Research Article

Fan Feng and Xiangrui Li*

Equational characterizations for some subclasses of domains

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Abstract: It is well known that the continuity of a poset can be seen as a special distributivity. There is an open problem: is there an equational characterization for continuous semilattices? Based on equational characterizations of continuous lattices, bounded complete domains and L-domains, we prove that a special class of domains can be characterized by an equation. As an application, an equational characterization for a subclass of continuous semilattices is given. Moreover, by using ideals instead of directed sets, we obtain a unified equational characterization for more subclasses of domains, including that of domains mentioned above. Unfortunately, even if using ideals, we still can not characterize all of the domains. Some examples are provided to illustrate it.

Keywords: domains; continuous semilattices; equational characterization; ideals; family of dcpos

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1 Introduction

In the late 1960s, Dana Scott found the semantic structure in computer science being close to partial order structures. Based on this observation, he established Domain Theory, which plays a central role in the field of theoretical computer science. It is well known that the continuity and the quasicontinity of posets are important concept in Domain Theory. These can be used to describe convergence and approximation in order theory [1–3]. Similar to the study of universal algebra, it raises the question as to whether domains are maintained under subalgebras, products and homomorphic images. For this question, it was shown in [4] that two special subclass of domains that continuous lattices and bounded complete domains can be characterized by the distributivity. This kind of characterization is called an equational characterization. The continuity on complete lattice can be viewed as an infinite distributive law. In [5], Marcel Erné shows the relationship between continuity and the other laws of infinite distribution. In fact, many infinite distributive laws can be applied to a broader range. For example, Wei Luan and Qingguo Li showed that quasi-continuity complete semilattice can be characterized by an equation in [6] and Paul Taylor provided an equational characterization for L-domains in [7]. There is an open problem about equational characterization in [8]: is there an equational characterization for continuous semilattices?

In this paper, we first introduce a concept of an $\mathcal{A}_{DM}$ dcpo. Then we prove that continuous $\mathcal{A}_{DM}$ dcpos can be characterized by an equation for some given $\mathcal{A}$ a family of some subsets of the dcpo. In particular, a subclass of continuous semilattices can be characterized in this way. This partially solves the problem presented in [8]. Moreover, we obtain an equational characterization for a special class of domains including $\mathcal{A}_{DM}$ domains, by

***Corresponding author: Xiangrui Li**, School of Mathematics, Hunan University, Changsha, Hunan, 410082, China, E-mail: 17731912783@163.com
Fan Feng, School of Mathematics, Hunan University, Changsha, Hunan, 410082, China, E-mail: Fengfan@hnu.edu.cn

using ideals instead of directed sets. However, we can not characterize general domains in this way, especially for FS-domains. At last, we give some examples about these characterizations.

2 Preliminaries

In this section, we recall some definitions and results related to the paper. A partially ordered set is a nonempty set equipped with a partial order $\leq$, where the partial order is a binary relation satisfying transitive, reflexive and antisymmetric. The term poset will be used to denote a partially ordered set.

Definition 2.1. [9]

Let L be a set equipped a partial order $\leq$ and A be a subset of L.

(1) A partial order $\leq_A$ on A is called the hereditary order, if $\leq_A = \leq \cap (A \times A)$.

(2) An element $x \in L$ is called an upper bound of A, if $a \leq x$ for all $a \in A$. Respectively, an element $y \in L$ is called a lower bound of A, if $y \leq a$ for all $a \in A$. For $x \in L$, we write $\downarrow x = \{y \in L : y \leq x\}$ and $\uparrow x = \{z \in L : x \leq z\}$.

(3) An element $x \in L$ is called the least upper bound of A, if $x \leq y$ for each upper bound y of A. And we write it as $\bigvee A$ or sup A. Respectively, p is called the greatest lower bound of A, if $q \leq p$ for each lower bound q of A. The greatest lower bound is written as $\bigwedge A$ or inf A.

(4) A is called a directed set, if A is nonempty and every finite subset of A has an upper bound in A. Respectively, A is called a filtered set, if A is nonempty and every finite subset of A has a lower bound in A.

(5) A is called a lower set, if $A = \{x \in L : x \leq a$ for some $a \in A\}$. Respectively, A is called an upper set, if $A = \{y \in L : a \leq y$ for some $a \in A\}$.

