DE GRUYTER Open Mathematics 2025; 23(1): 20250218

Research Article

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On I-convergence of nets of functions in fuzzy metric spaces

https://doi.org/10.1515/math-2025-0218
Received February 24, 2025; accepted October 30, 2025; published online December 22, 2025

Abstract: In this paper, we introduce ideal versions of semi-α convergence, semi-exhaustiveness and semi uniform convergence of nets of functions between two fuzzy metric spaces, and obtain some properties of them. Let $\mathcal{I}$ be a D-admissible ideal on the directed set $(D, \geq)$, X and Y be two fuzzy metric spaces, $(f_d)_{d\in D}$ be a net of functions in Y^X and $f \in Y^X$. We mainly show that: (1) If $(f_d)_{d\in D}$ is a net of continuous functions and $\mathcal{I}$-semi uniformly convergent to f, then f is continuous; (2) The following are equivalent: (a) the net $(f_d)_{d\in D}$ is $\mathcal{I}$-semi-exhaustive and pointwise $\mathcal{I}$-convergent to f; (b) the net $(f_d)_{d\in D}$ is $\mathcal{I}$-semi-α convergent to f; (c) the net $(f_d)_{d\in D}$ is $\mathcal{I}$-semi uniformly convergent to f and the function f is continuous.

Keywords and phrases: fuzzy metric spaces; ***I***-semi-α convergence; ***I***-semi-exhaustiveness; ***I***-semi uniform convergence; nets of functions

MSC 2020: 40A35; 54A20; 54A40; 40A99

1 Introduction

Fuzzy sets were initially introduced by Zadeh [1]. After that, fuzzy sets have been applied in many fields of mathematics. The concept of fuzzy metric space was introduced by many authors in different ways. In particular, George and Veeramani [2] modified the concept of fuzzy metric space introduced by Kramosil and Michálek [3] with the help of continuous t-norms and defined a Hausdorff topology on this modified fuzzy metric space. Moreover, the topology induced by this fuzzy metric is first countable [2] and metrizable [4]. It is worth noticing that numerous concepts and results from classical metric spaces have been generalized to fuzzy metric spaces [2,4–7].

The idea of statistical convergence is an extension of usual convergence, which was first given by Zygmund in the first edition of his monograph [8] in 1935. In 1951, Fast [9] and Steinhaus [10] independently gave the concept of statistical convergence of real number sequences based on the asymptotic density of subsets of positive integers. Although statistical convergence was introduced over nearly the last 90 years, it has become an active area of research for 40 years with the contributions of several authors, Šalát [11], Fridy [12,13], Di Maio and Kočinac [14]. Statistical convergence has been widely applied in different fields of mathematics. See [14–19] etc.

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The idea of statistical convergence has been extended to $\mathcal{I}$-convergence by Kostyrko et al. in [20] with the help of ideals. $\mathcal{I}$-convergence includes usual convergence and statistical convergence when $\mathcal{I}$ is, respectively, the ideal of all finite subsets of the set of positive integers and the ideal of all subsets of the set of positive integers with natural density zero. Over the last 20 years, a lot of work has been done on $\mathcal{I}$-convergence and associated topics, and it has turned out to be one of the most active research areas in Topology and Analysis. For more details see [21–28].

It is well known that the pointwise limit of a sequence of continuous functions is not necessarily a continuous function. So one of the central questions in analysis is what precisely must be added to pointwise convergence of a sequence of continuous functions to preserve the continuity of the limit function. In 1841, Weierstrass gave a sufficient condition called uniform convergence which yields the continuity of the limit function. In 1878, Dini [29] gave another sufficient condition, weaker than uniform convergence, for continuity of the limit function. In 1883–1884, Arzelà [30] discovered a necessary and sufficient condition under which the pointwise limit of a sequence of real valued continuous functions on a compact interval is continuous. He called this condition "uniform convergence by segment", which was called "quasi-uniform convergence" by Borel [31]. The concept of α-convergence (also named as continuous convergence) has been known at the beginning of the 20th century. Later, around 1950, Arens [32], Hahn [33], Stoilov [34] and Iséki [35] gave some characterizations of α-convergence. One of the interesting facts about α-convergence (proved by Stoilov [34]) is that it preserves the continuity of the limit function for sequences of functions in metric spaces. In 2008, Gregoriades and Papanastassiou [36] introduced the concept of exhaustiveness and established a connection between α-convergence and pointwise convergence by this concept. Based on the relevant concepts of statistical convergence, Caserta and Kocinač [16] considered statistical versions of exhaustiveness and α-convergence of sequences of functions between two metric spaces. Papachristodoulos, Papanastassiou and Wilczynski [37] introduced ideal versions of exhaustiveness and α-convergence, and Megaritis [38] studied ideal version of uniform convergence.

Recently, Papanastassiou [39] introduced the concepts of semi-α convergence, semi-exhaustiveness and semi uniform convergence of sequences of functions between metric spaces, which are strictly weaker than α-convergence, exhaustiveness and almost uniform convergence, respectively. Let (X, d) and (Y, ρ) be two metric spaces, $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in Y^X and $f \in Y^X$. The following facts are proved in [39]: (1) If $(f_n)_{n\in\mathbb{N}}$ is semi-α convergent to f, then f is continuous. (2) If $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent to f and semi-exhaustive, then f is continuous. (3) For a pointwise convergence of sequence of functions $(f_n)_{n\in\mathbb{N}}$, the concepts of semi-α convergence and semi-exhaustiveness are equivalent. (4) The sequence of functions $(f_n)_{n\in\mathbb{N}}$ is semi-α convergent to f if and only if $(f_n)_{n\in\mathbb{N}}$ is semi uniformly convergent to f and f is continuous.

