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Research Article

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Multiple solutions for the *p*-biharmonic equation with a polynomial nonlinearity and a sign-changing logarithmic nonlinearity

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Abstract: In this paper, we consider a boundary problem for a p-biharmonic equation $\Delta\left(|\Delta u|^{p-2}\Delta u\right) = a(x)|u|^{p-2}u\ \ln|u| + b(x)|u|^{p-2}u$ with a sign-changing logarithmic nonlinearity $a(x)|u|^{p-2}u\ \ln|u|$ and a polynomial nonlinearity $b(x)|u|^{p-2}u$ in a bounded domain. We derive a new result regarding the existence of solutions and illustrate how the weight functions $a(x)$ and $b(x)$ can affect the existence and multiplicity of solutions. Furthermore, the sign-changing weight function $a(x)$, associated with the logarithmic nonlinearity $a(x)|u|^{p-2}u\ \ln|u|$, plays a pivotal role in determining the existence and multiplicity of these solutions.

Keywords: p-biharmonic equation; polynomial nonlinearity; sign-changing logarithmic nonlinearity; multiple solutions

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1 Introduction

We consider the following boundary problem for a p-biharmonic equation

$$\begin{cases} \Delta\left(|\Delta u|^{p-2}\Delta u\right) = a(x)|u|^{p-2}u\ \ln|u| + b(x)|u|^{p-2}u, & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < +\infty$, $\Omega \subset \mathbb{R}^n(n > 1)$ with boundary $\partial\Omega$ is smooth, the weight functions $a(x), b(x) \in C(\overline{\Omega})$, and $a(x)$ changes sign in $\overline{\Omega}$, while $b(x)$ does not change sign in $\overline{\Omega}$. The term $a(x)|u|^{p-2}u\ \ln|u|$ is a sign-changing logarithmic nonlinearity, and $b(x)|u|^{p-2}u$ is a polynomial nonlinearity. Note that the sign-changing logarithmic nonlinearity possesses a unique structure that does not satisfy either the monotonicity condition or the Ambrosetti–Rabinowitz condition (see [1,2]), which brings some difficulties in applying variational methods. Therefore, the study of the existence of solutions to such problems with sign-changing logarithmic nonlinearity is both interesting and challenging.

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Recently, Tian [3] studied the following boundary problem for a semilinear elliptic equation with the sign-changing logarithmic nonlinearity

$$\begin{cases} -\Delta u = g(x)u \ln|u| + \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $g(x) \in C(\overline{\Omega})$ and changes sign in $\overline{\Omega}$. At least two nontrivial weak solutions were obtained by the Nehari manifold and logarithmic Sobolev inequality under the condition

$$\max_{x\in\overline{\Omega}} |g(x)| < 2\pi e^{2-\frac{4|\Omega|_n}{ne}}$$

and for small λ, where $|\Omega|_n$ is the Lebesgue measure of Ω in $\mathbb{R}^n$. When $\lambda = 0$ and $g(x)$ changes sign in $\overline{\Omega}$ to the problem (1.2), Shuai [4] obtained two sequences of solutions by using variational methods, and then showed that one sequence of solutions is with energy and $H_0^1(\Omega)$-norms diverging to positive infinity, while the other one is with energy and $H_0^1(\Omega)$-norms converging to zero. These results show that the sign-changing weight function $g(x)$ affects the existence and multiplicity of weak solutions. Noting that the unique structure of logarithmic nonlinearity in the problem (1.2) distinguishes it from the results obtained in the following polynomial case (see Brown and Zhang [5], Brown [6])

$$\begin{cases} -\Delta u = g(x)|u|^{p-1}u + \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $p \in (0, 1)$ or $p \in (1, 2^* - 1), 2_*$ is a critical Sobolev exponent, the problem (1.3) has only one solution for small λ.

Chen and Tang [7] considered the following boundary problem for a logarithmic Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = Q(x)|u|^{p-2}u \ln|u|^2, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $V(x)$ is a potential function that does not change sign in $\overline{\Omega}$, and the function $Q(x)$ changes sign in $\overline{\Omega}$. By the constraint variational method, quantitative deformation lemma, and some new energy inequalities, two ground state sign-changing solutions were obtained in two nodal domains, respectively. For more information on the logarithmic Schrödinger equation (1.4), we refer to [8–13] and their references.

Han, Cao, and Li [14] considered the following boundary problem for a fourth-order thin-film equation

$$\begin{cases} \Delta^2 u + c\Delta u = h(x)u \ln|u|, & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

At least two nontrivial weak solutions were shown by using the modified logarithmic Sobolev inequality and Nehari manifold provided that the sign-changing weight function $h(x)$ satisfies

$$\max_{x\in\overline{\Omega}} |h(x)| < 2\pi(\lambda_1 - c)e^{2-\frac{4|\Omega|_n}{ne}},$$

where $c < \lambda_1$ and λ_1 is the first eigenvalue of $-\Delta$ under the homogeneous boundary condition. When $h(x) = 1$ in the problem (1.5), Liu et al. [15] established the existence of ground state solutions by using the Linking theorem and logarithmic Sobolev inequality.