(6) A is called a ideal, if A is a directed lower set. A is called a filter, if A is a filtered upper set.

(7) A is called a principal idea, if A is an ideal with $\bigvee A \in A$. A is called a principal filter, if A is a filter with $\bigwedge A \in A$.

Definition 2.2. [10]

(1) A *complete lattice* is a poset in which every subset has a sup and an inf.

(2) A poset is called a *complete semilattice* if every nonempty subset has an inf and every directed subset has a sup.

(3) A poset is called a *dcpo* if every directed subset has a sup.

(4) A dcpo is called an *L-dcpo* if every principal ideal equipped with its hereditary order is a complete lattice.

Definition 2.3. [10]

(1) Let L be a poset. We say that x is *way-below* y, in symbols $x \ll y$, iff for all directed subsets $D \subseteq L$ for which sup D exists, the relation $y \leq \sup D$ always implies the existence of a $d \in D$ with $x \leq d$. For each $x \in L$, we denote by $\twoheaddownarrow x$ the set of all elements are *way-below* x.

(2) A poset L is called *continuous* if it satisfies the *axiom* of *approximation*:

$$(\forall x \in L) x = \bigvee^{\uparrow} \twoheaddownarrow x,$$

i.e. for all $x \in L$, the set $\twoheaddownarrow x = \{u \in L \mid u \ll x\}$ is directed and $x = \bigvee \{u \in L \mid u \ll x\}$.

(3) A dcpo which is continuous is called a *domain.*

(4) A domain which is a semilattice is called a *continuous semilattice.*

(5) A domain which is also an L-dcpo is called an *L-domain.*

(6) A domain which is a complete lattice is called a *continuous lattice.*

(7) A domain which is a complete semilattice is called a *bounded complete domain.*

(8) An element x of L is called a compact element, if for each directed set D of L with $\bigvee D$ exists, $x \leq \bigvee D$ always implies $x \leq d$ for some $d \in D$ (i.e., $x \ll x$). Denotes $K(L)$ as the set of all compact elements.

(9) A poset L is called *algebraic* if it satisfies the *axiom* of *approximation*:

$$(\forall x \in L) x = \bigvee^{\uparrow}(\Downarrow x \cap K(L)),$$

i.e. for all $x \in L$, the set $(\Downarrow x \cap K(L))$ is directed and $x = \bigvee(\Downarrow x \cap K(L))$.

Definition 2.4. [11] Let L be a poset. A topology τ on L is called the Alexandrov topology, if τ is the set of all upper set of L.

For a poset L and $x \in L$, let $\mathcal{J}(x) = \{I \in Id(L) \mid x \leq \sup I\}$ where $Id(L)$ is the set of all ideals of L. The following propositions are excerpted from [8, Proposition I-1.5, Proposition I-4.3].

Proposition 2.5. *If L is an algebraic domain, then L is a domain.*

Proposition 2.6. *Let L be a poset. Then the following conditions are equivalent:*
(1) $y \ll x$;
(2) $y \in \bigcap \mathcal{J}(x)$.

Theorem 2.7. *[4] Let L be a complete semilattice. Then the following conditions are equivalent.*
(1) *L is continuous.*
(2) *Let $\{x_{j,k} \mid j \in J, k \in K(j)\}$ be a nonempty family of elements in L such that $\{x_{j,k} \mid k \in K(j)\}$ is directed for each $j \in J$. Then the following identity holds:*

$$\bigwedge_{j\in J}\bigvee^{\uparrow}_{k\in K(j)} x_{j,k} = \bigvee^{\uparrow}_{f\in M}\bigwedge_{j\in J} x_{j,f(j)},$$

where $K(j)$ is a index set for any $j \in J$ and M is the set of all choice functions $f: J \to \bigcup_{j\in J} K(j)$ with $f(j) \in K(j)$ for all $j \in J$.

If L is a complete lattice, then these conditions are also equivalent to
(3) *Let $\{x_{j,k} \in J \times K\}$ be any family in L. Then the following identity holds:*

$$\bigwedge_{j\in J}\bigvee_{k\in K} x_{j,k} = \bigvee^{\uparrow}_{f\in N}\bigwedge_{j\in J}\bigvee_{k\in f(j)} x_{j,k},$$

where N denotes the set of all choice functions f from J into the finite subsets of K, i.e., $f: J \to fin(K)$.