Based on the analysis above, it is natural to consider ideal versions of semi-α convergence, semi-exhaustiveness and semi uniform convergence of nets of functions between two fuzzy metric spaces (in the sense of George and Veeramani), which is the main purpose of this paper. The outline of this paper is as follows. In Section 2, we review some necessary definitions and results. In Section 3, we define ideal versions of semi-α convergence, semi-exhaustiveness and semi uniform convergence of nets of functions, and clarify their relations with related concepts. Let $\mathcal{I}$ be a D-admissible ideal on the directed set $(D, \geq)$, X and Y be two fuzzy metric spaces, $(f_d)_{d\in D}$ be a net of functions in Y^X and $f \in Y^X$. We mainly show that: (1) If $(f_d)_{d\in D}$ is a net of continuous functions and $\mathcal{I}$-semi uniformly convergent to f, then f is continuous; (2) The following are equivalent: (a) the net $(f_d)_{d\in D}$ is $\mathcal{I}$-semi-exhaustive and pointwise $\mathcal{I}$-convergent to f; (b) the net $(f_d)_{d\in D}$ is $\mathcal{I}$-semi-α convergent to f; (c) the net $(f_d)_{d\in D}$ is $\mathcal{I}$-semi uniformly convergent to f and the function f is continuous. These results extend the corresponding results in [39].

2 Preliminaries

Throughout this paper, all fuzzy metric spaces are in the sense of George and Veeramani. $\mathbb{N}$ denotes the set of all positive integers with the natural ordering and Y^X (resp. $C(X, Y)$) denotes the set of all functions (resp. all continuous functions) from X to Y. Readers may consult [40] for notation and terminology not given here.

We begin with some basic concepts about the fuzzy metric spaces.

Definition 2.1. [2] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if
(1) $*$ is associative and commutative;
(2) $*$ is continuous;
(3) for all $a \in [0,1]$, $a * 1 = a$;
(4) for all $a, b, c, d \in [0,1]$, $a \leq c, b \leq d \Rightarrow a * b \leq c * d$.

Typical examples of continuous t-norms are $a * b = ab$, $a * b = \min\{a, b\}$, and $a * b = \max\{0, a + b - 1\}$.

Definition 2.2. [2] A triple $(X, M, *)$ is a *fuzzy metric space* if X is a set, $*$ is a continuous t-norm, and M is a function

$$M: X \times X \times (0, \infty) \to [0,1]$$

satisfying the following conditions for all $x, y, z \in X$ and all $s, t > 0$:
(1) $M(x, y, t) > 0$;
(2) $M(x, y, t) = 1$ if and only if $x = y$;
(3) $M(x, y, t) = M(y, x, t)$;
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(5) $M(x, y, \cdot): (0, \infty) \to [0,1]$ is continuous.

The adjective "Fuzzy" in the above definition comes from the fact that the function M is a fuzzy set. The value of $M(x, y, t)$ is usually understood as the degree of certainty that the distance between x and y is less than t. It was shown in [41] that $M(x, y, \cdot)$ is non-decreasing for all x, y in X.

Definition 2.3. [2] Let (X, d) be a metric space and $a * b = ab$ for any $a, b \in [0,1]$. Define

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, *)$ is fuzzy metric space. We call this fuzzy metric space the *standard fuzzy metric space* and M_d the *standard fuzzy metric* induced by the metric d.

It is worth noticing that there exist fuzzy metric spaces which are not standard fuzzy metric spaces for any metric (see [2, Example 2.11]).

In this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}_{M_{|\cdot|}}$ denotes the standard fuzzy metric space $(\mathbb{R}, M_{|\cdot|}, \cdot)$ where $|\cdot|$ is the Euclidean metric in the real line.

Definition 2.4. [2] Let $(X, M, *)$ be a fuzzy metric space. Then the ball $B(x, \varepsilon, t)$ with center $x \in X$ and $\varepsilon \in (0,1)$, $t > 0$ is defined by

$$B(x, \varepsilon, t) = \{y \in X \mid M(x, y, t) > 1 - \varepsilon\}.$$

It was shown in [2] that the collection of all balls in a fuzzy metric space $(X, M, *)$ induces a first countable and Hausdorff topology τ_M on X. In fact, this topology is always metrizable (see [4]). When the fuzzy metric space is the standard fuzzy metric space induced by the metric d, the topology induced by fuzzy metric coincides with the topology induced by the metric d [2].

A sequence in a fuzzy metric space $(X, M, *)$ is said to be *convergent* if it is convergent in (X, τ_M), where τ_M is the topology induced by the fuzzy metric M [2]. The following proposition gives a characterization of the convergent sequences in fuzzy metric spaces.

Proposition 2.5. [2] *Let $(X, M, *)$ be a fuzzy metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X converges to $x \in X$ if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.*

From Proposition 2.5, it is clear that $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if for each $\varepsilon \in (0, 1)$ and $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

We now recall the concept of $\mathcal{I}$-convergence of nets. For more details we refer the readers to [23,24].

Definition 2.6. [40] Recall some basic concepts about nets.

(1) Let D be a non-empty set and "$\geq$" be a binary relation on D. The pair $(D, \geq)$ is said to be a *directed set* if: (a) "$\geq$" is reflexive and transitive; (b) for any two elements $d_1, d_2 \in D$, there is an element $d_0 \in D$ such that $d_0 \geq d_1, d_0 \geq d_2$.

(2) Let $(D, \geq)$ be a directed set and X be a non-empty set. A mapping $s: D \to X$ is called a *net* in X, denoted by $(s_d: d \in D)$ or $(s_d)_{d\in D}$.

(3) A net $(s_d)_{d\in D}$ is said to be *eventually* in $A \subseteq X$ if there is $d_0 \in D$ such that $s_d \in A$ for each $d \in D$ with $d \geq d_0$.

(4) A net $(s_d)_{d\in D}$ in a topological space X is said to *converge* to $x \in X$ if $(s_d)_{d\in D}$ is eventually in every neighborhood of x and we write $\lim s_d = x$ or $(s_d) \to x$.