Inspired by the above literature, a natural question arises regarding whether the weight functions present in the polynomial nonlinearity and the sign-changing logarithmic nonlinearity influence the existence and multiplicity of weak solutions to the problem (1.1). Furthermore, it remains to be determined which of these weight functions exerts the dominant influence. In this paper, we will address this question. We derive a new result regarding the existence of solutions, two nontrivial weak solutions of the problem (1.1) are obtained by utilizing the Nehari manifold, fibering maps, and the improved logarithmic Sobolev inequality under appropriate conditions. We illustrate how the weight functions $a(x)$ and $b(x)$ can affect the existence and multiplicity of solutions. Furthermore, the sign-changing weight function $a(x)$, associated with the logarithmic nonlinearity $a(x)|u|^{p-2}u\ \ln|u|$, plays a pivotal role in determining the existence and multiplicity of these solutions.

Theorem 1.1. *Assume that the weight functions $a(x), b(x) \in C(\overline{\Omega})$, and $a(x)$ changes sign in $\overline{\Omega}$, while $b(x)$ does not change sign in $\overline{\Omega}$. Then the problem (1.1) has two nontrivial weak solutions provided that*

$$M_1 S_p^p e^{\frac{p^2}{n}\left(\frac{M_2}{M_1}+\frac{2|\Omega|_n}{pe}\right)-1-\ln\left(\frac{p}{nl_p}\right)} < p, \tag{1.6}$$

where $M_1 = \max\limits_{x\in\overline{\Omega}}|a(x)|, M_2 = \max\limits_{x\in\overline{\Omega}}|b(x)|$, S_p is an optimal embedding constant for the embedding $W_0^{2,p}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$, and l_p is a parameter of the improved logarithmic Sobolev inequality (see Lemma 2.5).

Corollary 1.1. *If $p = 2$ and $b(x) = 0$ in the problem (1.1), then the following problem for a biharmonic equation*

$$\begin{cases} \Delta^2 u = a(x)u\ \ln|u|, x \in \Omega, \\ \quad u = \Delta u = 0, \quad\ x \in \partial\Omega, \end{cases}$$

possesses two nontrivial weak solutions when $M_1 < \frac{2\pi e^{2-\frac{4|\Omega|_n}{ne}}}{S_2^2}$, which extends the previous existence result of solutions in [14].

The paper is organized as follows. In Section 2, we introduce the Nehari manifold, fibering maps, and the improved logarithmic Sobolev inequality, and then give some needed lemmas. In Section 3, we prove the energy functional $J(u)$ has two minimizers on N^+ and N^- by using the variational method, respectively.

2 Preliminaries

Associated with the problem (1.1), we can obtain the energy functional

$$J(u) = \frac{1}{p}\int\limits_\Omega |\Delta u|^p \mathrm{d}x - \frac{1}{p}\int\limits_\Omega a(x)|u|^p\ \ln|u|\mathrm{d}x + \frac{1}{p^2}\int\limits_\Omega a(x)|u|^p\mathrm{d}x - \frac{1}{p}\int\limits_\Omega b(x)|u|^p\mathrm{d}x. \tag{2.1}$$

Then $J(u) \in C^1\left(W_0^{2,p}(\Omega), \mathbb{R}\right)$ and its Fréchet derivative is given by

$$\left\langle J'(u), v\right\rangle = \int\limits_\Omega |\Delta u|^{p-2}\Delta u\Delta v\mathrm{d}x - \int\limits_\Omega a(x)|u|^{p-2}uv\ \ln|u|\mathrm{d}x - \int\limits_\Omega b(x)|u|^{p-2}uv\mathrm{d}x$$

for any $u, v \in W_0{}^{2,p}(\Omega)$. From (2.1), we know $J(u)$ is not bounded on $W_0^{2,p}(\Omega)$, but we can prove $J(u)$ is bounded from below on the Nehari manifold

$$N = \left\{u \in W_0^{2,p}(\Omega)\backslash\{0\} : \left\langle J'(u), u\right\rangle = 0\right\},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual duality between $W_0^{2,p}(\Omega)$ and $W_0^{-2,p'}(\Omega)$, $p' = \frac{p}{p-1}$. We equip $W_0^{2,p}(\Omega)$ with the norm $\|u\|_{W_0^{2,p}(\Omega)} = \|\Delta u\|_{L^p(\Omega)}$.