Next, we recall the definition of connectedness in order theory.

Definition 2.8. [12] Let P be a poset. Then P is called *connected*, if every two elements x, y can be connected by a zigzag in P, i.e. there is $n \in \mathbb{N}$ and there are $x_0, \dots, x_n, y_0, \dots, y_n \in P$ such that $x = x_0, y = x_n$ and $x_i \leq y_j$ whenever $0 \leq j - i \leqslant 1$. A subset A of P is called a *connected set*, if A is connected as a subspace of P.

The following theorem is an excerpt of [7, Proposition 1.3.3].

Theorem 2.9. *Let L be an L-dcpo. Then the following conditions are equivalent.*
(1) *L is continuous.*
(2) *Let $\{D_j\}_{j\in J} = \{\{x_{j,k} \mid k \in K(j)\}\}_{j\in J}$ be a family of directed subsets of L such that $\{\bigvee D_j \mid j \in J\}$ is a connected set. Then the following identity holds:*

$$\bigwedge_{j\in J}\bigvee^{\uparrow}_{k\in K(j)} x_{j,k} = \bigvee^{\uparrow}_{f\in M}\bigwedge_{j\in J} x_{j,f(j)},$$

where M is the set of all choice functions $f: J \to \bigcup_{j\in J} K(j)$ with $f(j) \in K(j)$ for all $j \in J$.

3 Main results

First, we propose some families of dcpos. For convenience, we denote by $PFS(L)$ the family of all principal filters and singleton sets of a poset L.

Definition 3.1. Let L be a dcpo and $\mathcal{A}$ be a family of sets with $PFS(L) \subseteq \mathcal{A}$.

(1) L is called an $\mathcal{A}_M$ dcpo if inf A exists for each $A \in \mathcal{A}$.

(2) L is called an $\mathcal{A}_D$ dcpo if for each family

$$\{D_j\}_{j\in J} = \{\{x_{j,k} \mid k \in K(j)\}\}_{j\in J}$$

of directed subsets of L with $\{\bigvee D_j \mid j \in J\} \in \mathcal{A}$, then there exists a choice function $f\colon J \to \bigcup_{j\in J} K(j)$ such that $f(j) \in K(j)$ for all $j \in J$ and $\{x_{j,f(j)} \mid j \in J\} \in \mathcal{A}$.

(3) L is called an $\mathcal{A}_{DM}$ dcpo, if L is both an $\mathcal{A}_M$ dcpo and an $\mathcal{A}_D$ dcpo.

Example 3.2.

(1) Let L be a complete lattice and $\mathcal{A}$ be the family of all subsets of L. Then L is an $\mathcal{A}_{DM}$ dcpo.

(2) Let L be a complete semilattice and $\mathcal{A}$ be the family of all nonempty subsets of L. Then L is an $\mathcal{A}_{DM}$ dcpo.

(3) Let L be an L-dcpo and $\mathcal{A}$ be the family of all connected sets of L. Then L is an $\mathcal{A}_{DM}$ dcpo.

(4) Let L be a dcpo and $\mathcal{A}$ be the family of all compact subsets of L with respect to Alexandrov topology. Then L is an $\mathcal{A}_M$ dcpo if and only if L is a semilattice.

Now we give the main result of this paper.

Theorem 3.3. *Let L be a dcpo and $\mathcal{A}$ be a family of sets with $PFS(L) \subseteq \mathcal{A}$. If L is an $\mathcal{A}_{DM}$ dcpo, then the following are equivalent.*

(1) *L is continuous.*

(2) *If $\{D_j\}_{j\in J} = \{\{x_{j,k} \mid k \in K(j)\}\}_{j\in J}$ is a family of directed sets with $\{\bigvee D_j \mid j \in J\} \in \mathcal{A}$, then the following identity holds:*

$$\bigvee\nolimits^{\uparrow}_{f\in M} \bigwedge_{j\in J} x_{j,f(j)} = \bigwedge_{j\in J} \bigvee\nolimits^{\uparrow}_{k\in K(j)} x_{j,k},$$

where M is the set of all choice functions $f\colon J \to \bigcup_{j\in J} K(j)$ such that $f(j) \in K(j)$ for all $j \in J$ and $\{x_{j,f(j)} \mid j \in J\} \in \mathcal{A}$.