Definition 2.7. [20] Let D be a non-empty set and $\mathcal{I}$ be a non-empty subfamily of the power set of D. $\mathcal{I}$ is said to be an *ideal on D* if:

(1) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;

(2) $A \in \mathcal{I}, B \subseteq A$ implies $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is said to be *non-trivial* if $\mathcal{I} \neq \{\emptyset\}$ and $D \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called *admissible* if it contains all finite subsets of D. Clearly, every non-trivial ideal $\mathcal{I}$ defines a *dual filter* $\mathcal{F}(\mathcal{I}) = \{A \subseteq D: D \backslash A \in \mathcal{I}\}$ on D.

Definition 2.8. [23,24] Let $\mathcal{I}$ be an ideal on a directed set D. A net $(s_d)_{d\in D}$ in a topological space X is said to be *$\mathcal{I}$-convergent* to a point $x \in X$ if for each neighborhood U of x, we have $\{d \in D: s_d \notin U\} \in \mathcal{I}$. In this case we write $\mathcal{I}$-$\lim s_d = x$ or $(s_d) \xrightarrow{\mathcal{I}} x$;

Actually, it is easy to verify that $(s_d)_{d\in D}$ $\mathcal{I}$-converges to x if and only if for each neighborhood U of x, there exists $A \in \mathcal{F}(\mathcal{I})$ such that $s_d \in U$ for all $d \in A$.

Remark 2.9. [23,24] Let $(D, \geq)$ be a directed set. For each $\alpha \in D$, define $D_\alpha = \{\gamma \in D: \gamma \geq \alpha\}$.

(1) A non-trivial ideal $\mathcal{I}$ of D is said to be *D-admissible* if $D_\alpha \in \mathcal{F}(\mathcal{I})$ for each $\alpha \in D$.

(2) Let $\mathcal{I}_0 = \{A \subseteq D: D\backslash A \supseteq D_\alpha \text{ for some } \alpha \in D\}$. It is easy to check that $\mathcal{I}_0$ is a D-admissible ideal on D.

(3) If $\mathcal{I}$ is a D-admissible ideal on D, $(s_d)_{d\in D}$ is a net in a topological space X and $x \in X$, then $(s_d) \to x$ implies $(s_d) \xrightarrow{\mathrm{I}} x$ and the converse holds if $\mathcal{I} = \mathcal{I}_0$.

(4) If $D = \mathbb{N}$, then the concepts of D-admissibility and admissibility coincide. In this case, $\mathcal{I}_0$ is the ideal of all finite subsets of $\mathbb{N}$.

Analogous to the definition of convergence of sequences in fuzzy metric spaces, a net in a fuzzy metric space $(X, M, *)$ is said to be *$\mathcal{I}$-convergent* if it is $\mathcal{I}$-convergent in (X, τ_M), and a net of functions $(f_d)_{d\in D}$ from a non-empty set X to a fuzzy metric space $(Y, M, *)$ is said to be *pointwise $\mathcal{I}$-convergent to $f \in Y^X$* if for each $x \in X$ the net $(f_d(x))_{d\in D}$ is $\mathcal{I}$-convergent in (Y, τ_M). In this case, we write $(f_d)_{d\in D} \xrightarrow{\mathcal{I}} f$ or simply $(f_d) \xrightarrow{\mathcal{I}} f$ and f is called the $\mathcal{I}$-limit function of $(f_d)_{d\in D}$.

Finally, we recall the definitions of semi-exhaustiveness, semi-α convergence, and semi uniform convergence of sequences of functions defined in [39], which are the main concepts discussed in this paper.

Definition 2.10. [39] Let (X, d), (Y, ρ) be metric spaces, $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in Y^X and $f \in Y^X$.

(1) The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be *semi-exhaustive at* $x \in X$ if for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x) > 0$ such that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m > n$ such that $\rho(f_m(y), f_m(x)) < \varepsilon$ for each $y \in B(x, \delta)$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be *semi-exhaustive* if it is semi-exhaustive at each $x \in X$.

(2) The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *semi-α* converge *to f at x* if
   (a) $(f_n(x)) \to f(x)$;
   (b) for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ such that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq n$ such that $\rho(f_m(y), f(x)) < \varepsilon$ for each $y \in B(x, \delta)$.

The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be *semi-α convergent* if it is semi-α convergent at each $x \in X$. A sequence of functions $(f_n)_{n\in\mathbb{N}}$ is said to have the *semi-α property* with respect to f if $(f_n)_{n\in\mathbb{N}}$ satisfies the condition (b).

Definition 2.11. [39] Let X be a topological space, (Y, ρ) be a metric space, $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in Y^X and $f \in Y^X$. The sequence $(f_n)_{n\in\mathbb{N}}$ is said to *semi uniformly converge to f at* $x \in X$ if

(1) $(f_n(x)) \to f(x)$;

(2) for each $\varepsilon > 0$ there exists a neighborhood O of x such that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq n$ such that $\rho(f_m(y), f(y)) < \varepsilon$ for each $y \in O$.

The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be *semi uniformly convergent* if it is semi uniformly convergent at each $x \in X$.

3 Main results

In this section, D denotes the directed set $(D, \geq)$, $\mathcal{I}$ denotes a D-admissible ideal, and $(X, N, \diamond)$ and $(Y, M, *)$ denote fuzzy metric spaces, which are simply denoted as X and Y. Nets of functions are always in Y^X, unless stated otherwise. A mapping from $(X, N, \diamond)$ to $(Y, M, *)$ is called continuous if it is continuous from (X, τ_N) to (Y, τ_M).

3.1 $\mathcal{I}$-semi-α convergence

In this subsection, we define the ideal version of semi-α convergence and clarify their relation. Firstly, the semi-α convergence of nets of functions in fuzzy metric spaces can be naturally defined as follows.