Inspired by [16–20], we can seek weak solutions to the problem (1.1) by using the Nehari manifold, fibering maps, and the improved logarithmic Sobolev inequality. As in Drabek and Pohozaev [21], who introduced fibering maps. We can consider a fibering maps $\Phi_u(t): \mathbb{R}^+ \to \mathbb{R}$, defined by

$$\begin{aligned}\Phi_u(t) &= J(tu)\\ &= \frac{t^p}{p}\int_\Omega |\Delta u|^p \mathrm{d}x - \frac{t^p}{p}\int_\Omega a(x)|u|^p \ln|tu| \mathrm{d}x + \frac{t^p}{p^2}\int_\Omega a(x)|u|^p \mathrm{d}x - \frac{t^p}{p}\int_\Omega b(x)|u|^p \mathrm{d}x,\end{aligned}$$

where $t > 0$, and by a direct calculation, we know

$$\begin{aligned}\Phi_u'(t) &= t^{p-1}\left(\int_\Omega |\Delta u|^p \mathrm{d}x - \int_\Omega a(x)|u|^p \ln|tu| \mathrm{d}x - \int_\Omega b(x)|u|^p \mathrm{d}x\right), \qquad (2.2)\\ \Phi_u''(t) &= (p-1)t^{p-2}\left(\int_\Omega |\Delta u|^p \mathrm{d}x - \int_\Omega a(x)|u|^p \ln|tu| \mathrm{d}x - \int_\Omega b(x)|u|^p \mathrm{d}x\right)\\ &\quad - t^{p-2}\int_\Omega a(x)|u|^p \mathrm{d}x.\end{aligned}$$

Lemma 2.1. *Let $u \in W_0^{2,p}(\Omega)\backslash\{0\}$ and $t > 0$. Then $tu \in N$ if and only if $\Phi_u'(t) = 0$.*

Proof. Since $t > 0$ and $tu \in N$, then from (2.2) we have

$$tu \in N \Leftrightarrow t^p\left(\int_\Omega |\Delta u|^p \mathrm{d}x - \int_\Omega a(x)|u|^p \ln|tu| \mathrm{d}x - \int_\Omega b(x)|u|^p \mathrm{d}x\right) = 0 \Leftrightarrow t\Phi_u'(t) = 0,$$

which implies $tu \in N$ if and only if $\Phi_u'(t) = 0$.

From Lemma 2.1, we know that if $u \in N$, then

$$\begin{aligned}\Phi_u'(1) &= \int_\Omega |\Delta u|^p \mathrm{d}x - \int_\Omega a(x)|u|^p \ln|u| \mathrm{d}x - \int_\Omega b(x)|u|^p \mathrm{d}x = 0,\\ \Phi_u''(1) &= -\int_\Omega a(x)|u|^p \mathrm{d}x.\end{aligned}$$

We can divide Nehari manifold N into three subsets N^+, N^-, and N^0, corresponding to the local minima, local maxima, and points of inflection of fibering maps, respectively. That is,

$$\begin{aligned}N^+ &= \{u \in N: \Phi''_u(1) > 0\} = \left\{u \in N: \int_\Omega a(x)|u|^p \mathrm{d}x < 0\right\},\\ N^- &= \{u \in N: \Phi''_u(1) < 0\} = \left\{u \in N: \int_\Omega a(x)|u|^p \mathrm{d}x > 0\right\},\\ N^0 &= \{u \in N: \Phi''_u(1) = 0\} = \left\{u \in N: \int_\Omega a(x)|u|^p \mathrm{d}x = 0\right\}.\end{aligned}$$

Lemma 2.2. *Nehari manifold N^+ and N^- are non-empty.*

Proof. Since $a(x)$ changes sign in Ω and $\Phi_u(t)$ has a turning point at

$$t(u) = \exp\left(\frac{\int_\Omega |\Delta u|^p \mathrm{d}x - \int_\Omega a(x)|u|^p \ln|u| \mathrm{d}x - \int_\Omega b(x)|u|^p \mathrm{d}x}{\int_\Omega a(x)|u|^p \mathrm{d}x}\right),$$

so we can select $u_1, u_2 \in N$ such that

$$\int_\Omega a(x)|u_1|^p \mathrm{d}x < 0 \Rightarrow t(u_1)u_1 \in N^+,$$
$$\int_\Omega a(x)|u_2|^p \mathrm{d}x > 0 \Rightarrow t(u_2)u_2 \in N^-,$$

which implies N^+ and N^- are non-empty.

Lemma 2.3. *If u_0 is a local minimizer for $J(u)$ on N and $u_0 \notin N^0$, then it is a critical point of $J(u)$, that is $J'(u_0) = 0$.*

Proof. Since u_0 is a local minimizer for $J(u)$ on N, so we can use the Lagrange multiplier method to find a constant $\gamma \in \mathbb{R}$ such that

$$J'(u_0) = \gamma g'(u_0),$$

where

$$g(u) = \int_\Omega |\Delta u|^p \mathrm{d}x - \int_\Omega a(x)|u|^p \ln|u| \mathrm{d}x - \int_\Omega b(x)|u|^p \mathrm{d}x.$$

Because of $u_0 \in N$ with $u_0 \notin N^0$, so

$$0 = \langle J'(u_0), u_0 \rangle = \gamma \langle g'(u_0), u_0 \rangle,$$

where

$$\langle g'(u_0), u_0 \rangle = \Phi''_{u_0}(1) = -\int_\Omega a(x)|u_0|^p \mathrm{d}x \neq 0.$$

This implies $\gamma = 0$ and $J'(u_0) = 0$.

