}\right)$$

$$=2\left[\frac{1}{3}(L^2(2L^2-1)-1)+\left(\frac{1}{8}-\frac{n}{2}\right)\right]\left[\frac{L(2L+1)(2L-1)}{3}-1\right]$$

$$+\left(\frac{1}{4}n^2+\frac{1}{4}n-\frac{1}{3}\right)(L^2-1)+\left(-\frac{1}{4}n^2+\frac{n}{4}-\frac{1}{8}\right)(L-1)$$

where the number of terms are denoted by L. Putting $L=\frac{n-1}{2}$, we get:

$$2\sum_{l=3\text{lisodd}}^{n-1}\sum_{m=1}^{\frac{l-1}{2}}lam-m^2+m+\frac{1}{2}\{n^2-2ln+l^2+l-n\}$$

$$=\frac{1}{24}n^4+\frac{1}{24}n^3-\frac{1}{24}n^2-\frac{1}{24}n.$$

$$\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}N_{e^x_{lm}}=\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}\frac{1}{2}[2l^2+2m^2-2lm+2(1-n)l+n^2-n]$$

$$=\sum_{l=1\text{lisodd}}^{n-1}\frac{1}{2}\left[\frac{3}{2}l^2+(2-2n)l+\frac{1}{2}+n^2-n\right]$$

$$=\frac{1}{2}\left[\frac{2}{3}\left(\frac{L(2L+1)(2L-1)}{3}\right)+(2-2n)L^2\right.$$

$$\left.+\left(\frac{1}{2}+n^2-n\right)L\right]$$

where the number of terms are denoted by L, putting $L=\frac{n-1}{2}$, we get:

$$\sum_{l=1\text{lisodd}}^{n-1}\sum_{m=\frac{l+1}{2}}^{\frac{l+1}{2}}N_{e^x_{lm}}=\frac{1}{2}\left(\frac{1}{4}n^3-\frac{1}{4}n^2-\frac{1}{4}n+\frac{1}{4}\right).$$

$$2\sum_{l=2\text{liseven}}^{n-1}\sum_{m=1}^{\frac{l}{2}}N_{e^x_{lm}}=2\sum_{l=2\text{llseven}}^{n-1}\sum_{m=1}^{\frac{l}{2}}lam-m^2+m+\frac{1}{2}\{n^2-2ln+l^2+l-n\}$$

$$=2\sum_{l=2\text{llseven}}^{n-1}\frac{1}{3}l^3+\left(\frac{1}{2}-\frac{n}{2}\right)l^2+\left(\frac{1}{6}+\frac{n^2}{4}-\frac{n}{4}\right)l$$

$$=2\left[\frac{1}{3}(2L^2(L+1)^2)+\left(\frac{1}{2}-\frac{n}{2}\right)\frac{2L(L+1)(2L+1)}{3}l^2\right.$$

$$\left.+\left(\frac{1}{6}+\frac{n^2}{4}-\frac{n}{4}\right)(L+1)\right]$$

where the number of terms are denoted by L. Putting $L=\frac{n-1}{2}$ we get:

$$2\sum_{l=2\text{liseven}}^{n-1}\sum_{m=1}^{\frac{l}{2}}N_{e^x_{lm}}=\frac{1}{24}n^4-\frac{1}{6}n^3+\frac{1}{24}n^2+\frac{1}{6}n-\frac{1}{8}$$

Adding above equations, we get:

$$|N_{e^x}|=\frac{1}{12}n^4-\frac{1}{12}n^2.$$

Corollary 4.1

$$|N_e|=\frac{1}{4}n^4-\frac{1}{4}n^2.$$

Proof. It is easy to see that $|N_{e^x}|=|N_{e^y}|=|N_{e^z}|$, we get:

$$|N_e|=3|Le^x|=\frac{1}{4}n^4-\frac{1}{4}n^2.$$

Theorem 4.1

If G is a graph of triangular mesh Tn then $PI_v(G)=\frac{1}{2}n^4-\frac{1}{2}n^2$.