Definition 3.1. A net of functions $(f_d)_{d\in D}$ is said to *semi-α converge to f at x* if

(1) $(f_d(x)) \to f(x)$;

(2) for each $\varepsilon \in (0, 1)$ and $t > 0$, there exists $r \in (0, 1)$ and $s > 0$ such that for every $d \in D$ there exists $m \in D$, $m \geq d$ such that $M(f_m(y), f(x), t) > 1 - \varepsilon$ for each $y \in B_N(x, r, s)$.

A net of functions $(f_d)_{d\in D}$ is said to be *semi-α convergent* if it is semi-α convergent at each $x \in X$. A net of functions $(f_d)_{d\in D}$ is said to have the *semi-α property* with respect to f if $(f_d)_{d\in D}$ satisfies the condition (2).

Similarly, we can define the semi-exhaustiveness and semi uniform convergence of nets of functions in fuzzy metric spaces.

Let (X, d) and (Y, ρ) be metric spaces. Papanastassiou proved that if a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in Y^X is pointwise convergent to f and satisfies the semi-α property with respect to f at $x \in X$, then f is continuous at x [39, Proposition 4.3]. It is easy to prove that [39, Proposition 4.3] still holds when X and Y are fuzzy metric spaces and $(f_d)_{d\in D}$ is a net of functions in Y^X. We may define $\mathcal{I}$-semi-α convergence by replacing $(f_d) \to f$ with $(f_d) \xrightarrow{\mathcal{I}} f$ in Definition 3.1. However, the following example shows that it fails to preserve the continuity of the pointwise $\mathcal{I}$-limit function in this case.

Example 3.2. Let $D = \mathbb{N}$ and $\mathcal{I} \neq \mathcal{I}_0$. Then there exists a sequence of functions which is pointwise $\mathcal{I}$-convergent to a function f and satisfies the semi-α property with respect to f, but f is not continuous.

Proof. Since $D = \mathbb{N}$ and $\mathcal{I} \neq \mathcal{I}_0$, there exists an infinite set $A \in \mathcal{I}$. Consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$, where $f_n\colon \mathbb{R}_{M_{|\cdot|}} \to \mathbb{R}_{M_{|\cdot|}}$ for each $n \in \mathbb{N}$, defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } n \in A; \\ 1, & \text{if } n \notin A,\ x = 0; \\ 1/n, & \text{if } n \notin A,\ x \neq 0. \end{cases}$$

Let $f(x) = \begin{cases} 1, & x = 0; \\ 0, & x \neq 0. \end{cases}$ It is clear that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ is pointwise $\mathcal{I}$-convergent to f. Next we show that $(f_n)_{n\in\mathbb{N}}$ satisfies the semi-α property with respect to f. For each $x \in X$, if $x = 0$, fix arbitrarily $\varepsilon > 0$, $t > 0$ and take $r = \frac{1}{2}$, $s = 1$. For every $n \in \mathbb{N}$, since the set A is infinite, there exists $m \in A$ with $m \geq n$ such that $M_{|\cdot|}(f_m(y), f(0), t) = M_{|\cdot|}(1, 1, t) = 1 > 1 - \varepsilon$ whenever $y \in B_{M_{|\cdot|}}(x, \frac{1}{2}, 1)$. If $x \neq 0$, then for each $\varepsilon > 0$ and $t > 0$, there is $r \in (0, 1)$ and $s > 0$ such that $0 \notin B_{M_{|\cdot|}}(x, r, s)$. For every $n \in \mathbb{N}$, since the set $\mathbb{N}\backslash A$ is infinite, there is $m \in \mathbb{N}\backslash A$ with $m \geq \max\{[\frac{1-\varepsilon}{\varepsilon\cdot t}] + 1, n\}$ such that $M_{|\cdot|}(f_m(y), f(x), t) = \frac{t}{t+\frac{1}{m}} > 1 - \varepsilon$ for each $y \in B_{M_{|\cdot|}}(x, r, s)$. It follows that $(f_n)_{n\in\mathbb{N}}$ satisfies the semi-α property with respect to f. However, f is not continuous at $x = 0$. □

For this reason, we define the ideal version of semi-α convergence of nets of functions in fuzzy metric spaces as follows.

Definition 3.3. A net of functions $(f_d)_{d\in D}$ is said to *$\mathcal{I}$-semi-α converge to f at x* if

(1) $(f_d(x)) \xrightarrow{\mathcal{I}} f(x)$;

(2) for each $\varepsilon \in (0, 1)$ and $t > 0$, there exists $r \in (0, 1)$ and $s > 0$ such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that $M(f_m(y), f(x), t) > 1 - \varepsilon$ for each $y \in B_N(x, r, s)$.

A net of functions $(f_d)_{d\in D}$ is said to be *$\mathcal{I}$-semi-α convergent* if it is $\mathcal{I}$-semi-α convergent at each $x \in X$. A net of functions $(f_d)_{d\in D}$ is said to have the *$\mathcal{I}$-semi-α property* with respect to f if it satisfies the condition (2).

For an arbitrary directed set D, $\mathcal{I}$-semi-α convergence coincides with semi-α convergence whenever $\mathcal{I} = \mathcal{I}_0$. However, there exists a directed set D such that for each $\mathcal{I} \neq \mathcal{I}_0$, they are different. As the following examples show.

Example 3.4. Let $D = \mathbb{N}$ and $\mathcal{I} \neq \mathcal{I}_0$. Then there exists a sequence of functions which is semi-α convergent but not $\mathcal{I}$-semi-α convergent.