Lemma 2.4. ([3]) *Let $\{u_m\}$ be a sequence in $W_0^{2,p}(\Omega)$. If $u_m \rightharpoonup u_0$ and $u_m \nrightarrow u_0$ in $W_0^{2,p}(\Omega)$, then $J(u_0) < \underline{\lim}_{m\to\infty} J(u_m)$. If $u_m \to u_0$ in $W_0^{2,p}(\Omega)$, then $J(u_0) = \underline{\lim}_{m\to\infty} J(u_m)$.*

Lemma 2.5. (Improved logarithmic Sobolev inequality, [22–25]) *Let $u \in W_0^{2,p}(\Omega)$ and $1 < p < +\infty$, $\mu > 0$ be any number. Then*

$$p\int_\Omega |u|^p \ln \frac{|u|}{\|u\|_{L^p(\Omega)}} \mathrm{d}x + \frac{n}{p} \ln\left(\frac{p\mu e}{nl_p}\right) \int_\Omega |u|^p \mathrm{d}x \leq \mu S_p^p \int_\Omega |\Delta u|^p \mathrm{d}x, \tag{2.3}$$

where $l_p = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p-1}{p}+1\right)}\right]^{\frac{p}{n}}$, Γ is a Gamma function, S_p is an optimal embedding constant for the embedding $W_0^{2,p}(\Omega) \hookrightarrow W_0^{1,p}(\Omega)$.

Lemma 2.6. *Assume that $u \in W_0^{2,p}(\Omega)$ and $\int_\Omega a(x)|u|^p \mathrm{d}x = 0$. Let $M_1 = \max_{x\in\overline{\Omega}} |a(x)|$, $M_2 = \max_{x\in\overline{\Omega}} |b(x)|$. Then*

$$J(u) \geq \frac{1}{p}\left(1 - \frac{M_1 S_p^p}{p} e^{\frac{p^2}{n}\left(\frac{M_2}{M_1} + \frac{2|\Omega|_n}{pe}\right) - 1 - \ln\frac{p}{nl_p}}\right) \|\Delta u\|^p_{L^p(\Omega)}. \tag{2.4}$$

Proof. Since $\int_\Omega a(x)|u|^p \mathrm{d}x = 0$, it follows from (2.1) that

$$J(u) = \frac{1}{p}\left(\int_\Omega |\Delta u|^p \mathrm{d}x - \int_\Omega a(x)|u|^p \ln \frac{|u|}{\|u\|_{L^p(\Omega)}} \mathrm{d}x - \int_\Omega b(x)|u|^p \mathrm{d}x\right). \tag{2.5}$$

Taking $\overline{u} = \frac{u}{\|u\|_{L^p(\Omega)}}$, then

$$\int_\Omega a(x)|u|^p \ln \frac{|u|}{\|u\|_{L^p(\Omega)}} \mathrm{d}x = \int_{\Omega_1} a(x)|u|^p \ln|\overline{u}|\mathrm{d}x + \int_{\Omega_2} a(x)|u|^p \ln|\overline{u}|\mathrm{d}x, \tag{2.6}$$

where $\Omega_1 = \{x \in \Omega, |\overline{u}| < 1\}$, $\Omega_2 = \{x \in \Omega, |\overline{u}| \geq 1\}$. It follows that

$$\begin{aligned}
\int_{\Omega_1} a(x)|u|^p \ln|\bar{u}|\mathrm{d}x &\leq \frac{\|u\|^p_{L^p(\Omega)}}{p}\left|\int_{\Omega_1} a(x)\frac{|u|^p}{\|u\|^p_{L^p(\Omega)}} \ln \frac{|u|^p}{\|u\|^p_{L^p(\Omega)}}\mathrm{d}x\right| \\
&\leq \frac{M_1\|u\|^p_{L^p(\Omega)}}{p}\int_{\Omega_1}\left|\frac{|u|^p}{\|u\|^p_{L^p(\Omega)}} \ln \frac{|u|^p}{\|u\|^p_{L^p(\Omega)}}\right|\mathrm{d}x \\
&\leq \frac{M_1|\Omega|_n\|u\|^p_{L^p(\Omega)}}{pe}.
\end{aligned} \tag{2.7}$$

Using the improved logarithmic Sobolev inequality (2.3) and (2.7), we have

$$\begin{aligned}
&\int_{\Omega_2} a(x)|u|^p \ln|\overline{u}|\mathrm{d}x \\
&\leq M_1\left(\int_\Omega |u|^p \ln|\bar{u}|\mathrm{d}x + \frac{|\Omega|_n\|u\|^p_{L^p(\Omega)}}{pe}\right) \\
&\leq M_1\left(\frac{\mu S_p^p}{p}\|\Delta u\|^p_{L^p(\Omega)} - \left(\frac{n}{p^2}\ln\left(\frac{p\mu e}{nl_p}\right) - \frac{|\Omega|_n}{pe}\right)\|u\|^p_{L^p(\Omega)}\right).
\end{aligned} \tag{2.8}$$