Proof. We prove that (1) implies (2).

To this end, we first claim that $\{\bigwedge_{j\in J} x_{j,f(j)} \mid f \in M\}$ is directed. Assume $f_1, f_2 \in M$. For each $j \in J$, there is $x_{j,k_j} \in D_j$ with $x_{j,f_1(j)}, x_{j,f_1(j)} \leq x_{j,k_j}$. Then $\{\uparrow x_{j,k_j} \cap D_j\}_{j\in J}$ is a family of directed sets with

$$\left\{\bigvee(\uparrow x_{j,k_j} \cap D_j) \mid j \in J\right\} \in \mathcal{A}.$$

Take a choice function $f\colon J \to \bigcup_{j\in J} K(j)$ such that $x_{j,f(j)} \in \uparrow x_{j,k_j} \cap D_j$ and $\{x_{j,f(j)} \mid j \in J\} \in \mathcal{A}$. Then $\bigwedge_{j\in J} x_{j,f_1(j)}, \bigwedge_{j\in J} x_{j,f_2(j)} \leq \bigwedge_{j\in J} x_{j,f(j)}$ and $f \in M$. Thus $\{\bigwedge_{j\in J} x_{j,f(j)} \mid f \in M\}$ is directed.

Assume that $\{D_j\}_{j\in J} = \{x_{j,k} \mid k \in K(j)\}_{j\in J}$ is a family of directed sets with $\{\bigvee D_j \mid j \in J\} \in \mathcal{A}$. For convenience, let *lhs* denote the left side of the equation of (2) and *rhs* denote the right one. For each $f_0 \in M$ and $j_0 \in J$, $x_{j_0,f_0(j_0)} \leq \bigvee^{\uparrow}_{k\in K(j_0)} x_{j_0,k}$. Thus it is clear that $lhs \leq rhs$.

Assume $y \ll rhs$. Given $j \in J$, there is $k_j \in K(j)$ such that $y \leq x_{j,k_j}$. Because $\{\uparrow x_{j,k_j} \cap D_j \mid j \in J\}$ is a family of directed sets with

$$\left\{\bigvee(\uparrow x_{j,k_j} \cap D_j) \mid j \in J\right\} = \left\{\bigvee D_j \mid j \in J\right\} \in \mathcal{A},$$

there is a choice function $f \in M$ such that $x_{j,k_j} \leq x_{j,f(j)}$ for each $j \in J$ and $\{x_{j,f(j)} \mid j \in J\} \in \mathcal{A}$. It follows that $y \leq \bigwedge_{j\in J} x_{j,f(j)}$. So $rhs \leq lhs$.

Now we show that (2) implies (1).

Fix $x \in L$, let $\{I_j\}_{j\in J} = \{\{x_{j,k} \mid k \in K(j)\}\}_{j\in J}$ denote all of ideals with sup in $\uparrow x$. The equation of (2) holds, because $\{\bigvee I_j \mid j \in J\} = \uparrow x \in \mathcal{A}$. We claim that

$$\left\{\bigwedge_{j\in J} x_{j,f(j)} \mid f \in M\right\} = \Downarrow x.$$

Since $z \in \Downarrow x$ implies $z \in \bigcap_{j\in J} I_j$, there is $f \in M$ such that $x_{j,f(j)} = z$ for each $j \in J$, due to the fact that $\mathcal{A}$ includes all the singleton sets. That is $\Downarrow x \subseteq \{\bigwedge_{j\in J} x_{j,f(j)} \mid f \in M\}$. Conversely, for each $j_0 \in J$ and all $f \in M$, we have $\bigwedge_{j\in J} x_{j,f(j)} \leq x_{j_0,f(j_0)}$. It follows that $\bigwedge_{j\in J} x_{j,f(j)} \in I_{j_0}$. Hence, $\bigwedge_{j\in J} x_{j,f(j)} \in \cap_{j\in J} I_j = \Downarrow x$. In the equation, $lhs = \bigvee^{\uparrow} \Downarrow x$ and $rhs = \bigwedge \uparrow x$. Thus, $x = \bigvee^{\uparrow} \Downarrow x$. Therefore, L is continuous.

