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Research Article

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Morse index of circular solutions for repulsive central force problems on surfaces

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Abstract: The classical theory of repulsive central force problem on the standard (flat) Euclidean plane can be generalized to surfaces by reformulating the basic underlying physical principles by means of differential geometry. The aim of the present paper is to compute the Morse index of the circular periodic orbits in the case of repulsive power-law potentials of the Riemannian distance on revolution's surfaces.

Keywords: conformal surfaces; circular orbits; Morse index; Maslov index

MSC 2020: 37B30; 53D12; 53C22; 58J30

1 Description of the problem and main results

Central-force dynamics and orbital motion are fundamental topics in advanced mechanics. In particular, Newtonian mechanics on non-flat spaces can be naturally formulated within the framework of Riemannian geometry. A mechanical system is described by a triple (M, g, V), where M is a smooth manifold representing the configuration space, g is a Riemannian metric that determines the kinetic energy, and V is the potential function.

In Physics and Classical Mechanics, many interactions are modeled by using potentials depending on the distance alone, such as when one studies systems of many particles interacting with each other or when one considers a single particle that interacts with a source. Thus, it seems natural to investigate systems on manifolds M whose potential is a function of the distance from a point.

For attractive central force problem, the Keplerian problem is a classical model. In the last decades, several authors provided a generalization of the gravitational Keplerian potential in the constant curvature case, starting with the well-known manuscript of Harin & Kozlov [1]. Some generalizations can be found in refs. [2–4] and references therein. Inspired by the approach in ref. [5], in ref. [6] we compute the Morse index of the circular orbits under some attractive power-law potentials of the Riemannian distance on revolution's surfaces by using the normal form provided in ref. [7].

But some classical repulsive central force problems are excluded, for example, the dynamics of electrons with the same charge. Therefore, in the rest of the paper, we aim to compute the Morse index of the circular periodic orbits in the case of repulsive power-law potentials of the Riemannian distance on revolution's surfaces which are conformal to the flat one, namely, the sphere and hyperbolic plane. The computation of the

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Morse index can be reduced to the calculation of the Maslov index associated with the periodic orbit of the corresponding linear Hamiltonian system. For detailed accounts of the Maslov index and its applications to stability problem, we refer the reader to refs. [8–21], among others.

In polar coordinates ξ, θ and up to rescaling time and normalizing the radial variable, we end up considering the following Lagrangian function

$$L_\alpha(\xi, \vartheta, \dot{\xi}, \dot{\vartheta}) = \frac{1}{2} p(\xi)\left[\dot{\xi}^2 + \xi^2 \dot{\vartheta}^2\right] + q(\xi).$$

Here p denotes the conformal factor and q the potential. For the sphere and the hyperbolic plane with constant curvature metrics the conformal factors potentials are

– (Sphere case)

$$p(\xi) := \frac{2}{(1+\xi^2)^2}, \qquad q(\xi) := -\arctan^\alpha \xi. \tag{1.1}$$

– (Hyperbolic plane case)

$$p(\xi) := \frac{2}{(1-\xi^2)^2}, \qquad q(\xi) := -\ln^\alpha\left(\frac{1+\xi}{1-\xi}\right). \tag{1.2}$$

(Here α is non-vanishing and ξ is non-negative). T-periodic solutions of the associated Euler–Lagrange equation can be seen as critical points of the Lagrangian action function

$$A(x) = \int_0^T L\big(\xi(t), \vartheta(t), \xi'(t), \vartheta'(t)\big)\, \mathrm{d}t,$$

where $x = (\xi, \vartheta)$, $T > 0$ denotes the prime period of the orbit and A is defined on the Hilbert space of the H^1 loops (of period T) in the punctured plane. We let x be a T-periodic circular orbit, i.e. a solution of the form $x(t) = (\xi_0, \theta_0 + t\omega)$ for $t \in [0, T/\omega]$. Denoting by $m^-(x)$ the Morse index of the critical point x, our results read as follows.

Theorem 1. (***Sphere Case***). *Let p and q be as in* (1.1)*, and let x denote a circular solution. Then the Morse index of x is given by*

$$m^-(x) = \begin{cases} 1 & \text{if } (\xi_0, \alpha) \in \Omega^-_{1,0}, \\ 2 & \text{if } (\xi_0, \alpha) \in \Omega^+_1 \cup \Omega^0_1 \cup \Omega^+_2 \cup \Omega^0_2, \\ 2k+1 & \text{if } (\xi_0, \alpha) \in \Omega^-_{1,k} \cup \Omega^-_{2,k}, \quad k = 1, 2, \ldots \end{cases}$$

Here, Ω^+_1 and Ω^0_1 are defined in (3.6)*, $\Omega^-_{1,k}$ in* (3.7)*, Ω^+_2 and Ω^0_2 in* (3.8)*, and $\Omega^-_{2,k}$ in* (3.9)*. The regions are illustrated in* Figure 1.

Theorem 2. (***Hyperbolic plane Case***). *Let p and q be as in* (1.2)*, and let x denote a circular solution. Then the Morse index of x is given by*

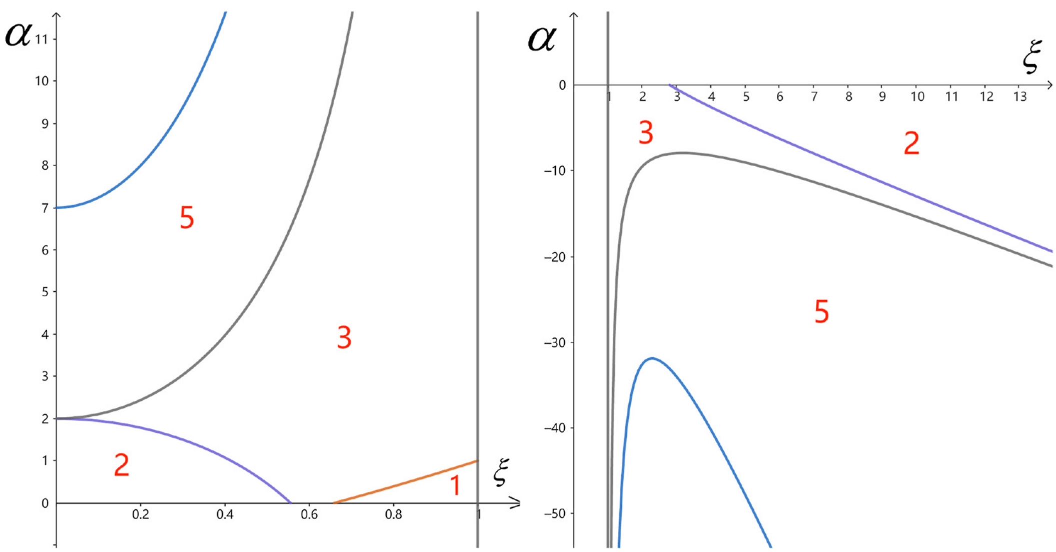
$$m^-(x) = \begin{cases} 2k & \text{if } (\xi_0, \alpha) \in \Omega^+_{3,k} \cup \Omega^0_{3,k}, \\ 2k+1 & \text{if } (\xi_0, \alpha) \in \Omega^-_{3,k}, \end{cases} \quad \text{for } k = 1, 2, \ldots,$$

where $\Omega^\pm_{3,k}$ and $\Omega^0_{3,k}$ are defined in (3.11)*. These regions are illustrated in* Figure 2.