Combining (2.5), (2.6), (2.7) and (2.8), we get

$$J(u) \geq \frac{1}{p}\left(1 - \frac{\mu M_1 S_p^p}{p}\right)\|\Delta u\|^p_{L^p(\Omega)} + \frac{1}{p}\left(\frac{nM_1}{p^2}\ln\left(\frac{p\mu e}{nl_p}\right) - \frac{2M_1|\Omega|_n}{pe} - M_2\right)\|u\|^p_{L^p(\Omega)}. \tag{2.9}$$

By $\frac{1}{p}\left(\frac{nM_1}{p^2}\ln\left(\frac{p\mu e}{nl_p}\right) - \frac{2M_1|\Omega|_n}{pe} - M_2\right) = 0$, we have

$$\mu = e^{\frac{p^2}{n}\left(\frac{M_2}{M_1} + \frac{2|\Omega|_n}{pe}\right) - 1 - \ln\left(\frac{p}{nl_p}\right)}. \tag{2.10}$$

Substituting (2.10) into (2.9), we can see that (2.4) holds.

3 Multiple solutions

In this section we will show that the energy functional $J(u)$ has two minimizers on N^+ and N^- by the variational method, respectively.

Lemma 3.1. *$J(u)$ is bounded below on N^+.*

Proof. Since $u \in N^+$, we know

$$J(u) = \frac{1}{p^2}\int_\Omega a(x)|u|^p\mathrm{d}x \geq \frac{1}{p^2}\min_{x\in\overline{\Omega}} a(x)\|u\|^p_{L^p(\Omega)} \geq \frac{B_p^p}{p^2}\min_{x\in\overline{\Omega}} a(x)\|u\|^p_{W_0^{2,p}(\Omega)},$$

where B_p is an optimal embedding constant for the embedding $W_0^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$. If N^+ is bounded, then $J(u)$ is bounded below on N^+. So we prove N^+ is bounded by contradiction.

Suppose that N^+ is not bounded, then there exists a sequence $\{u_m\} \subseteq N^+$ such that $\|u_m\|_{W_0^{2,p}(\Omega)} \to \infty$ as $m \to \infty$. Let $v_m = \frac{u_m}{\|u_m\|_{W_0^{2,p}(\Omega)}}$, we may assume that $v_m \rightharpoonup v_0$ in $W_0^{2,p}(\Omega)$, and so $v_m \to v_0$ in $L^p(\Omega)$.

Since $u_m \in N^+$, we have

$$\int_\Omega a(x)|u_m|^p \mathrm{d}x < 0, \int_\Omega a(x)|v_m|^p \mathrm{d}x < 0,$$

so $\int_\Omega a(x)|v_0|^p \mathrm{d}x \le 0$. On the other hand, $u_m \in N$, we have

$$\int_\Omega |\Delta u_m|^p \mathrm{d}x - \int_\Omega a(x)|u_m|^p \ln|u_m| \mathrm{d}x - \int_\Omega b(x)|u_m|^p \mathrm{d}x = 0,$$

which implies

$$\begin{aligned}&\int_\Omega |\Delta v_m|^p \mathrm{d}x - \int_\Omega a(x)|v_m|^p \ln|v_m| \mathrm{d}x - \int_\Omega b(x)|v_m|^p \mathrm{d}x\\&= \ln \|u_m\|_{W_0^{2,p}(\Omega)} \int_\Omega a(x)|v_m|^p \mathrm{d}x < 0.\end{aligned} \tag{3.1}$$

Because $v_m = \frac{u_m}{\|u_m\|_{W_0^{2,p}(\Omega)}}$, we obtain

$$\|v_m\|_{W_0^{2,p}(\Omega)} = \left(\int_\Omega |\Delta v_m|^p \mathrm{d}x\right)^{\frac{1}{p}} = \frac{\left(\int_\Omega |\Delta u_m|^p \mathrm{d}x\right)^{\frac{1}{p}}}{\|u_m\|_{W_0^{2,p}(\Omega)}} = 1.$$

Similar to (2.6), (2.7) and (2.8), we can get

$$\begin{aligned}&\left|\int_\Omega |\Delta v_m|^p \mathrm{d}x - \int_\Omega a(x)|v_m|^p \ln|v_m| \mathrm{d}x - \int_\Omega b(x)|v_m|^p \mathrm{d}x\right|\\&\le 1 + \left|\int_\Omega a(x)|v_m|^p \ln|v_m| \mathrm{d}x\right| + \left|\int_\Omega b(x)|v_m|^p \mathrm{d}x\right|\\&\le c,\end{aligned} \tag{3.2}$$

where c is independent of m. Because $\|u_m\|_{W_0^{2,p}(\Omega)} \to \infty$ as $m \to \infty$, it follows from (3.1) and (3.2) that

$$\int_\Omega a(x)|v_0|^p \mathrm{d}x = 0. \tag{3.3}$$

If $v_m \nrightarrow v_0$ in $W_0^{2,p}(\Omega)$, by (1.6) and (2.4),

$$J(v_0) \ge \frac{1}{p}\left(1 - \frac{M_1 S_p^p}{p} e^{\frac{p^2}{n}\left(\frac{M_2}{M_1} + \frac{2|\Omega|_n}{pe}\right) - 1 - \ln\frac{p}{nl_p}}\right)\|\Delta v_0\|_{L^p(\Omega)}^p \ge 0.$$