Proof. Since $D = \mathbb{N}$ and $\mathcal{I} \neq \mathcal{I}_0$, there exists an infinite set $A \in \mathcal{I}$. Consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$, where $f_n\colon \mathbb{R}_{M_{|\cdot|}} \to \mathbb{R}_{M_{|\cdot|}}$ for each $n \in \mathbb{N}$, defined as follows:

$$f_n(x) = \begin{cases} 1/n, & \text{if } n \in A; \\ 0, & \text{if } n \notin A,\ x \in (-\infty, -1/n] \cup [0, +\infty); \\ 2nx + 2, & \text{if } n \notin A,\ x \in (-1/n, -1/2n]; \\ -2nx, & \text{if } n \notin A,\ x \in (-1/2n, 0). \end{cases}$$

Let $f(x) = 0$ for each $x \in \mathbb{R}$. For each $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_1 \in \mathbb{N}$ such that $M_{|\cdot|}(\frac{1}{n}, 0, t) > 1 - \varepsilon$ for every $n \geq n_1$. For each $x \in \mathbb{R}$, if $x \geq 0$, then for every $n \geq n_1$ we have $M_{|\cdot|}(f_n(x), f(x), t) > 1 - \varepsilon$; if $x < 0$, then there exists $n_2 \in \mathbb{N}$ such that $x < -\frac{1}{n}$ for every $n \geq n_2$. Take $n_3 = \max\{n_1, n_2\}$. Then for each $n \geq n_3$ we have $M_{|\cdot|}(f_n(x), f(x), t) > 1 - \varepsilon$. It follows that $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent to f, which implies that it is pointwise $\mathcal{I}$-convergent to f.

In addition, let $x \in \mathbb{R}$, $\varepsilon > 0$ and $t > 0$ be given. Choose $r \in (0,1)$ and $s > 0$. For every $n \in \mathbb{N}$, since the set A is infinite, there exists $m \in A$ with $m \geq n$ such that $M_{|\cdot|}(f_m(y), f(x), t) = M_{|\cdot|}(\frac{1}{m}, 0, t) > 1 - \varepsilon$ for each $y \in B_{M_{|\cdot|}}(x, r, s)$. Thus, $(f_n)_{n\in\mathbb{N}}$ is semi-α convergent to f.

However, $(f_n)_{n\in\mathbb{N}}$ does not $\mathcal{I}$-semi-α converge to f at $x_0 = 0$. Indeed, take $\varepsilon_0 = 1/2$ and $t_0 = 1$. Then for each $r \in (0,1)$ and $s > 0$, there exists $n_0 \in \mathbb{N}$ such that $-1/2n \in B_{M_{|\cdot|}}(0, r, s)$ for every $n \geq n_0$. For every $n \in B = (\mathbb{N}\backslash A) \cap \{n \in \mathbb{N}: n \geq n_0\} \in \mathcal{F}(\mathcal{I})$, pick $y = -1/2n \in B_{M_{|\cdot|}}(0, r, s)$. Then we have $M_{|\cdot|}(f_n(y), f(0), t_0) = 1/2 \leq 1/2$. It follows that $(f_n)_{n\in\mathbb{N}}$ does not $\mathcal{I}$-semi-α converge to f at $x_0 = 0$. □

Example 3.5. Let $D = \mathbb{N}$ and $\mathcal{I} \neq \mathcal{I}_0$. Then there exists an $\mathcal{I}$-semi-α convergent sequence of functions which is not semi-α convergent.

Proof. Since $D = \mathbb{N}$ and $\mathcal{I} \neq \mathcal{I}_0$, there exists an infinite set $A \in \mathcal{I}$. Consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$, where $f_n\colon \mathbb{R}_{M_{|\cdot|}} \to \mathbb{R}_{M_{|\cdot|}}$ for each $n \in \mathbb{N}$, defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } n \in A; \\ 1/n, & \text{if } n \notin A. \end{cases}$$

Let $f(x) = 0$ for each $x \in \mathbb{R}$. It is easy to check that $(f_n)_{n\in\mathbb{N}}$ is pointwise $\mathcal{I}$-convergent to f. For each $x \in \mathbb{R}$, $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M_{|\cdot|}(\frac{1}{n}, 0, t) > 1 - \varepsilon$ for every $n \geq n_0$. Fix arbitrary $r \in (0,1)$ and $s > 0$. For every $B \in \mathcal{F}(\mathcal{I})$, let $B_1 = B \cap (\mathbb{N}\backslash A) \cap \{n \in \mathbb{N}: n \geq n_0\}$. Since $B_1 \neq \emptyset$, we can choose $m \in B_1$. Then for each $y \in B_{M_{|\cdot|}}(x, r, s)$, $M_{|\cdot|}(f_m(y), f(x), t) = M_{|\cdot|}(\frac{1}{m}, 0, t) > 1 - \varepsilon$. It follows that the sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{I}$-semi-α convergent to f. However, since the set A is infinite, the sequence of functions $(f_n)_{n\in\mathbb{N}}$ is not pointwise convergent to f. Thus it is not semi-α convergent. □

Remark 3.6. Example 3.4 also shows that a net of functions pointwise $\mathcal{I}$-converging to a continuous function is not necessarily $\mathcal{I}$-semi-α convergent.

3.2 $\mathcal{I}$-semi-exhaustiveness

In this subsection, we consider the ideal version of semi-exhaustiveness.

Definition 3.7. A net of functions $(f_d)_{d\in D}$ is said to be *$\mathcal{I}$-semi-exhaustive* at $x \in X$ if for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $r \in (0,1)$ and $s > 0$ such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that $M(f_m(y), f_m(x), t) > 1 - \varepsilon$ for each $y \in B_N(x, r, s)$. The net $(f_d)_{d\in D}$ is said to be *$\mathcal{I}$-semi-exhaustive* if it is $\mathcal{I}$-semi-exhaustive at each $x \in X$.