Remark 3.4. Note that we can characterize many domains by changing $\mathcal{A}$. From the examples in Example 3.2, we obtain that Theorem 2.7 and Theorem 2.9 are two special cases of Theorem 3.3.

Taking bounded complete domains as an example, let L be a dcpo and $\mathcal{A}$ be the set of all nonempty subsets. Then L is a bounded complete domain if and only if L is an $\mathcal{A}_{DM}$ dcpo satisfying Theorem 3.3.

For a special subclass of continuous semilattices, we have the following equational characterization.

Corollary 3.5. *Suppose that dcpo L is also a semilattice, and $\mathcal{A}$ is the family of all compact subsets of L with respect to Alexandrov topology. If L is an $\mathcal{A}_D$ dcpo, then the following are equivalent.*

(1) *L is continuous.*

(2) *If $\{D_j\}_{j\in J} = \{\{x_{j,k} \mid k \in K(j)\}\}_{j\in J}$ is a family of directed sets with $\{\bigvee D_j \mid j \in J\} \in \mathcal{A}$, then the following identity holds:*

$$\bigvee^{\uparrow}_{f\in M} \bigwedge_{j\in J} x_{j,f(j)} = \bigwedge_{j\in J} \bigvee^{\uparrow}_{k\in K(j)} x_{j,k},$$

where M is the set of all choice functions $f\colon J \to \bigcup_{j\in J} K(j)$ such that $f(j) \in K(j)$ for all $j \in J$ and $\{x_{j,f(j)} \mid j \in J\} \in \mathcal{A}$.

Proof. Since a compact set of the semilattice L with the Alexandrov topology always has an inf, L is an $\mathcal{A}_M$ dcpo. Thus, we conclude that (1) equivalent to (2) by Theorem 3.3.

Meet continuous is a distributivity on directed complete semilattice. In [13], Hui Kou stated that the property can be extended to general dcpos and the characterization uses sets to replace points. By using ideals instead of directed sets, we will obtain an equational characterization for a subclass of domains, including that of domains mentioned above. For convenience, we denote by $PF(L)$ the family of all principal filters of a poset L.

Definition 3.6. Let L be a dcpo and $\mathcal{A}$ be a family of sets with $PF(L) \subseteq \mathcal{A}$. Then L is called an $\mathcal{A}_{IM}$ dcpo, if

$$\bigcap_{j\in J} I_j \in Id(L)$$

for each $\{I_j\}_{j\in J} \subseteq Id(L)$ with $\{\bigvee I_j \mid j \in J\} \in \mathcal{A}$.

Proposition 3.7. *Let L be a dcpo and $\mathcal{A}$ be a family of sets with $PFS(L) \subseteq \mathcal{A}$. If L is an $\mathcal{A}_{DM}$ dcpo, then L is an $\mathcal{A}_{IM}$ dcpo.*

Proof. Assume that $\{I_j\}_{j\in J} = \{\{x_{j,k} \mid k \in K(j)\}\}_{j\in J}$ is a family of ideals with $\{\bigvee I_j \mid j \in J\} \in \mathcal{A}$. Since L is an $\mathcal{A}_D$ dcpo, there exists $f\colon J \to \bigcup_{j\in J} K(j)$ such that $f(j) \in K(j)$ for all $j \in J$ and $\{x_{j,f(j)} \mid j \in J\} \in \mathcal{A}$. Thus, $\bigwedge_{j\in J} x_{j,f(j)}$ exists and $\bigwedge_{j\in J} x_{j,f(j)} \in \bigcap_{j\in J} I_j$. That is $\bigcap_{j\in J} I_j \neq \emptyset$.

Fix $a, b \in \bigcap_{j\in J} I_j$. Then there exists $k_j \in K(j)$ such that $a, b \leq x_{j,k_j}$ for each $j \in J$. So $\{\uparrow x_{j,k_j} \cap I_j \mid j \in J\}$ is a family of directed sets with $\{\bigvee(\uparrow x_{j,k_j} \cap I_j) \mid j \in J\} \in \mathcal{A}$. And there exists a choice function $f\colon J \to \bigcup_{j\in J} K(j)$ such that $x_{j,f(j)} \in (\uparrow x_{j,k_j} \cap I_j)$ for each $j \in J$ and $\{x_{j,f(j)} \mid j \in J\} \in \mathcal{A}$. It follows that $\bigwedge_{j\in J} x_{j,f(j)} \in \bigcap_{j\in J} I_j$ and $a, b \leq \bigwedge_{j\in J} x_{j,f(j)}$. Therefore, $\bigcap_{j\in J} I_j$ is an ideal.