Lemma 2.4, (3.1) and (3.3) imply

$$\begin{aligned} pJ(v_0) &= \int_\Omega |\Delta v_0|^p \mathrm{d}x - \int_\Omega a(x)|v_0|^p \ln|v_0| \mathrm{d}x - \int_\Omega b(x)|v_0|^p \mathrm{d}x \\ &< \varliminf_{m\to\infty}\left[\int_\Omega |\Delta v_m|^p \mathrm{d}x - \int_\Omega a(x)|v_m|^p \ln|v_m| \mathrm{d}x - \int_\Omega b(x)|v_m|^p \mathrm{d}x\right] \\ &= \varliminf_{m\to\infty} \ln \|u_m\|_{W_0^{2,p}(\Omega)} \int_\Omega a(x)|v_m|^p \mathrm{d}x \le 0, \end{aligned}$$

which is a contradiction.

If $v_m \to v_0$ in $W_0^{2,p}(\Omega)$, $\|v_0\|_{W_0^{2,p}(\Omega)} = \|\Delta v_0\|_{L^p(\Omega)}^p = 1$, from (1.6) and (2.4),

$$J(v_0) \ge \frac{1}{p}\left(1 - \frac{M_1 S_p^p}{p} e^{\frac{p^2}{n}\left(\frac{M_2}{M_1} + \frac{2|\Omega|_n}{pe}\right) - 1 - \ln\frac{p}{nl_p}}\right) > 0.$$

By Lemma 2.4, (3.1) and (3.3),

$$\begin{aligned} pJ(v_0) &= \int_\Omega |\Delta v_0|^p \mathrm{d}x - \int_\Omega a(x)|v_0|^p \ln|v_0| \mathrm{d}x - \int_\Omega b(x)|v_0|^p \mathrm{d}x \\ &= \lim_{m\to\infty}\left[\int_\Omega |\Delta v_m|^p \mathrm{d}x - \int_\Omega a(x)|v_m|^p \ln|v_m| \mathrm{d}x - \int_\Omega b(x)|v_m|^p \mathrm{d}x\right] \\ &= \lim_{m\to\infty} \ln \|u_m\|_{W_0^{2,p}(\Omega)} \int_\Omega a(x)|v_m|^p \mathrm{d}x \le 0, \end{aligned}$$

which is also a contradiction.

Hence N^+ is bounded and then $J(u)$ is bounded below on N^+.

Lemma 3.2. *$J(u)$ has a minimizer on N^+.*

Proof. Since $J(u)$ is bounded below on N^+, there exists a minimizing sequence $\{u_m\} \subseteq \mathrm{N}^+$ such that $\lim\limits_{m\to\infty} J(u_m) = \inf\limits_{u\in\mathrm{N}^+} J(u) < 0$. From Lemma 3.1 we know N^+ is bounded, without loss of generality we can assume that $u_m \rightharpoonup u_0$ in $W_0^{2,p}(\Omega)$, and so $u_m \to u_0$ in $L^p(\Omega)$.

By $u_m \in N^+$, we have

$$\int_\Omega a(x)|u_m|^p \mathrm{d}x < 0, \int_\Omega a(x)|u_0|^p \mathrm{d}x = \lim_{m\to\infty} \int_\Omega a(x)|u_m|^p \mathrm{d}x = \lim_{m\to\infty} p^2 J(u_m) < 0.$$

Because $u_m \in N$, we know

$$\int_\Omega |\Delta u_m|^p \mathrm{d}x - \int_\Omega a(x)|u_m|^p \ln|u_m| \mathrm{d}x - \int_\Omega b(x)|u_m|^p \mathrm{d}x = 0.$$

Suppose $u_m \nrightarrow u_0$ in $W_0^{2,p}(\Omega)$, then

$$\begin{aligned}&\int\limits_\Omega |\Delta u_0|^p \mathrm{d}x - \int\limits_\Omega a(x)|u_0|^p \ln|u_0|\mathrm{d}x - \int\limits_\Omega b(x)|u_0|^p \mathrm{d}x\\&< \underline{\lim}_{m\to\infty}\left[\int\limits_\Omega |\Delta u_m|^p \mathrm{d}x - \int\limits_\Omega a(x)|u_m|^p \ln|u_m|\mathrm{d}x - \int\limits_\Omega b(x)|u_m|^p \mathrm{d}x\right]\\&= 0.\end{aligned}$$