For an arbitrary directed set D, it is easy to check that $\mathcal{I}$-semi-exhaustiveness implies semi-exhaustiveness, and $\mathcal{I}$-semi-exhaustiveness coincides with semi-exhaustiveness whenever $\mathcal{I} = \mathcal{I}_0$. However, there exists a directed set D such that for each $\mathcal{I} \neq \mathcal{I}_0$, there is a net of functions which is semi-exhaustive but not $\mathcal{I}$-semi-exhaustive.

Example 3.8. Let $D = \mathbb{N}$ and $\mathcal{I} \neq \mathcal{I}_0$. Then there is a semi-exhaustive sequence of functions which is not $\mathcal{I}$-semi-exhaustive.

Proof. Since $D = \mathbb{N}$ and $\mathcal{I} \neq \mathcal{I}_0$, there exists an infinite set $A \in \mathcal{I}$. Consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$, where $f_n \colon \mathbb{R}_{M_{|\cdot|}} \to \mathbb{R}_{M_{|\cdot|}}$ for each $n \in \mathbb{N}$, defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } n \in A; \\ 1, & \text{if } n \notin A,\ x < 0; \\ 1/n, & \text{if } n \notin A,\ x \geq 0. \end{cases}$$

Let $x \in \mathbb{R}$, $\varepsilon > 0$ and $t > 0$ be given. Choose $r \in (0,1)$ and $s > 0$. For every $n \in \mathbb{N}$, since the set A is infinite, there is $m \in A$ with $m \geq n$ such that $M_{|\cdot|}(f_m(y), f_m(x), t) = 1 > 1 - \varepsilon$ for each $y \in B_{M_{|\cdot|}}(x, r, s)$. It follows that $(f_n)_{n\in\mathbb{N}}$ is semi-exhaustive. However, take $\varepsilon = 1/2$ and $t_0 = 1/2$. For each $r \in (0,1)$, $s > 0$ and $n \in \mathbb{N}\backslash(A \cup \{1\})$, pick $y_0 \in B_{M_{|\cdot|}}(0, r, s) \cap (-\infty, 0)$. Then, $M_{|\cdot|}(f_n(y_0), f_n(0), \frac{1}{2}) = \frac{n}{3n-2} \leq 1/2$. Since the set $\mathbb{N}\backslash(A \cup \{1\}) \in \mathcal{F}(\mathcal{I})$, it follows that $(f_n)_{n\in\mathbb{N}}$ is not $\mathcal{I}$-semi-exhaustive at $x_0 = 0$. □

The following example shows that there is a net of functions which is pointwise $\mathcal{I}$-convergent to a continuous function but not $\mathcal{I}$-semi-exhaustive.

Example 3.9. For an arbitrary ideal $\mathcal{I}$, there is a net of functions which is pointwise $\mathcal{I}$-convergent to a continuous function but not $\mathcal{I}$-semi-exhaustive.

Proof. Let $D = \mathbb{N}$, consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$, where $f_n \colon \mathbb{R}_{M_{|\cdot|}} \to \mathbb{R}_{M_{|\cdot|}}$ for each $n \in \mathbb{N}$, defined as follows:

$$f_n(x) = \begin{cases} 1/4n, & \text{if } x \leq 0; \\ n, & \text{if } x = 1/n; \\ 1/n, & \text{if } x > 0, x \neq 1/n. \end{cases}$$

It is easy to check that $(f_n)_{n\in\mathbb{N}}$ is pointwise $\mathcal{I}$-convergent to $f \equiv 0$. However, take $\varepsilon = 1/4$ and $t_0 = 1$. For each $r \in (0,1)$ and $s > 0$, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n} \in B_{M_{|\cdot|}}(0, r, s)$ for each $n \geq n_0$. Let $A = \{n \in \mathbb{N} \colon n \geq n_0\}$. Then for each $n \in A$, there is $y = \frac{1}{n} \in B_{M_{|\cdot|}}(0, r, s)$ such that

$$M_{|\cdot|}(f_n(y), f_n(0), t_0) = \frac{1}{1 + n - \frac{1}{4n}} \leq \frac{3}{4}.$$

It follows that $(f_n)_{n\in\mathbb{N}}$ is not $\mathcal{I}$-semi-exhaustive at $x_0 = 0$. □

Let $\mathcal{D}$ be the class of all D-admissible ideals on $(D, \geq)$. The class $\mathcal{D}$ is partially ordered by inclusion. If $\mathcal{D}_0 \subseteq \mathcal{D}$ is a non-empty linearly ordered subset of $\mathcal{D}$, then it is easy to verify that $\cup\mathcal{D}_0$ is a D-admissible ideal on $(D, \geq)$ which is an upper bound of $\mathcal{D}_0$. It follows from the Zorn's Lemma that there is a maximal D-admissible ideal on $(D, \geq)$. The following lemma gives a characterization of a maximal D-admissible ideal.

Lemma 3.10. *Let $\mathcal{I}$ be a D-admissible ideal on $(D, \geq)$. Then $\mathcal{I}$ is a maximal D-admissible ideal if and only if*

$$(A \in \mathcal{I}) \vee (D\backslash A \in \mathcal{I})$$

holds for each $A \subseteq D$.

Proof. ($\Leftarrow$) Let a D-admissible ideal $\mathcal{I}$ fulfill the condition for each $A \subseteq D$. We show that $\mathcal{I}$ is a maximal D-admissible ideal. On the contrary, let $\mathcal{I}$ be a proper subset of $\mathcal{I}_1$ where $\mathcal{I}_1$ is a D-admissible ideal on D. Then, there is $A \subseteq D$ such that $A \in (\mathcal{I}_1 \backslash \mathcal{I})$. Since $A \notin \mathcal{I}$, by assumption we have $(D\backslash A) \in \mathcal{I}$. Consequently $A \in \mathcal{I}_1$ and $(D\backslash A) \in \mathcal{I}_1$, so $D \in \mathcal{I}_1$, which is a contradiction.