Theorem 3.8. *Let L be a dcpo and $\mathcal{A}$ be a family of sets with $PF(L) \subseteq \mathcal{A}$. If L is an $\mathcal{A}_{IM}$ dcpo, then the following conditions are equivalent.*
(1) *L is continuous.*
(2) *If $\{I_j\}_{j\in J} \subseteq Id(L)$ with $\{\bigvee I_j \mid j \in J\} = \uparrow x$ for some $x \in L$, then the following identity holds:*

$$\downarrow\Big(\bigvee \bigcap_{j\in J} I_j\Big) = \bigcap_{j\in J} \downarrow\Big(\bigvee I_j\Big).$$

Proof. (1) implies (2): Let $\{I_j\}_{j\in J} \subseteq Id(L)$ with $\{\bigvee I_j \mid j \in J\} \in \mathcal{A}$. We have $\downarrow(\bigvee \bigcap_{j\in J} I_j) \subseteq \bigcap_{j\in J} \downarrow(\bigvee I_j)$, since $\bigvee \bigcap_{j\in J} I_j \leq \bigvee I_{j_0}$ for each $j_0 \in J$. Conversely, for each $y \in \bigcap_{j\in J} \downarrow(\bigvee I_j)$. Assume $z \in \twoheaddownarrow y$. Then $z \in I_j$ for each $j \in J$, due to $y \leq \bigvee I_j$. It follows that $\bigvee_{z\in \twoheaddownarrow y} z \leq \bigvee \bigcap_{j\in J} I_j$. Since L is a domain, we have that $y = \bigvee \twoheaddownarrow y$, Thus, $y \in \downarrow(\bigvee \bigcap_{j\in J} I_j)$.

(2) implies (1): Assume $x \in L$. Let $\{I_j\}_{j\in J} = \{I \in Id(L) \mid x \leq \bigvee I\}$. By Proposition 2.6, $\bigcap_{j\in J} I_j = \twoheaddownarrow x$. So we only need to prove that $x \in \downarrow(\bigvee \bigcap_{j\in J} I_j)$. It is clear that $x \in \bigcap_{j\in J} \downarrow(\bigvee I_j)$ by the definition of $\{I_j\}_{j\in J}$. Since $\{\bigvee I_j \mid j \in J\} \in \mathcal{A}$, we have

$$\downarrow\left(\bigvee \bigcap_{j\in J} I_j\right) = \bigcap_{j\in J} \downarrow\left(\bigvee I_j\right).$$

Then the conclusion is proved.

4 Some examples

In this section, we will give some examples. First, we illustrate that there is a poset which belongs to the subclass of continuous semilattice discussed in Corollary 3.5. Before this, we give the following concept of an A-domain.

Definition 4.1. A poset L is called an A-domain if $\uparrow x$ is a finite lattice for each $x \in L$ under the hereditary order.

Lemma 4.2. *If L is an A-domain, then L is an algebraic domain.*

Proof. Assume D is a directed set of L. Fix $\mathrm{d} \in D$, then $\uparrow \mathrm{d} \cap D$ is a finite directed set. Hence, $\bigvee(\uparrow \mathrm{d} \cap D)$ exists in D. Moreover, all elements in L are compact elements because $\bigvee D$ exists in D for each directed set D. It follows that L is algebraic.

Example 4.3. Let $L = \{F \subseteq \mathbb{N} \mid F \text{ is finite}\}$ be ordered by reverse inclusion and $\mathcal{A}$ be all of compact subsets of L with respect to Alexandrov topology. Then L is an $\mathcal{A}_{DM}$ dcpo and an A-domain.