Proof. Let e be the edge of G then:

$$PI_v(G)=|V(G)||E(G)|-|N_e|$$

Using $|V(G)|=n(n+1)/2$, $|E(G)|=3n(n-1)/2$ and $|N_e|=\frac{1}{4}n^4-\frac{1}{4}n^2$, we obtain:

$$PI_v(G)=\frac{n(n+1)}{2}\frac{3n(n-1)}{2}-\frac{1}{4}n^4-\frac{1}{4}n^2$$

$$=\frac{1}{2}n^4-\frac{1}{2}n^2.$$

5 The vertex PI index of the graph derived from enhanced mesh

An $m \times n$ mesh is a graph $M(m,n)$ with vertex set $V = \{(l,m): 1 \le l \le m, 1 \le m \le n\}$ and edge set $E = \{((l,m), (l,m+1)): 1 \le l \le m, 1 \le m \le n-1\} \cup \{((l,m),(l+1,m)): 1 \le l \le m-1, 1 \le m \le n\}$. The graph of enhanced mesh $EL(m,n)$ is derived by replacing each 4-cycle of $M(m,n)$ by a wheel. Let $h_{lm}, 1 \le l \le m-1, 1 \le m \le n-1$ be the newly added vertices of wheel called hub vertices. In Figure 3, the graph of enhanced mesh $EL(5,5)$ is shown.

Lemma 5.1
If G is a graph of enhanced mesh $EL(m,n)$ and e^h be the horizontal edge then $|N_{e^h_{lm}}| = m-1$.

Proof. The only vertices which are equidistant from e^h_{lm} are the hub vertices in the m^{th} column. This lipless that $|N_{e^h_{lm}}| = m-1$.

Corollary 5.1
If G is a graph of enhanced mesh $EL(m,n)$ and e^h be the vertical edge then $|N_{e^v_{lm}}| = n-1$.

Proof. The only vertices which are equidistant from e^v_{lm} are the hub vertices in the l^{th} row. This lipless that $|N_{e^v_{lm}}| = n-1$.

Lemma 5.2
If G is a graph of enhanced mesh $EL(m,n)$ and e be the edge between $x_{l,m}$ and $h_{l,m}$ then $|N_{e'_{l,m}}| = (n^2-n)(m-\frac{1}{2}) - \frac{1}{2}m(n-1)(m-1)$.

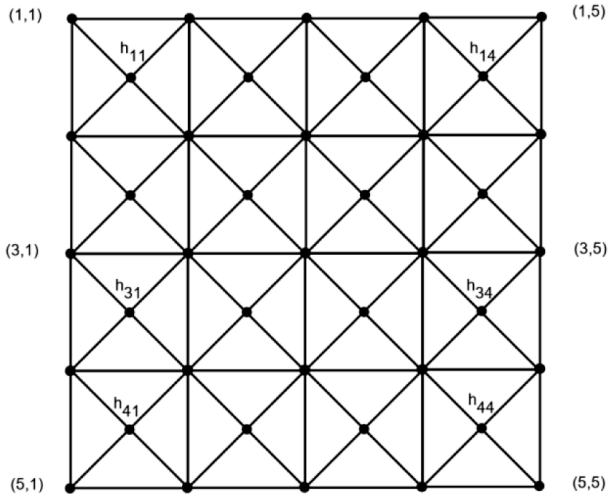

Figure 3: An enhanced mesh $EM(5,5)$.

Proof. Let $e_{l,m} = x_{l,m}h_{l,m}$ be the edge of $EL(m,n)$ then the vertices of the graph that are equidistance to end vertices of edge $e_{l,m}$ is $l(n-m)+l(n-m-1)+m(m-l)+m(m-l-1)$. Therefore:

$$|N_{e_{l,m}}| = \sum_{l=1}^{m-1}\sum_{m=1}^{n-1} l(n-m)+l(n-m-1)+m(m-l)+m(m-l-1)$$
$$= (n^2-n)\left(m-\frac{1}{2}\right) - \frac{1}{2}m(n-1)(m-1).$$

Lemma 5.3
If G is a graph of enhanced mesh $EL(m,n)$ and e be the edge between $h_{l,m}$ and $x_{l+1,m+1}$ then $|N_{e''_{l,m}}| = (n-1)(m-1)\left(mn-\frac{1}{2}n-\frac{1}{2}m\right)$.