For Euclidean case we have the following result.

Theorem 3. *Let $p(\xi) = 1$, $q(\xi) = -\xi^\alpha$ and let x be a circular solution. Then the Morse index of x is given by*

$$m^-(x) = \begin{cases} 2 & \text{if } 0 < \alpha \leq 2, \\ 2k+1 & \text{if } k^2 - 2 < \alpha \leq (k+1)^2 - 2 \quad \text{for} \quad k \geq 2. \end{cases}$$



(a) (b)

Figure 1: (Sphere case) The subregions of Ω_1, Ω_2 corresponding to the jumps of the Morse index. (a) (Sphere case α positive) In this figure are displayed the subregions of the $\widehat{\xi O \alpha}$-region $\Omega_1 := (0,1) \times (0,+\infty)$ labeled by the Morse index of the corresponding circular orbit. (b) (Sphere case α negative) In this figure are displayed the subregions of the $\widehat{\xi O \alpha}$-region $\Omega_2 := (1,+\infty) \times (-\infty, 0)$ labeled by the Morse index of the corresponding circular orbit.

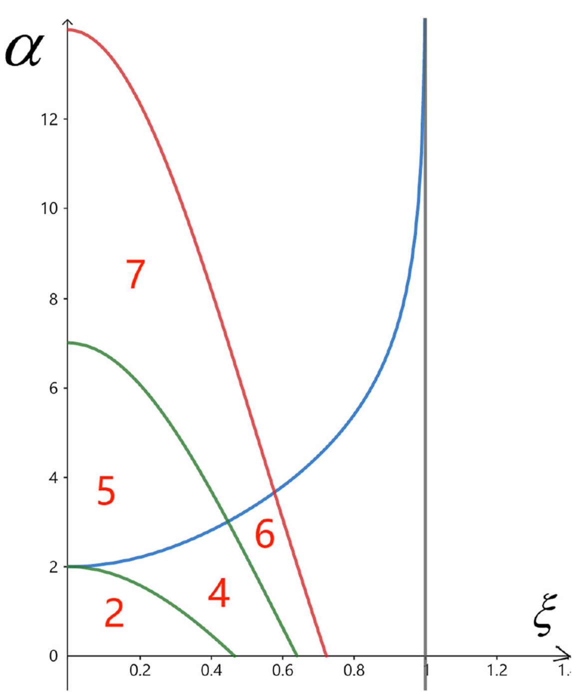


Figure 2: (Hyperbolic plane case) In the figure we represent the subregions of the Ω_3 corresponding to the jumps of the Morse index.

2 Central force problem on constant curvature surfaces

We start by considering the configuration space $(\mathbb{R}^2, g)$ equipped with polar coordinates (r, ϑ), where g is a conformally flat metric and we denote by $\mathbb{S}^2_R$ (resp. $\mathbb{H}^2_R$) the sphere (resp. the pseudo-sphere) of radius R. The conformal factor is given by

$$\mu_R(r) := \begin{cases} \dfrac{2R^2}{R^2 + r^2} & \text{for } \mathbb{S}^2_R, \\ \dfrac{2R^2}{R^2 - r^2} & \text{for } \mathbb{H}^2_R. \end{cases}$$

In terms of the curvature κ, the conformal factor can be written at once as

$$\mu_R(r) = \frac{2}{1 + \kappa r^2}, \quad \text{where} \quad \kappa = \begin{cases} 1/R^2 & \text{for the sphere,} \\ -1/R^2 & \text{for the pseudo} - \text{sphere.} \end{cases}$$

We now take the origin as the center of the central force and we consider the simple mechanical system (M, g, V) where $M := \mathbb{R}^2 \setminus \{(0,0)\}$ and $V: M \to \mathbb{R}$ is a power law potential energy (independent on ϑ) depending only on the Riemannian distance from the origin. By a direct integration of the conformal factor, we get that the distance of the point $P(r, \vartheta)$ to the origin is

$$d_R(r) = \begin{cases} 2R \arctan(r/R) & \text{for } \mathbb{S}^2_R, \\ R \ln\left(\dfrac{R + r}{R - r}\right) & \text{for } \mathbb{H}^2_R. \end{cases}$$

Given $\alpha \in \mathbb{R}^*$, we let

$$V_\alpha : M \to \mathbb{R} \quad \text{defined by} \quad V_\alpha(r, \vartheta) = m\,[d_R(r)]^\alpha, \qquad m \in (0, +\infty)$$

and we consider the Lagrangian $\widetilde{L}_\alpha$ of the mechanical system (M, g, V_α) on the state space TM given by

$$\widetilde{L}_\alpha(r, \vartheta, v_r, v_\vartheta) = \frac{1}{2}\mu_R^2(r)(v_r^2 + r^2 v_\vartheta^2) - V_\alpha(r, \vartheta).$$

By introducing the change of variables $\xi := r/R$, the Lagrangian function can be rewritten as follows:

$$\widetilde{L}_\alpha(\xi, \vartheta, v_\xi, v_\vartheta) := \begin{cases} \dfrac{2R^2}{(1 + \xi^2)^2}\left[v_\xi^2 + \xi^2 v_\vartheta^2\right] - m\left[2R \arctan \xi\right]^\alpha & \text{for } \mathbb{S}^2_R, \\ \dfrac{2R^2}{(1 - \xi^2)^2}\left[v_\xi^2 + \xi^2 v_\vartheta^2\right] - m\left[R \ln\left(\dfrac{1 + \xi}{1 - \xi}\right)\right]^\alpha & \text{for } \mathbb{H}^2_R. \end{cases}$$

Now, computing $\widetilde{L}_\alpha$ along a smooth curve and rescaling time by setting

$$t := \begin{cases} \left(m\, 2^{\alpha - 1} R^{\alpha - 2}\right)^{-1/2} \tau & \text{in the case of } \mathbb{S}^2_R, \\ \left(m\, 2^{-1} R^{\alpha - 2}\right)^{-1/2} \tau & \text{in the case of } \mathbb{H}^2_R, \end{cases}$$

and denoting by $\cdot$ the τ derivative as well, we get that $\widetilde{L}_\alpha = C_\alpha L_\alpha$ where

$$L_\alpha(\xi, \vartheta, \dot{\xi}, \dot{\vartheta}) := \begin{cases} \dfrac{1}{(1 + \xi^2)^2}\left[\dot{\xi}^2 + \xi^2 \dot{\vartheta}^2\right] - \arctan^\alpha \xi & \text{in the case of } \mathbb{S}^2_R, \\ \dfrac{1}{(1 - \xi^2)^2}\left[\dot{\xi}^2 + \xi^2 \dot{\vartheta}^2\right] - \ln^\alpha\left(\dfrac{1 + \xi}{1 - \xi}\right) & \text{in the case of } \mathbb{H}^2_R, \end{cases}$$

and $C_\alpha := mR^\alpha 2^\alpha$ (resp. $C := mR^\alpha$) in the case of $\mathbb{S}^2_R$ (resp. $\mathbb{H}^2_R$).