By the fibering maps $\Phi_u(t)$, there exists a

$$t(u_0) = \exp\left(\frac{\int_\Omega |\Delta u_0|^p \mathrm{d}x - \int_\Omega a(x)|u_0|^p \ln|u_0|\mathrm{d}x - \int_\Omega b(x)|u_0|^p \mathrm{d}x}{\int_\Omega a(x)|u_0|^p \mathrm{d}x}\right) > 1,$$

such that $t(u_0)u_0 \in N^+$, and $J(u)$ attains a minimum at $t(u_0)u_0$. So

$$J(t(u_0)u_0) < J(u_0) \le \lim_{m\to\infty} J(u_m) = \inf_{u\in N^+} J(u),$$

which is impossible. So $u_m \to u_0$ in $W_0^{2,p}(\Omega)$ and $u_0 \in N^+$, this means u_0 is a minimizer for $J(u)$ on N^+, namaly, $J(u_0) = \inf\limits_{u\in N^+} J(u) < 0.$

Lemma 3.3. *Every minimizing sequence $\{u_m\} \subseteq N^-$ is bounded for $J(u)$ on N^-.*

Proof. Let $\{u_m\} \subseteq N^-$ be a minimizing sequence, then we have

$$\int\limits_\Omega a(x)|u_m|^p \mathrm{d}x \to c, c \ge 0.$$

Assume by contradiction that $\{u_m\} \subseteq N^-$ is not bounded in $W_0^{2,p}(\Omega)$, that is, $\|u_m\|_{W_0^{2,p}(\Omega)} \to \infty$ as $m \to \infty$. Let $v_m = \frac{u_m}{\|u_m\|_{W_0^{2,p}(\Omega)}}$, we may assume that $v_m \rightharpoonup v_0$ in $W_0^{2,p}(\Omega)$, and so $v_m \to v_0$ in $L^p(\Omega)$.

Since $u_m \in N^-$, we have

$$\int\limits_\Omega a(x)|u_m|^p \mathrm{d}x > 0, \int\limits_\Omega a(x)|v_m|^p \mathrm{d}x > 0,$$

so $\int_\Omega a(x)|v_0|^p \mathrm{d}x \ge 0$. On the other hand, $u_m \in N$, we have

$$\int\limits_\Omega |\Delta u_m|^p \mathrm{d}x - \int\limits_\Omega a(x)|u_m|^p \ln|u_m|\mathrm{d}x - \int\limits_\Omega b(x)|u_m|^p \mathrm{d}x = 0.$$

Then we have

$$\begin{aligned}&\int\limits_\Omega |\Delta v_m|^p \mathrm{d}x - \int\limits_\Omega a(x)|v_m|^p \ln|v_m|\mathrm{d}x - \int\limits_\Omega b(x)|v_m|^p \mathrm{d}x\\&= \ln\|u_m\|_{W_0^{2,p}(\Omega)} \int\limits_\Omega a(x)|v_m|^p \mathrm{d}x \ge 0.\end{aligned} \tag{3.4}$$

Similar to (3.2), we can get

$$\left|\int\limits_\Omega |\Delta v_m|^p \mathrm{d}x - \int\limits_\Omega a(x)|v_m|^p \ln|v_m|\mathrm{d}x - \int\limits_\Omega b(x)|v_m|^p \mathrm{d}x\right| \le c, \tag{3.5}$$

where c is independent of m. Because $\|u_m\|_{W_0^{2,p}(\Omega)} \to \infty$ as $m \to \infty$, it follows from (3.4) and (3.5) that

$$\int_\Omega a(x)|v_0|^p \mathrm{d}x = 0. \tag{3.6}$$

If $v_m \nrightarrow v_0$ in $W_0^{2,p}(\Omega)$, by (1.6) and (2.4),

$$J(v_0) \geq \frac{1}{p}\left(1 - \frac{M_1 S_p^p}{p} e^{\frac{p^2}{n}\left(\frac{M_2}{M_1} + \frac{2|\Omega|_n}{pe}\right) - 1 - \ln\frac{p}{nl_p}}\right)\|\Delta v_0\|_{L^p(\Omega)}^p \geq 0.$$

Lemma 2.4, (3.4) and (3.6) imply

$$\begin{aligned} pJ(v_0) &= \int_\Omega |\Delta v_0|^p \mathrm{d}x - \int_\Omega a(x)|v_0|^p \ln|v_0| \mathrm{d}x - \int_\Omega b(x)|v_0|^p \mathrm{d}x \\ &< \underline{\lim}_{m\to\infty}\left[\int_\Omega |\Delta v_m|^p \mathrm{d}x - \int_\Omega a(x)|v_m|^p \ln|v_m| \mathrm{d}x - \int_\Omega b(x)|v_m|^p \mathrm{d}x\right] \\ &= \underline{\lim}_{m\to\infty} \ln\|u_m\|_{W_0^{2,p}(\Omega)} \int_\Omega a(x)|v_m|^p \mathrm{d}x = 0, \end{aligned}$$

which is a contradiction.