Proof. Let $e_{lm} = h_{l,m}x_{l+1,m+1}$ be the edge of $EL(m,n)$ then the vertices of the graph that are equidistance to end vertices of edge e_{lm} is $(2l-1)(n-m)+(2m-1)(m-l)$. Therefore:

$$|N_{e_{lm}}| = \sum_{l=1}^{m-1}\sum_{m=1}^{n-1}(2l-1)(n-m)+(2m-1)(m-l)$$
$$= mn(m-1)(n-1)+\frac{1}{2}m(m-1)(n-1)+(m-1)$$
$$(n-1)\left(\frac{n}{2}-n-m\right)$$
$$= (n-1)(m-1)\left(Mon-\frac{1}{2}n-\frac{1}{2}m\right).$$

Theorem 5.1
If G is a graph of enhanced mesh $EL(m,n)$ then $PI_v(G)$ is:

$$m(n-1)[mn+(m-1)(n-2)]+n(m-1)[mn+(n-1)(m-2)]$$
$$+(m-1)(n-1)[mn+(n-1)(m-1)]$$
$$-(n^2-n)\left(m-\frac{1}{2}\right)+\frac{1}{2}m(n-1)(m-1)+(m-1)(n-1)$$
$$[mn+(n-1)(m-1)]-(n-1)(m-1)\left(mn+\frac{1}{2}n-\frac{1}{2}m\right)$$

Proof. To prove the statement, we partition the edges into four sets, horizontal edges E^h, vertical edges E^v, $x_{l,m}h_{l,m}$ edges E', and $h_{l,m}x_{l+1,m+1}$ edges E'':

$$PI_v(G) = \sum_{e \ln E^h} n_e(G) + \sum_{e \ln E^v} n_e(G) + \sum_{e \ln E'} n_e(G) + \sum_{e \ln E''} n_e(G) \quad (2)$$

The number of vertices of G is:

$$|V(G)| = mn+(m-1)(n-1). \quad (3)$$

$$|E^h| = m(n-1), \quad |N_{e^h}| = m(n-1)(m-1). \tag{4}$$

Similarly:

$$|E^v| = n(m-1), \quad |N_{e^v}| = m(m-1)(n-1). \tag{5}$$

$$|E'| = (m-1)(n-1), \quad |N_{e'}| = (n^2-n)\left(m-\frac{1}{2}\right) - \frac{1}{2}m(n-1)(m-1). \tag{6}$$

$$|E''| = (m-1)(n-1), \quad |N_{e''}| = (n-1)(m-1)\left(Mon-\frac{1}{2}n-\frac{1}{2}m\right). \tag{7}$$

Using Eq. 3-7 in Eq. 2, we get:

$$\begin{aligned} PI_v(G) = {} & m(n-1)[Mon+(m-1)(n-2)]+n(m-1)\\ & [Mon+(n-1)(m-2)]+(m-1)(n-1)\\ & [Mon+(n-1)(m-1)]-(n^2-n)\left(m-\frac{1}{2}\right)\\ & +\frac{1}{2}m(n-1)(m-1)+\\ & (m-1)(n-1)[Mon+(n-1)(m-1)]-(n-1)(m-1)\\ & \left(Mon+\frac{1}{2}n-\frac{1}{2}m\right). \end{aligned}$$

6 Conclusion

In this paper, we have solved vertex PI Index of certain triangular tessellations networks. We have considered hexagonal networks, triangular grids and enhanced meshes and analytical closed from results for these triangular tessellations networks were obtained. These results will be useful to understand the molecular topology of these important classes of networks.

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