2.1 Euler–Lagrange equation and Sturm-Liouville problem

We let $p_\pm(\xi) := 2(1 \pm \xi^2)^{-2}$, $q_+(\xi) := -\arctan^\alpha \xi$ and $q_-(\xi) := -\ln^\alpha\left(\frac{1+\xi}{1-\xi}\right)$.

Notation 2.1. In shorthand notation, with abuse of notation, we use $p(\xi)$ for denoting either $p_+(\xi)$ or $p_-(\xi)$ and $q(\xi)$ for denoting either $q_+(\xi)$ or $q_-(\xi)$. Furthermore, we denote by p' (resp. q') the ξ derivative of p (resp. q).

Now we can consider the Lagrangian L_α instead of $\widetilde{L}_\alpha$ which is given by

$$L_\alpha(\xi, \vartheta, \dot\xi, \dot\vartheta) = \frac{1}{2} p(\xi)\big[\dot\xi^2 + \xi^2 \dot\vartheta^2\big] + q(\xi),$$

and by a direct calculation we get that the associated Euler–Lagrangian equation is

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}(p\,\dot\xi) = \dfrac{1}{2} p'(\dot\xi^2 + \xi^2\dot\vartheta^2) + p\,\xi\dot\vartheta^2 + q' & \text{on } [0, T], \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}(p\,\xi^2\dot\vartheta) = 0. \end{cases} \tag{2.1}$$

A special class of solutions of the Euler–Lagrange equation is provided by the circular solutions pointwise defined by $x(t) = \big(\xi_0, \vartheta(t)\big)$, where by the first equation of (2.1), we immediately get

$$\dot\vartheta^2 = \frac{-2q'}{(p\,\xi^2)'}\bigg|_{\xi=\xi_0}.$$

Notation 2.2. We set

$$\begin{aligned} &p_0 := p(\xi_0), && p_0' := p'(\xi_0), && p_0'' := p''(\xi_0), \\ &q_0 := q(\xi_0), && q_0' := q'(\xi_0), && q_0'' := q''(\xi_0), \\ &\eta_0 := p_0\,\xi_0^2, && \eta_0' := \big(p_0\,\xi_0^2\big)' = p_0'\xi_0^2 + 2p_0\,\xi_0, && \dot\vartheta_0^2 = -2q_0'\,{\eta'}_0^{-1}. \end{aligned} \tag{2.2}$$

So, the period T is given by

$$T = 2\pi\,\omega_0, \quad \text{where} \quad \omega_0 := \sqrt{\frac{\eta_0'}{-2q_0'}}.$$

By linearizing along x we get the Sturm-Liouville equation given by

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\big(P\dot y + Qy\big) + Q^\mathsf{T}\dot y + Ry = 0 \quad \text{on} \quad [0, T],$$

where

$$P = \begin{bmatrix} p_0 & 0 \\ 0 & \eta_0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 \\ \zeta_0 & 0 \end{bmatrix}, \quad \text{where} \quad \zeta_0 := \begin{cases} \sqrt{-2q_0'\,\eta_0'}, & \text{if} \quad \eta_0' \geq 0 \\ -\sqrt{-2q_0'\,\eta_0'}, & \text{if} \quad \eta_0' < 0 \end{cases} \quad \text{and finally}$$

$$R = \begin{bmatrix} R_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for} \quad R_{11} := \eta_0' \cdot \left[\frac{q_0'}{\eta_0'}\right]'.$$

We set $B = \begin{bmatrix} P^{-1} & -P^{-1}Q \\ -Q^\mathsf{T}P^{-1} & Q^\mathsf{T}P^{-1}Q - R \end{bmatrix}$. By a direct computation, we get

$$P^{-1} = \begin{bmatrix} p_0^{-1} & 0 \\ 0 & \eta_0^{-1} \end{bmatrix}, \qquad P^{-1}Q = \begin{bmatrix} 0 & 0 \\ \eta_0^{-1} \cdot \zeta_0 & 0 \end{bmatrix},$$

$$Q^{\mathsf{T}}P^{-1}Q = \begin{bmatrix} \eta_0^{-1} \cdot \zeta_0^2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q^{\mathsf{T}}P^{-1}Q - R = \begin{bmatrix} \eta_0^{-1} \cdot \zeta_0^2 - \eta_0' \cdot (q_0'/\eta_0')' & 0 \\ 0 & 0 \end{bmatrix}.$$

We observe that

$$\eta_0^{-1}\,\zeta_0^2 = -2\,q_0'\,(\ln \eta_0)', \qquad \eta_0^{-1}\,\zeta_0 = \sqrt{2\,q_0'\,(\eta_0^{-1})'}, \qquad \text{and} \qquad \eta_0^{-1}\,\zeta_0^2 - R_{11} = -2q_0'\,\frac{\eta_0'}{\eta_0} - \eta_0'\left(\frac{q_0'}{\eta_0'}\right)'.$$

In conclusion, we get

$$B = \begin{bmatrix} p_0^{-1} & 0 & 0 & 0 \\ 0 & \eta_0^{-1} & -\eta_0^{-1} \cdot \zeta_0 & 0 \\ 0 & -\eta_0^{-1} \cdot \zeta_0 & -2q_0'\,\frac{\eta_0'}{\eta_0} - \eta_0'\left(\frac{q_0'}{\eta_0'}\right)' & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad JB = \begin{bmatrix} 0 & \eta_0^{-1} \cdot \zeta_0 & 2q_0'\,\frac{\eta_0'}{\eta_0} + \eta_0'\left(\frac{q_0'}{\eta_0'}\right)' & 0 \\ 0 & 0 & 0 & 0 \\ p_0^{-1} & 0 & 0 & 0 \\ 0 & \eta_0^{-1} & -\eta_0^{-1} \cdot \zeta_0 & 0 \end{bmatrix},$$

where $J := \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}$ denotes the standard complex structure.

Notation 2.3. We let

$$a := p_0^{-1}, \quad b := \eta_0^{-1} \cdot \zeta_0, \quad c := \eta_0^{-1}, \quad d := 2q_0'\,\frac{\eta_0'}{\eta_0} + \eta_0'\left(\frac{q_0'}{\eta_0'}\right)'. \tag{2.3}$$

Bearing this notation in mind the linear autonomous Hamiltonian system $z'(t) = JBz(t)$ reads as

$$\dot{z}(t) = Az(t) \qquad t \in [0, T]$$

where

$$A = \begin{bmatrix} 0 & b & d & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & c & -b & 0 \end{bmatrix}.$$

Then we introduce the following formula to compute the Morse index.