(⇒) Let $\mathcal{I}$ be a maximal D-admissible ideal. If possible, let $A \subseteq D$ be such that

$$(A \notin \mathcal{I}) \wedge (D \backslash A \notin \mathcal{I}).$$

Define $\mathcal{K} = \{E \subseteq D: E \cap A \in \mathcal{I}\}$. For each $B \in \mathcal{I}$, since $B \cap A \subseteq B \in \mathcal{I}$, it follows that $B \cap A \in \mathcal{I}$, and hence $B \in \mathcal{K}$. Thus $\mathcal{I} \subseteq \mathcal{K}$. Obviously $D \notin \mathcal{K}$. It follows immediately that $\mathcal{K}$ is a non-trivial D-admissible ideal containing $\mathcal{I}$. However, $\mathcal{I}$ is maximal, so $\mathcal{I} = \mathcal{K}$. Now since $(D \backslash A) \cap A = \emptyset \in \mathcal{I}$, this shows that $(D \backslash A) \in \mathcal{I}$ which is again a contradiction. This proves the lemma. □

The ideal version of exhaustiveness of sequences of functions is defined in [37] between metric spaces, and it can be extended to nets of functions between fuzzy metric spaces naturally as follows.

Definition 3.11. A net of functions $(f_d)_{d \in D}$ is said to be *$\mathcal{I}$-exhaustive* at $x \in X$ if for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $r \in (0,1)$, $s > 0$ and $A \in \mathcal{F}(\mathcal{I})$ such that for every $m \in A$ and $y \in B_N(x, r, s)$ we have $M(f_m(y), f_m(x), t) > 1 - \varepsilon$. The net $(f_d)_{d \in D}$ is said to be *$\mathcal{I}$-exhaustive* if it is $\mathcal{I}$-exhaustive at each $x \in X$.

Next we show that $\mathcal{I}$-exhaustiveness and $\mathcal{I}$-semi-exhaustiveness are equivalent when $\mathcal{I}$ is a maximal D-admissible ideal.

Theorem 3.12. *If $\mathcal{I}$ is a maximal D-admissible ideal, then a net of functions $(f_d)_{d \in D}$ is $\mathcal{I}$-semi-exhaustive if and only if it is $\mathcal{I}$-exhaustive.*

Proof. (⇐) It is clear by the definition of $\mathcal{I}$-semi-exhaustiveness and $\mathcal{I}$-exhaustiveness.

(⇒) Since $(f_d)_{d \in D}$ is $\mathcal{I}$-semi-exhaustive, for each $x \in X$, $\varepsilon \in (0,1)$ and $t > 0$, there exists $r \in (0,1)$ and $s > 0$ such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that $M(f_m(y), f_m(x), t) > 1 - \varepsilon$ for each $y \in B_N(x, r, s)$. Let

$$H = \{d \in D: M(f_d(y), f_d(x), t) > 1 - \varepsilon \text{ for each } y \in B_N(x, r, s)\}.$$

Then $M(f_d(y), f_d(x), t) > 1 - \varepsilon$ for each $d \in H$ and $y \in B_N(x, r, s)$. We claim that $H \in \mathcal{F}(\mathcal{I})$. Suppose that $H \notin \mathcal{F}(\mathcal{I})$, since $\mathcal{I}$ is maximal, this means that $D \backslash H \in \mathcal{F}(\mathcal{I})$. Therefore, there exists $m \in D \setminus H$ such that $M(f_m(y), f_m(x), t) > 1 - \varepsilon$ for all $y \in B_N(x, r, s)$. By the construction of H, we can conclude that $m \in H$, which is a contradiction. Hence, $(f_d)_{d \in D}$ is $\mathcal{I}$-exhaustive at x, and then $(f_d)_{d \in D}$ is $\mathcal{I}$-exhaustive. □

3.3 $\mathcal{I}$-semi uniform convergence

In this subsection, we define the ideal version of semi uniform convergence of nets of functions, and prove the main results of this paper.

Definition 3.13. A net of functions $(f_d)_{d \in D}$ is said to *$\mathcal{I}$-semi uniformly converge to f at x* if

(1) $(f_d(x)) \xrightarrow{\mathcal{I}} f(x)$;

(2) for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $r \in (0,1)$ and $s > 0$ such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that $M(f_m(y), f(y), t) > 1 - \varepsilon$ for each $y \in B_N(x, r, s)$.

The net $(f_d)_{d \in D}$ is said to be *$\mathcal{I}$-semi uniformly convergent* if it is $\mathcal{I}$-semi uniformly convergent at each $x \in X$.

For an arbitrary directed set D, from the definitions of $\mathcal{I}$-semi uniform convergence and semi uniform convergence, it is easy to check that when $\mathcal{I} = \mathcal{I}_0$, $\mathcal{I}$-semi uniform convergence coincides with semi uniform convergence.

Next we will show that $\mathcal{I}$-semi uniform convergence preserves the continuity of the pointwise $\mathcal{I}$-limit of a net of continuous functions.

Theorem 3.14. *Let $(f_d)_{d\in D}$ be a net of functions in $C(X,Y)$ and $f\in Y^X$. If $(f_d)_{d\in D}$ is $\mathcal{I}$-semi uniformly convergent to f, then f is continuous.*