Lemma 4.4. *Let L be an A-domain, and $\mathcal{A}$ be the family of all the principal filters and singleton sets. Then L is an $\mathcal{A}_{DM}$ dcpo.*

Proof. Clearly, L is an $\mathcal{A}_M$ dcpo. Assume $\{D_j\}_{j\in J} = \{\{x_{j,k} \mid k \in K(j)\}\}_{j\in J}$ is a family of directed sets with $\{\bigvee D_j \mid j \in J\} \in \mathcal{A}$. We can choose $x_{j,k_j} = \bigvee D_j$ for each $j \in J$. And there is a choice function $f\colon J \to \bigcup_{j\in J} K(j)$

given by $f(j) = k_j$ for each $j \in J$. Then

$$\{x_{j,f(j)} \mid j \in J\} = \left\{\bigvee D_j \mid j \in J\right\} \in \mathcal{A}.$$

Thus, L is an $\mathcal{A}_{DM}$ dcpo.

Thus each A-domain L can seen be as an $\mathcal{A}_{DM}$ dcpo where $\mathcal{A} = PFS(L)$. The following example indicates that there is an $\mathcal{A}_{DM}$ dcpo, which may be not an L-domain or a continuous semilattice.

Example 4.5. Let $L = \{a, b, c\}$ be a set. Define a relation $\leq$ on L by

$$x \leq y \text{ iff } x = y \text{ } or \text{ } y = c.$$

Obviously, $\leq$ is a partial order on L and $(L, \leq)$ is an A-domain. But $(L, \leq)$ is neither a continuous semilattice nor an L-domain.

Finally, we propose the following example to reveal that the method of using ideals can not be used to characterize all of domains.

Example 4.6. Let $F = \{\frac{n}{n+1} \mid n \in \mathbb{N}\} \bigcup \{1\}$ and $L = \{\{0,1\} \times F\} \bigcup \{F \times \{0\}\}$. We define a partial order $\leq^*$ on L below:

$$(a_1, b_1) \leq^* (a_2, b_2) \text{iff} \begin{cases} a_2 \notin F, \ b_1 \leq b_2, & a_1 = 0 \cdots \leq_1^*, \\ a_2 = 0, a_1 \leq b_2, & a_1 \in F \cdots \leq_2^*, \\ a_2 \in F, a_1 \leq a_2, & a_1 \in F \cdots \leq_3^*, \\ a_2 = 1, & a_1 \in F \cdots \leq_4^*, \\ a_2 = 1, \ b_1 \leq b_2, & a_1 = 1 \cdots \leq_5^*, \end{cases}$$

where “ $\leq$ ” is the usual order on $\mathbb{R}$.

We now prove that the relation $\leq^*$ forms a partial order.

We first prove that $\leq^*$ is reflexive. Given $(a, b) \in L$.

(i) $a = 0, b \in F$. Then $(a, b) \leq^* (a, b)$ by $\leq_1^*$.
(ii) $a = 1, b \in F$. Then $(a, b) \leq^* (a, b)$ by $\leq_5^*$.
(iii) $a \in F, b = 0$. Then $(a, b) \leq^* (a, b)$ by $\leq_3^*$.

Secondly, we prove that $\leq^*$ is antisymmetric. Given $(a, b), (c, d) \in L$ with $(a, b) \leq^* (c, d)$ and $(c, d) \leq^* (a, b)$.

(i) $(a, b) \leq_1^* (c, d)$. Then $c = 0$ or 1. By $\leq_4^*, \leq_5^*$, we have $c \neq 1$. Hence $c = 0$ and it follows that $(a, b) \leq_1^* (c, d)$ and $(c, d) \leq_1^* (a, b)$. It implies that $a = c = 0, b = d$.
(ii) $(a, b) \leq_2^* (c, d)$. Then $c = 0$. Because $(c, d) \leq^* (a, b)$, we have $(c, d) \leq_1^* (a, b)$ and $a \notin F$, a contradiction.
(iii) $(a, b) \leq_3^* (c, d)$. Then $b = d = 0$. And $c = d$, by $(c, d) \leq_3^* (a, b)$. That is $a = c$.
(iv) $(a, b) \leq_4^* (c, d)$. Then $c = 1$. By $(c, d) \leq_5^* (a, b)$, we have $a = 1$, a contradiction.
(v) $(a, b) \leq_5^* (c, d)$. Then $a = c = 1$. Since $(c, d) \leq_5^* (a, b)$, it holds that $b = d$.

Finally, we prove the transitivity of $\leq^*$. Given $(a, b) \leq^* (c, d) \leq^* (e, f)$.

(i) $(a, b) \leq_1^* (c, d), (c, d) \leq_1^* (e, f)$.