Lemma 2.4. *Let $x = (\xi_0, \vartheta(t))$ be a circular T-periodic solution of the Euler–Lagrange equation. Then, the Morse index of x is given by*

$$m^-(x) = \begin{cases} 2k & \text{if} \quad d < 0 \text{ and } cd + b^2 \geq 0, \\ 2k+1 & \text{if} \quad d < 0 \text{ and } cd + b^2 < 0, \\ 0 & \text{if} \quad \begin{cases} d = 0,\ b = 0 \text{ and } c > 0, \\ d = 0 \text{ and } b \neq 0, \\ d > 0 \text{ and } cd + b^2 > 0, \end{cases} \end{cases}$$

where $k \in \mathbb{N}$ is given by $k \cdot 2\pi < \sqrt{-ad} \cdot T \leq (k+1) \cdot 2\pi$.

Proof. It is referred to [6, Theorem 1]. ☐

3 Computations of Morse index for circular orbits

This section is devoted to compute the Morse index of the circular orbits on the sphere, pseudo-sphere and Euclidean plane. We start by recalling that, in this case, the Lagrangian of the problem is given by

$$L(\xi, \vartheta, \dot{\xi}, \dot{\vartheta}) = \frac{1}{2} p(\xi)\left[\dot{\xi}^2 + \xi^2 \dot{\vartheta}^2\right] + q(\xi).$$

3.1 Circular orbits on the sphere

In this case $p(\xi) = \frac{2}{(1+\xi^2)^2}$ and $q(\xi) = -\arctan^{\alpha} \xi$. In shorthand notation, we set $F(\xi) := \arctan \xi$. Bearing in mind the notation given in Equation (2.2), we get

$$p_0 = 2\left(1+\xi_0^2\right)^{-2}, \quad p_0' = -8\xi_0\left(1+\xi_0^2\right)^{-3}, \quad p_0'' = 8\left(5\xi_0^2 - 1\right)\left(1+\xi_0^2\right)^{-4},$$

$$q_0, = -F^{\alpha}(\xi_0), \quad q_0' = -\alpha \cdot F^{\alpha-1}(\xi_0)(1+\xi_0^2)^{-1}, \quad q_0'' = \frac{-\alpha(\alpha-1)F^{\alpha-2}(\xi_0) + 2\alpha\xi_0 F^{\alpha-1}(\xi_0)}{\left(1+\xi_0^2\right)^2},$$

$$\eta_0 = 2\xi_0^2\left(1+\xi_0^2\right)^{-2}, \quad \eta_0' = 4\xi_0\left(1-\xi_0^2\right)\left(1+\xi_0^2\right)^{-3}, \quad \dot{\vartheta}_0^2 = \frac{\alpha F^{\alpha-1}(\xi_0)(1+\xi_0^2)^2}{2\xi_0\left(1-\xi_0^2\right)}.$$

In order for the (RHS) of $\dot{\vartheta}_0^2$ to be positive, we have to impose the following restriction on ξ_0:

$$\xi_0 \in \begin{cases} (0,1) & \text{if } \alpha > 0, \\ (1,+\infty) & \text{if } \alpha < 0. \end{cases} \tag{3.1}$$

By a direct computation, we get

$$\begin{aligned} a &= \frac{\left(1+\xi_0^2\right)^2}{2}, \quad b = \frac{2\left(1-\xi_0^2\right)}{\xi_0} \cdot \sqrt{\frac{\alpha F^{\alpha-1}(\xi_0)}{2\xi_0\left(1-\xi_0^2\right)}}, \\ c &= \frac{\left(1+\xi_0^2\right)^2}{2\xi_0^2}, \quad d = \frac{-\alpha F^{\alpha-2}(\xi_0)\left\{\left(3\xi_0^4 - 2\xi_0^2 + 3\right) \cdot F(\xi_0) + (\alpha-1)\xi_0\left(1-\xi_0^2\right)\right\}}{\xi_0\left(1+\xi_0^2\right)^2\left(1-\xi_0^2\right)}. \end{aligned} \tag{3.2}$$

In order to determine the Morse index, we have to discuss the sign of d according to α and ξ_0.

3.1.1 First case: α positive

Since in this case $\xi_0 \in (0,1)$, then we get that the sign of d is minus the sign of

$$f_1(\xi) = (3\xi^4 - 2\xi^2 + 3) \cdot \arctan \xi + (\alpha - 1)\xi(1-\xi^2). \tag{3.3}$$

We let

$$\Omega_1 := (0,1) \times (0,+\infty)$$

and it is easy to check that $f_1(\xi) > 0$ for all $(\xi, \alpha) \in \Omega_1$ and consequently there always holds $d < 0$ in Ω_1.

Next we need to determine the sign of $b^2 + cd$. Taking into account Equation (3.2) and after some algebraic manipulations, we have

$$b^2 + cd = \frac{\alpha F^{\alpha-2}(\xi_0)\left\{\left(\xi_0^4 - 6\xi_0^2 + 1\right) \cdot F(\xi_0) - (\alpha-1)\xi_0\left(1-\xi_0^2\right)\right\}}{2\xi_0^3\left(1-\xi_0^2\right)}. \tag{3.4}$$

We observe that, since $F(\xi_0)$ is strictly positive and $1-\xi_0^2>0$, then the sign of b^2+cd coincides with that of $\left(\xi_0^4-6\xi_0^2+1\right)\cdot F(\xi_0)-(\alpha-1)\xi_0\left(1-\xi_0^2\right)$. Let

$$f_2(\xi)=(\xi^4-6\xi^2+1)\arctan\xi-(\alpha-1)\xi(1-\xi^2). \tag{3.5}$$

According to the sign of $f_2(\xi)$ we can split Ω_1 into the following three subregions

$$\begin{aligned}\Omega_1^+ &:=\{(\xi,\alpha)\in\Omega_1\mid f_2(\xi)>0\}=\left\{(\xi,\alpha)\in\Omega_1\;\middle|\;0<\alpha<1+\frac{(\xi^4-6\xi^2+1)\arctan\xi}{\xi(1-\xi^2)}\right\},\\ \Omega_1^- &:=\{(\xi,\alpha)\in\Omega_1\mid f_2(\xi)<0\}=\left\{(\xi,\alpha)\in\Omega_1\;\middle|\;\alpha>1+\frac{(\xi^4-6\xi^2+1)\arctan\xi}{\xi(1-\xi^2)}\right\},\\ \Omega_1^0 &:=\{(\xi,\alpha)\in\Omega_1\mid f_2(\xi)=0\}=\left\{(\xi,\alpha)\in\Omega_1\;\middle|\;\alpha=1+\frac{(\xi^4-6\xi^2+1)\arctan\xi}{\xi(1-\xi^2)}\right\}.\end{aligned} \tag{3.6}$$