If $v_m \to v_0$ in $W_0^{2,p}(\Omega)$, $\|v_0\|_{W_0^{2,p}(\Omega)} = \|\Delta v_0\|_{L^p(\Omega)}^p = 1$, from (1.6) and (2.4),

$$J(v_0) \geq \frac{1}{p}\left(1 - \frac{M_1 S_p^p}{p} e^{\frac{p^2}{n}\left(\frac{M_2}{M_1} + \frac{2|\Omega|_n}{pe}\right) - 1 - \ln\frac{p}{nl_p}}\right) > 0.$$

By Lemma 2.4, (3.4) and (3.6),

$$\begin{aligned} pJ(v_0) &= \int_\Omega |\Delta v_0|^p \mathrm{d}x - \int_\Omega a(x)|v_0|^p \ln|v_0| \mathrm{d}x - \int_\Omega b(x)|v_0|^p \mathrm{d}x \\ &= \lim_{m\to\infty}\left[\int_\Omega |\Delta v_m|^p \mathrm{d}x - \int_\Omega a(x)|v_m|^p \ln|v_m| \mathrm{d}x - \int_\Omega b(x)|v_m|^p \mathrm{d}x\right] \\ &= \lim_{m\to\infty} \ln\|u_m\|_{W_0^{2,p}(\Omega)} \int_\Omega a(x)|v_m|^p \mathrm{d}x = 0, \end{aligned}$$

which is also a contradiction.

This means that every minimizing sequence $\{u_m\} \subseteq N^-$ is bounded for $J(u)$ on N^-.

Lemma 3.4. *$J(u)$ has a minimizer on N^-.*

Proof. From Lemma 3.3, a minimizing sequence $\{u_m\} \subseteq N^-$ is bounded for $J(u)$ on N^-, and $\lim_{m\to\infty} J(u_m) = \inf_{u\in N^-} J(u) > 0$. Assume that $u_m \rightharpoonup u_0$ in $W_0^{2,p}(\Omega)$, and so $u_m \to u_0$ in $L^p(\Omega)$.

Using $u_m \in N^-$, we have

$$\int_\Omega a(x)|u_0|^p \mathrm{d}x = \lim_{m\to\infty}\int_\Omega a(x)|u_m|^p \mathrm{d}x = \lim_{m\to\infty} p^2 J(u_m) > 0.$$

Suppose $u_m \nrightarrow u_0$ in $W_0^{2,p}(\Omega)$, then

$$\begin{aligned}&\int_\Omega |\Delta u_0|^p \mathrm{d}x - \int_\Omega a(x)|u_0|^p \ln|u_0| \mathrm{d}x - \int_\Omega b(x)|u_0|^p \mathrm{d}x\\&< \underline{\lim}_{m\to\infty}\left[\int_\Omega |\Delta u_m|^p \mathrm{d}x - \int_\Omega a(x)|u_m|^p \ln|u_m| \mathrm{d}x - \int_\Omega b(x)|u_m|^p \mathrm{d}x\right]\\&= 0\end{aligned}$$

By the fibering maps $\Phi_u(t)$, there exists a

$$t(u_0) = \exp\left(\frac{\int_\Omega |\Delta u_0|^p \mathrm{d}x - \int_\Omega a(x)|u_0|^p \ln|u_0| \mathrm{d}x - \int_\Omega b(x)|u_0|^p \mathrm{d}x}{\int_\Omega a(x)|u_0|^p \mathrm{d}x}\right) < 1$$

such that $t(u_0)u_0 \in N^-$, $t(u_0)u_m \rightharpoonup t(u_0)u_0$. But $t(u_0)u_m \nrightarrow t(u_0)u_0$ in $W_0^{2,p}(\Omega)$. Then

$$J(t(u_0)u_0) < \underline{\lim}_{m\to\infty} J(t(u_0)u_m).$$

Since the fibering maps $\Phi_{u_m}(t) = J(tu_m)$ attains a maximum at $t = 1$, so

$$\underline{\lim}_{m\to\infty} J(t(u_0)u_m) \le \lim_{m\to\infty} J(u_m) = \inf_{u\in N^-} J(u),$$

this implies $J(t(u_0)u_0) < \inf_{u\in N^-} J(u)$, which is a contradiction. Hence $u_m \to u_0$ in $W_0^{2,p}(\Omega)$ and $u_0 \in N^-$, this means that u_0 is a minimizer for $J(u)$ on N^-. Namaly, $J(u_0) = \inf_{u\in N^-} J(u) > 0$.

Proof of Theorem 1.1 From Lemmas 3.2 and 3.4, we obtain the energy functional $J(u)$ has two minimizers on N^+ and N^-, respectively. By Lemma 2.3, we know $J(u)$ has two critical points u_1 and u_2 in $W_0^{2,p}(\Omega)$, which means the problem (1.1) has two nontrivial weak solutions.

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