Then $a = c = 0$, $e = 0$ or 1, $b \leq d \leq f$. And $(a, b) \leq^* (e, f)$, by $\leq_1^*$.

(ii) $(a, b) \leq_1^* (c, d), (c, d) \leq_5^* (e, f)$.

Then $a = 0$, $c = e = 1$, $b \leq d \leq f$. And $(a, b) \leq^* (e, f)$, by $\leq_1^*$.

(iii) $(a, b) \leq_2^* (c, d), (c, d) \leq_1^* (e, f)$.

Then $a \le d$, $c = 0$, $e = 0$ or 1, $d \le f$.
If $e = 0$, then $(a, b) \le_2^* (e, f)$.
If $e = 1$, then $(a, b) \le_4^* (e, f)$.
(iv) $(a, b) \le_3^* (c, d)$.

If $(c, d) \le_2^* (e, f)$, then $e = 0$ and $a \le c \le f$, i.e., $(a, b) \le_2^* (e, f)$.
If $(c, d) \le_3^* (e, f)$, then $a \le c \le e$ and $(a, b) \le_3^* (e, f)$.
If $(c, d) \le_4^* (e, f)$, then $e = 1$ and $(a, b) \le_4^* (e, f)$.
(v) $(a, b) \le_4^* (c, d)$, $(c, d) \le_5^* (e, f)$.

Then $c = e = 1$ and $(a, b) \le_4^* (e, f)$.
(vi) $(a, b) \le_5^* (c, d)$, $(c, d) \le_5^* (e, f)$.

Then $a = c = e = 1$ and $b \le d \le f$. That is $(a, b) \le_5^* (e, f)$.
Hence $\le^*$ is a partial order.
To understand the above dcpo L more intuitively, we can see Figure 1.

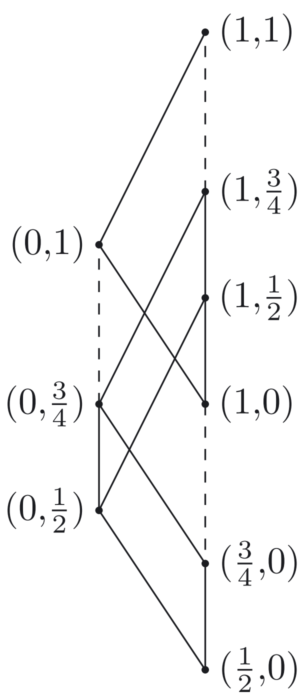

Figure 1: The poset L in Example 4.6.

For $r \in F$ and $(a, b) \in L$, we set $\downarrow r = \{x \in \mathbb{R} : x \le r \text{ in } \mathbb{R}\}$, $\downarrow^*(a, b) = \{(c, d) \in L : (c, d) \le^* (a, b)\}$ and $\uparrow^*(a, b) = \{(c, d) \in L : (a, b) \le^* (c, d)\}$.

There is an approximate identity $\{h_n\}_{n\in\mathbb{N}}$ consisting of finitely separating functions defined below.

$$h_n((a, b)) = \begin{cases} \left(0,\ \max\left\{\downarrow b \cap \downarrow \frac{n}{n+1}\right\}\right), & a = 0, \\ \left(\max\left\{\downarrow a \cap \downarrow \frac{n}{n+1}\right\}, 0\right), & b = 0, \\ \left(1,\ \max\left\{\downarrow b \cap \downarrow \frac{n}{n+1}\right\}\right), & \text{else.} \end{cases}$$

Thus L is an FS-domain, and also a *domain*.

Let $\mathcal{A}$ be all of principal filters of L. Then $L\setminus\{(1,1),(0,1)\}$ and $\downarrow^*(0,1)$ are two ideals, and

$$\uparrow^*(0,1) = \{\bigvee(L\setminus\{(1,1),(0,1)\}), \bigvee\downarrow^*(0,1)\} \in \mathcal{A}.$$

However, $L\setminus\{(1,1),(0,1)\} \cap \downarrow(0,1)$ is not an ideal. It shows that L is not an $\mathcal{A}_{IM}$ dcpo. Therefore, L is not an $\mathcal{A}_{DM}$ dcpo, even though L is an FS-domain.

Example 4.6 also tells us that an equational characterization for FS-domains has not been established. So it is a good follow-up effort to resolve the above question.

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