By this discussion, we finally get that

$$b^2+cd \text{ is }\begin{cases}\text{positive} & \text{for}\quad (\xi,\alpha)\in\Omega_1^+,\\ \text{negative} & \text{for}\quad (\xi,\alpha)\in\Omega_1^-.\end{cases}$$

In this case for computing the Morse index, we need to calculate the integer k defined by $k\cdot 2\pi<\sqrt{-ad}\,T\le(k+1)\cdot 2\pi$. Since

$$T=\frac{2\pi}{\dot\vartheta}=2\pi\sqrt{\frac{2\xi_0\left(1-\xi_0^2\right)}{\alpha F^{\alpha-1}(\xi_0)(1+\xi_0^2)^2}}$$

and using Equation (3.2), we get

$$\sqrt{-ad}\cdot T=2\pi f_3(\xi),\qquad \text{where}\qquad f_3(\xi):=\sqrt{\frac{(3\xi^4-2\xi^2+3)\cdot\arctan\xi+(\alpha-1)\xi(1-\xi^2)}{\arctan\xi\cdot(1+\xi^2)^2}}.$$

Indeed, we can check that for every $k\in\mathbb{N}$ there exists $(\xi,\alpha)\in\Omega_1$ such that $k<f_3(\xi)\le k+1$. Therefore, for every $k\in\mathbb{N}$ we define

$$\begin{aligned}\Omega_{1,k}^{\pm} &:=\left\{(\xi,\alpha)\in\Omega_1^{\pm}\mid k<f_3(\xi)\le k+1\right\},\\ \Omega_{1,k}^{0} &:=\left\{(\xi,\alpha)\in\Omega_1^{0}\mid k<f_3(\xi)\le k+1\right\}.\end{aligned} \tag{3.7}$$

For example, by a direct computation, we infer that

$$k=0\iff 0<\alpha\le 1-\frac{2(1-\xi^2)\arctan\xi}{\xi}.$$

So, denoting these regions as follows

$$\begin{aligned}\Omega_{1,0}^{\pm} &:=\left\{(\xi,\alpha)\in\Omega_1^{\pm}\;\middle|\;0<\alpha\le 1-\frac{2(1-\xi^2)\arctan\xi}{\xi}\right\},\\ \Omega_{1,0}^{0} &:=\left\{(\xi,\alpha)\in\Omega_1^{0}\;\middle|\;0<\alpha\le 1-\frac{2(1-\xi^2)\arctan\xi}{\xi}\right\}.\end{aligned}$$

we finally get that for any $(\xi,\alpha)\in\Omega_{1,0}^{\pm}\bigcup\Omega_{1,0}^{0}$, we have $k=0$.

Similarly, direct computations show that

$$\begin{aligned}k=1 &\iff 1-\frac{2(1-\xi^2)\arctan\xi}{\xi}<\alpha\le 1+\frac{(\xi^4+10\xi^2+1)\arctan\xi}{\xi(1-\xi^2)},\\ k=2 &\iff 1+\frac{(\xi^4+10\xi^2+1)\arctan\xi}{\xi(1-\xi^2)}<\alpha\le 1+\frac{(6\xi^4+20\xi^2+6)\arctan\xi}{\xi(1-\xi^2)}\end{aligned}$$

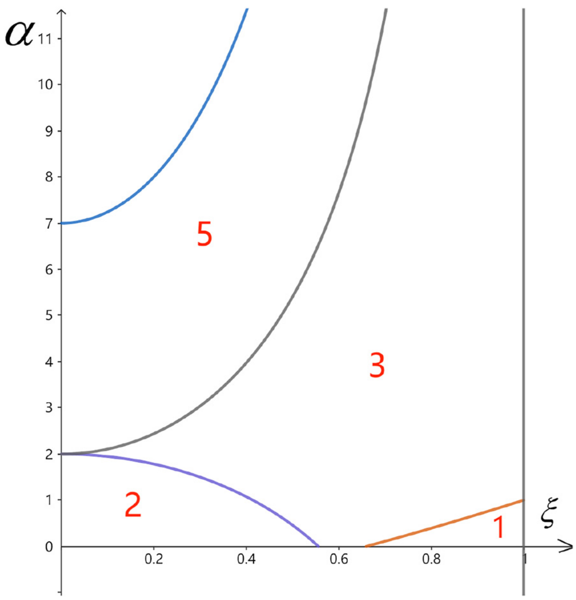


Figure 3: (Sphere case α positive) The subregions of the $\widehat{\xi O \alpha}$-region $\Omega_1 := (0,1) \times (0,+\infty)$ labeled by the Morse index of the corresponding circular orbit.

and so on. Figure 3 are displayed all indices in these involved regions.

By invoking Lemma 2.4 we finally get

$$m^-(x) = \begin{cases} 2 & \text{if } (\xi_0, \alpha) \in \Omega_1^+ \bigcup \Omega_1^0, \\ 2k+1 & \text{if } (\xi_0, \alpha) \in \Omega_{1,k}^-, \quad k = 0, 1, \dots \end{cases}$$

3.1.2 Second case: α negative

In this case, by taking into account the restrictions provided at Equation (3.1), we have $\xi_0 \in (1, +\infty)$ and consequently $1 - \xi_0^2 < 0$. Arguing precisely as before, we need to establish the signs of d and $b^2 + cd$ as well as the value of k. Since all expressions are precisely as before with the only difference about the range of ξ and α.

We let

$$\Omega_2 := (1, +\infty) \times (-\infty, 0).$$

and we start observing that the sign of $f_1(\xi)$ defined in (3.3) for $(\xi, \alpha) \in \Omega_2$ is opposite to that of d. Precisely as before, we can check that $f_1(\xi) > 0$ and consequently $d < 0$ for all $(\xi, \alpha) \in \Omega_2$.

Next we need to determine the sign of $b^2 + cd$. By Equation (3.4) the sign of $b^2 + cd$ coincides with that of $f_2(\xi)$ defined at Equation (3.5) for $(\xi, \alpha) \in \Omega_2$. Then by an algebraic manipulation we have

$$\begin{aligned} \Omega_2^+ &:= \{(\xi, \alpha) \in \Omega_2 \mid f_2(\xi) > 0\} = \left\{ (\xi, \alpha) \in \Omega_2 \,\middle|\, \alpha > 1 + \frac{(\xi^4 - 6\xi^2 + 1) \cdot \arctan \xi}{\xi(1 - \xi^2)} \right\}, \\ \Omega_2^- &:= \{(\xi, \alpha) \in \Omega_2 \mid f_2(\xi) < 0\} = \left\{ (\xi, \alpha) \in \Omega_2 \,\middle|\, \alpha < 1 + \frac{(\xi^4 - 6\xi^2 + 1) \cdot \arctan \xi}{\xi(1 - \xi^2)} \right\}, \\ \Omega_2^0 &:= \{(\xi, \alpha) \in \Omega_2 \mid f_2(\xi) = 0\} = \left\{ (\xi, \alpha) \in \Omega_2 \,\middle|\, \alpha = 1 + \frac{(\xi^4 - 6\xi^2 + 1) \cdot \arctan \xi}{\xi(1 - \xi^2)} \right\}. \end{aligned} \tag{3.8}$$

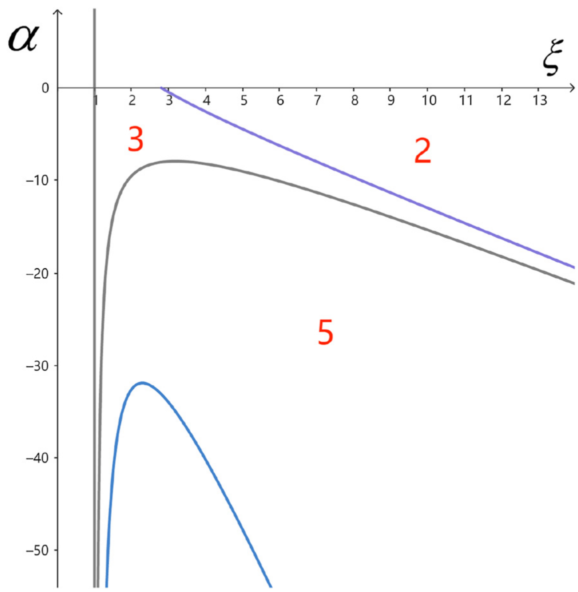


Figure 4: (Sphere case α negative) The subregions of the $\widehat{\xi O\alpha}$-region $\Omega_1 := (1, +\infty) \times (-\infty, 0)$ labeled by the Morse index of the corresponding circular orbit.

So, there holds that

$$cd + b^2 \text{ is } \begin{cases} \text{positive} & \text{for } (\xi, \alpha) \in \Omega_2^+, \\ \text{negative} & \text{for } (\xi, \alpha) \in \Omega_2^-, \\ \text{zero} & \text{for } (\xi, \alpha) \in \Omega_2^0. \end{cases}$$

Firstly, we can check that $f_3(\xi) > 1$ holds for all $(\xi, \alpha) \in \Omega_2$ and for every $k \in \mathbb{N}^+$ there exists $(\xi, \alpha) \in \Omega_2$ such that $k < f_3(\xi) \le k + 1$. Moreover, it is straightforward to check that

$$k = 1 \Longleftrightarrow 1 + \frac{(\xi^4 + 10\xi^2 + 1)\arctan\xi}{\xi(1 - \xi^2)} \le \alpha < 0,$$

$$k = 2 \Longleftrightarrow 1 + \frac{(6\xi^4 + 20\xi^2 + 6)\arctan\xi}{\xi(1 - \xi^2)} \le \alpha < 1 + \frac{(\xi^4 + 10\xi^2 + 1)\arctan\xi}{\xi(1 - \xi^2)}$$

and so on. Some simple computations show that $k = 1$ for all $(\xi, \alpha) \in \Omega_2^+ \cup \Omega_2^0$. Now we define the following planar regions

$$\Omega_{2,k}^- := \{(\xi, \alpha) \in \Omega_2^- | k < f_3(\xi) \le k + 1\}. \tag{3.9}$$

Therefore, by invoking Lemma 2.4, we get

$$m^-(x) = \begin{cases} 2 & \text{if } (\xi, \alpha) \in \Omega_2^+ \bigcup \Omega_2^0, \\ 2k + 1 & \text{if } (\xi, \alpha) \in \Omega_{2,k}^-, \quad k = 1, 2, \dots \end{cases}$$

It is shown in Figure 4.

We finally are in position to summarize the involved discussion in the following conclusive result for the sphere.

Theorem 3.1. *Under the above notations, the indices of a circular orbit on sphere are given by*

$$m^-(x) = \begin{cases} 1 & \text{if } (\xi,\alpha) \in \Omega^-_{1,0}, \\ 2 & \text{if } (\xi_0,\alpha) \in \Omega^+_1 \bigcup \Omega^0_1 \cup \Omega^+_2 \bigcup \Omega^0_2, \\ 2k+1 & \text{if } (\xi_0,\alpha) \in \Omega^-_{1,k} \bigcup \Omega^-_{2,k}, \quad k = 1,2,\ldots. \end{cases}$$

A direct consequence of Theorem 3.1 concerns the Morse index of the circular orbits in one physically interesting cases: $\alpha = 2$ corresponding to a elastic like potential.

Corollary 3.2. *For $\alpha = 2$, then we get $m^-(x) = 3$.*

3.2 Circular orbits on the hyperbolic plane

This subsection is devoted to compute the Morse index for circular orbits on the pseudo-sphere. In this case

$$p(\xi) = \frac{2}{(1-\xi^2)^2} \qquad \text{and} \qquad q(\xi) = -\ln^\alpha\left(\frac{1+\xi}{1-\xi}\right) \qquad \xi \in (0,1).$$

We let

$$G(\xi) := \ln\left(\frac{1+\xi}{1-\xi}\right) \quad \text{and} \quad G'(\xi) := \frac{dG}{d\xi}(\xi).$$

By a direct computations we have

$$p'(\xi) = \frac{8\xi}{(1-\xi^2)^3}, \qquad p''(\xi) = \frac{8+40\xi^2}{(1-\xi^2)^4},$$

$$q'(\xi) = -\frac{2\alpha \cdot G^{\alpha-1}(\xi)}{1-\xi^2}, \qquad q''(\xi) = -\frac{4\alpha(\alpha-1)G^{\alpha-2}(\xi) + 4\alpha\xi G^{\alpha-1}(\xi)}{(1-\xi^2)^2}.$$

Notation 3.3. Abusing notation, let us now introduce the following notation similar to that of Equation (2.2), we have

$$p_0 = \frac{2}{\left(1-\xi_0^2\right)^2}, \qquad p_0' = \frac{8\xi_0}{\left(1-\xi_0^2\right)^3}, \qquad p_0'' = \frac{8+40\xi_0^2}{\left(1-\xi_0^2\right)^4},$$

$$q_0 = -G^\alpha(\xi_0), \qquad q_0' = -\frac{2\alpha G^{\alpha-1}(\xi_0)}{1-\xi_0^2}, \qquad q_0'' = -\frac{4\alpha(\alpha-1)G^{\alpha-2}(\xi_0) + 4\alpha\xi_0 G^{\alpha-1}(\xi_0)}{\left(1-\xi_0^2\right)^2},$$

$$\eta_0 = \frac{2\xi_0^2}{\left(1-\xi_0^2\right)^2}, \qquad \eta_0' = \frac{4\xi_0^3+4\xi_0}{\left(1-\xi_0^2\right)^3}, \qquad \dot\vartheta^2 = \frac{\alpha G^{\alpha-1}(\xi_0)(1-\xi_0^2)^2}{\xi_0\left(1+\xi_0^2\right)}.$$

Since the (RHS) of the equation defining $\dot\vartheta^2$ should be positive, we only restrict to the case

$$\alpha > 0.$$

By a straightforward computation, we get

$$\zeta_0 = \frac{4\xi_0\left(1+\xi_0^2\right)}{\left(1-\xi_0^2\right)^2} \cdot \sqrt{\frac{\alpha G^{\alpha-1}(\xi_0)}{\xi_0\left(1+\xi_0^2\right)}}, \qquad \frac{1}{2}\eta_0'' = \frac{6\xi_0^4+16\xi_0^2+2}{\left(1-\xi_0^2\right)^4}, \qquad 2q_0'\frac{\eta_0'}{\eta_0} = \frac{-8\alpha G^{\alpha-1}(\xi_0)(1+\xi_0^2)}{\xi_0\left(1-\xi_0^2\right)^2},$$

$$\eta_0' \cdot \left[\frac{q_0'}{\eta_0'}\right]' = \frac{2\alpha G^{\alpha-2}(\xi_0)\left[\left(\xi_0^4+6\xi_0^2+1\right)\cdot G(\xi_0) - 2(\alpha-1)\xi_0\left(1+\xi_0^2\right)\right]}{\xi_0\left(1-\xi_0^2\right)^2\left(1+\xi_0^2\right)}.$$

Then we have

$$a = p_0^{-1} = \frac{(1-\xi_0^2)^2}{2}, \qquad b = \eta_0^{-1}\zeta_0 = \frac{2(1+\xi_0^2)}{\xi_0} \cdot \sqrt{\frac{\alpha G^{\alpha-1}(\xi_0)}{\xi_0(1+\xi_0^2)}}, \qquad c = \eta_0^{-1} = \frac{(1-\xi_0^2)^2}{2\xi_0^2},$$
$$d = 2q_0'(\ln \eta_0)' + \eta_0'\left(\frac{q_0'}{\eta_0'}\right)' = \frac{-2\alpha G^{\alpha-2}(\xi_0)\{(3\xi_0^4+2\xi_0^2+3)\cdot G(\xi_0) + 2(\alpha-1)\xi_0(1+\xi_0^2)\}}{\xi_0(1-\xi_0^2)^2(1+\xi_0^2)}. \tag{3.10}$$

Similar to the sphere case, we start determining the sign of d ranging in the parameter region

$$\Omega_3 := (0,1)\times(0,\infty).$$

By the explicit computation of d given at Equation (3.10), we infer that the sign of d is minus the sign of $(3\xi_0^4+2\xi_0^2+3)\cdot G(\xi_0) + 2(\alpha-1)\xi_0(1+\xi_0^2)$. We let

$$g_1(\xi) = (3\xi^4+2\xi^2+3)\cdot \ln\left(\frac{1+\xi}{1-\xi}\right) + 2(\alpha-1)\xi(1+\xi^2) \qquad (\xi,\alpha)\in\Omega_3.$$

We can check that $g_1(\xi) > 0$ and consequently there holds $d < 0$ for all $(\xi,\alpha)\in\Omega_3$.

Next we have to study the sign of $b^2 + cd$. By Equation (3.10) and by a direct computation, we get

$$b^2 + cd = \frac{\alpha G^{\alpha-2}(\xi_0)\{(\xi_0^4+6\xi_0^2+1)\cdot G(\xi_0) - 2(\alpha-1)\xi_0(1+\xi_0^2)\}}{\xi_0^3(1+\xi_0^2)}.$$

Now, we observe that since $\alpha > 0$ and $G(\xi_0) > 0$, then the sign of $b^2 + cd$ is equal to the sign of the function

$$g_2(\xi) := (\xi^4+6\xi^2+1)\ln\left(\frac{1+\xi}{1-\xi}\right) - 2(\alpha-1)\xi(1+\xi^2).$$

So we have following three subregions of Ω_3 according to the sign of $g_2(\xi)$.

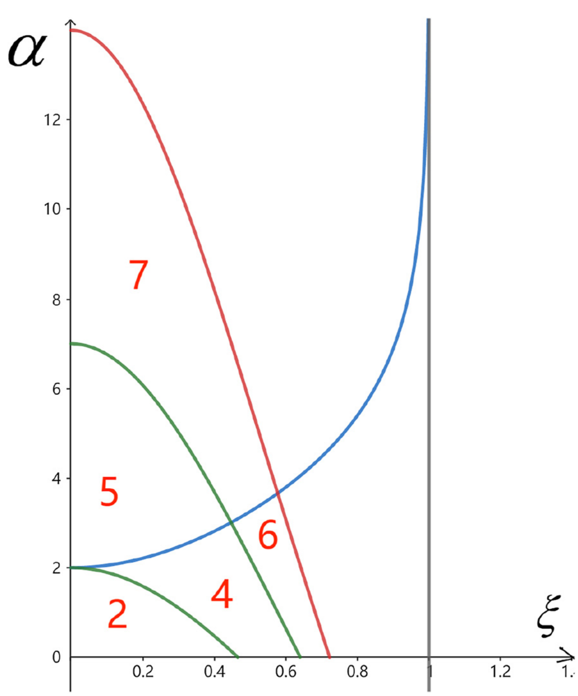


Figure 5: (Hyperbolic case α positive) In this figure are displayed the subregions of the $\widehat{\xi O\alpha}$-region $\Omega_1 := (0,1) \times (0,+\infty)$ labeled by the Morse index of the corresponding circular orbit.

$$\Omega_3^+ := \{(\xi,\alpha)\in\Omega_3 \mid g_2(\xi)>0\} = \left\{(\xi,\alpha)\in\Omega_3 \,\middle|\, \alpha < 1+\frac{(\xi^4+6\xi^2+1)\cdot\ln((1+\xi)/(1-\xi))}{2\xi(1-\xi^2)}\right\},$$
$$\Omega_3^- := \{(\xi,\alpha)\in\Omega_3 \mid g_2(\xi)<0\} = \left\{(\xi,\alpha)\in\Omega_3 \,\middle|\, \alpha > 1+\frac{(\xi^4+6\xi^2+1)\cdot\ln((1+\xi)/(1-\xi))}{2\xi(1-\xi^2)}\right\},$$
$$\Omega_3^0 := \{(\xi,\alpha)\in\Omega_3 \mid g_2(\xi)=0\} = \left\{(\xi,\alpha)\in\Omega_3 \,\middle|\, \alpha = 1+\frac{(\xi^4+6\xi^2+1)\cdot\ln((1+\xi)/(1-\xi))}{2\xi(1-\xi^2)}\right\}.$$

So, there holds that

$$cd+b^2 \text{ is } \begin{cases} \text{positive} & \text{for } (\xi,\alpha)\in\Omega_3^+,\\ \text{negative} & \text{for } (\xi,\alpha)\in\Omega_3^-,\\ \text{zero} & \text{for } (\xi,\alpha)\in\Omega_3^0. \end{cases}$$

In order to compute the index by Lemma 2.4 we have to determine the value of k. Recall that the value of k is determined by $k\cdot 2\pi < \sqrt{-ad}T \le (k+1)\cdot 2\pi$. Indeed, we have

$$T = \frac{2\pi}{\dot\vartheta} = 2\pi\sqrt{\frac{\xi_0(1+\xi_0^2)}{\alpha G^{\alpha-1}(\xi_0)(1-\xi_0^2)^2}}.$$

Moreover, by Equation (3.10) we have

$$\sqrt{-ad} = \sqrt{\frac{\alpha G^{\alpha-2}(\xi_0)(3\xi_0^4+2\xi_0^2+3)\cdot G(\xi_0)+2(\alpha-1)\xi_0(1+\xi_0^2)}{\xi_0(1+\xi_0^2)}}.$$

Therefore,

$$\sqrt{-ad}\cdot T = 2\pi\cdot\sqrt{\frac{(3\xi_0^4+2\xi_0^2+3)\cdot G(\xi_0)+2(\alpha-1)\xi_0(1+\xi_0^2)}{G(\xi_0)(1-\xi_0^2)^2}}.$$

Let

$$g_3(\xi) := \sqrt{\frac{(3\xi^4+2\xi^2+3)\cdot\ln\left(\frac{1+\xi}{1-\xi}\right)+2(\alpha-1)\xi(1+\xi^2)}{(1-\xi^2)^2\,\ln\left(\frac{1+\xi}{1-\xi}\right)}}.$$

Direct computations show that

$$k=0 \iff 0<g_3(\xi)\le 1 \iff 0<\alpha\le 1-\frac{1+\xi^2}{\xi}\cdot\ln\frac{1+\xi}{1-\xi}.$$

But it is impossible since the right-hand side of above inequality is negative for $\xi\in(0,1)$. Moreover, we can check that for every fixed α and $k\in\mathbb{N}^+$ there exists $(\xi,\alpha)\in\Omega_3^+$ such that $k<g_3(\xi)\le k+1$. Then we can define the subregions

$$\Omega_{3,k}^{\pm} := \{(\xi,\alpha)\in\Omega_3^{\pm} \mid k<g_3(\xi)\le k+1\},\quad \Omega_{3,k}^0 := \left\{(\xi,\alpha)\in\Omega_3^0 \mid k<g_3(\xi)\le k+1\right\}. \tag{3.11}$$

All these subregions are displayed in Figure 5. Invoking Lemma 2.4, then we have the following result.

Theorem 3.4. *Under the above notations, the Morse index of the circular orbit on the hyperbolic plane is given by*

$$m^-(x) = \begin{cases} 2k & \text{if} \quad (\xi,\alpha)\in\Omega_{3,k}^+\cup\Omega_{3,k}^0,\\ 2k+1 & \text{if} \quad (\xi,\alpha)\in\Omega_{3,k}^-, \end{cases} \quad k=1,2,\dots$$

As direct consequence of Theorem 3.4, in the special case of $\alpha = 2$ we get the following result.

Corollary 3.5. *For $\alpha = 2$, the Morse index of the circular solution is given by*

$$m^-(x) = 2k \qquad \text{if } (\xi_0, 2) \in \Omega^+_{3,k}, \quad k = 2, 3, \ldots$$

3.3 Euclidean case

The last case is provided by the Euclidean one. We start letting

$$p(\xi) = 1, \quad q(\xi) = -\xi^\alpha,$$

and by a direct computation, we get

$$p'(\xi) = p''(\xi) = 0, \qquad q'(\xi) = -\alpha\xi^{\alpha-1}, \qquad q''(\xi) = -\alpha(\alpha-1)\xi^{\alpha-2}.$$

By Equation (2.2), we get

$$\begin{aligned} &p_0 = 1, && p_0' = 0, && p_0'' = 0, \\ &q_0 = -\xi_0^\alpha, && q_0' = -\alpha\xi_0^{\alpha-1}, && q_0'' = -\alpha(\alpha-1)\xi_0^{\alpha-2}, \\ &\eta_0 = \xi_0^2, && \eta_0' = 2\xi_0, && \dot\vartheta^2 = \alpha\xi_0^{\alpha-2}. \end{aligned} \tag{3.12}$$

Since the (RHS) of the equation defining $\dot\vartheta^2$ should be positive, we only consider the case

$$\alpha > 0.$$

By a direct computation, we get

$$\zeta_0 = 2\xi_0 \cdot \sqrt{\alpha\xi_0^{\alpha-2}}, \qquad 2q_0'(\ln\eta_0)' = -4\alpha\xi_0^{\alpha-2}, \qquad \eta_0' \cdot \left[\frac{q_0'}{\eta_0'}\right]' = -\alpha(\alpha-2)\xi_0^{\alpha-2}.$$

By this, we get that the four constants appearing at Equation (2.3) are the following

$$\begin{aligned} &a = 1, && b = \frac{2}{\xi_0}\sqrt{\alpha\xi_0^{\alpha-2}}, \\ &c = \frac{1}{\xi_0^2}, && d = -\alpha(\alpha+2)\xi_0^{\alpha-2}. \end{aligned} \tag{3.13}$$

Now, we observe that $d < 0$ for all $\alpha > 0$ and $\xi_0 > 0$.

Next we need to determine the sign of the term $b^2 + cd$. By a direct computing we get

$$b^2 + cd = -\alpha(\alpha-2)\xi_0^{\alpha-4} \quad \text{is} \quad \begin{cases} \text{negative} & \text{if} \quad \alpha > 2, \\ 0 & \text{if} \quad \alpha = 2, \\ \text{positive} & \text{if} \quad 0 < \alpha < 2. \end{cases}$$

By taking into account Equations (3.12) and (3.13) we finally get

$$T = \frac{2\pi}{\dot\vartheta} = 2\pi\sqrt{\frac{1}{\alpha\xi_0^{\alpha-2}}}, \qquad \sqrt{-ad} = \sqrt{\alpha(\alpha+2)\xi_0^{\alpha-2}}.$$

Therefore

$$\sqrt{-ad} \cdot T = 2\pi \cdot \sqrt{\alpha+2}$$

and we get that $k \geq 1$. Indeed, for every $k \in \mathbb{N}^+$ there exists $\alpha > 0$ such that $k < \sqrt{\alpha + 2} \leq k + 1$. For example, there hold

$$k = 1 \Longleftrightarrow 0 < \alpha \leq 2, \quad k = 2 \Longleftrightarrow 2 < \alpha \leq 7, \quad k = 3 \Longleftrightarrow 7 < \alpha \leq 14$$

and so on.

Then we have the following result.

Theorem 3.6. *Under the above notations, the Morse index of a circular orbit in the Euclidean plane is given by*

$$m^-(x) = \begin{cases} 2 & \text{if} \quad 0 < \alpha \leq 2, \\ 2k + 1 & \text{if} \quad k^2 - 2 < \alpha \leq (k+1)^2 - 2 \quad \textit{for} \quad k \geq 2. \end{cases}$$

By Theorem 3.6 we have the following direct corollary.

Corollary 3.7. *For $\alpha = 2$, we get*

$$m^-(x) = 2.$$

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