Proof. For each $\varepsilon\in(0,1)$, there is $\varepsilon_1\in(0,\varepsilon)$ such that $(1-\varepsilon_1)*(1-\varepsilon_1)*(1-\varepsilon_1)>1-\varepsilon$. For each $x\in X$ and $t>0$, since $(f_d)_{d\in D}$ is $\mathcal{I}$-semi uniformly convergent to f at x, there exists $r_1\in(0,1)$ and $s_1>0$ such that for every $A\in\mathcal{F}(\mathcal{I})$ there exists $m\in A$ such that for each $y\in B_N(x,r_1,s_1)$ we have

$$M\Big(f_m(y),f(y),\frac{t}{3}\Big)>1-\varepsilon_1.$$

As the net of functions $(f_d)_{d\in D}$ pointwise $\mathcal{I}$-converges to f at x, there is $A'\in\mathcal{F}(\mathcal{I})$ such that $M(f_d(x),f(x),\frac{t}{3})>1-\varepsilon_1$ for every $d\in A'$. Thus there is $m\in A'$ such that

$$M\Big(f_m(y),f(y),\frac{t}{3}\Big)>1-\varepsilon_1,\ M\Big(f_m(x),f(x),\frac{t}{3}\Big)>1-\varepsilon_1.$$

By the continuity of f_m, there exists $r_2\in(0,1)$ and $s_2>0$ such that

$$M\Big(f_m(y),f_m(x),\frac{t}{3}\Big)>1-\varepsilon_1$$

for each $y\in B_N(x,r_2,s_2)$. Let $r=\min\{r_1,r_2\}$ and $s=\min\{s_1,s_2\}$. Then for each $y\in B_N(x,r,s)$ we have

$$\begin{aligned}M(f(y),f(x),t)&\geq M\Big(f_m(y),f(y),\frac{t}{3}\Big)*M\Big(f_m(x),f(x),\frac{t}{3}\Big)*M\Big(f_m(y),f_m(x),\frac{t}{3}\Big)\\&\geq(1-\varepsilon_1)*(1-\varepsilon_1)*(1-\varepsilon_1)>1-\varepsilon.\end{aligned}$$

Hence f is continuous at x, and then f is continuous. □

The condition $(f_d)_{d\in D}$ in $C(X,Y)$ is necessary in Theorem 3.14 as the following example states.

Example 3.15. For an arbitrary ideal $\mathcal{I}$, there is a net of functions $(f_d)_{d\in D}$ $\mathcal{I}$-semi uniformly converging to a function f, which is not continuous.

Proof. Let $D=\mathbb{N}$. Consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$, where $f_n\colon\mathbb{R}_{M_{|\cdot|}}\to\mathbb{R}_{M_{|\cdot|}}$ for each $n\in\mathbb{N}$, defined as follows:

$$f_n(x)=\begin{cases}1, & \text{if } x=0;\\ 1/n, & \text{if } x\neq 0.\end{cases}$$

Let $f(x)=\begin{cases}1, & x=0;\\ 0, & x\neq 0.\end{cases}$ It is clear that $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent to f, which implies that it is pointwise $\mathcal{I}$-convergent to f. For each $x\in X$, $\varepsilon,r\in(0,1)$, $t,s>0$ and $A\in\mathcal{F}(\mathcal{I})$, since A is infinite, there exists $m\in A$ such that $M_{|\cdot|}(\frac{1}{m},0,t)>1-\varepsilon$. Then for each $y\in B_{M_{|\cdot|}}(x,r,s)$, if $y=0$, we have $M_{|\cdot|}(f_m(y),f(y),t)=1>1-\varepsilon$; if $y\neq0$, we have $M_{|\cdot|}(f_m(y),f(y),t)=M_{|\cdot|}(\frac{1}{m},0,t)>1-\varepsilon$. Thus, $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{I}$-semi uniformly convergent to f. However, f is not continuous. □

The following result reveals the relations among $\mathcal{I}$-semi-exhaustiveness, $\mathcal{I}$-semi-α convergence, and $\mathcal{I}$-semi uniform convergence of a net of functions.

Theorem 3.16. *Let $(f_d)_{d\in D}$ be a net of functions in Y^X and $f\in Y^X$. Then the following are equivalent:*
(1) *The net $(f_d)_{d\in D}$ is $\mathcal{I}$-semi-exhaustive and $(f_d)\xrightarrow{\mathcal{I}}f$.*
(2) *The net $(f_d)_{d\in D}$ $\mathcal{I}$-semi-α converges to f.*
(3) *The net $(f_d)_{d\in D}$ $\mathcal{I}$-semi uniformly converges to f and the function f is continuous.*

Proof. (1) ⇒ (2) For each $\varepsilon \in (0,1)$, there is $\varepsilon_1 \in (0,\varepsilon)$ such that $(1-\varepsilon_1)*(1-\varepsilon_1) > 1-\varepsilon$. For each $x \in X$ and $t > 0$, since $(f_d)_{d\in D}$ is $\mathcal{I}$-semi-exhaustive at x, there exists $r \in (0,1)$ and $s > 0$ such that for every $A \in \mathcal{F}(\mathcal{I})$ there is $m \in A$ such that

$$M\Big(f_m(y), f_m(x), \frac{t}{2}\Big) > 1-\varepsilon_1$$

for each $y \in B_N(x,r,s)$. By assumption, $(f_d)_{d\in D}$ is pointwise $\mathcal{I}$-convergent to f, then there exists $A' \in \mathcal{F}(\mathcal{I})$ such that

$$M\Big(f_d(x), f(x), \frac{t}{2}\Big) > 1-\varepsilon_1$$

for every $d \in A'$. Hence, for every $A \in \mathcal{F}(\mathcal{I})$, since $A \cap A' \in \mathcal{F}(\mathcal{I})$, there is $m \in A \cap A'$ such that

$$M\Big(f_m(y), f_m(x), \frac{t}{2}\Big) > 1-\varepsilon_1,\ M\Big(f_m(x), f(x), \frac{t}{2}\Big) > 1-\varepsilon_1$$

for each $y \in B_N(x,r,s)$. It follows that

$$M(f_m(y), f(x), t) \geq M\Big(f_m(y), f_m(x), \frac{t}{2}\Big) * M\Big(f_m(x), f(x), \frac{t}{2}\Big)$$

$$\geq (1-\varepsilon_1)*(1-\varepsilon_1) > 1-\varepsilon$$

for each $y \in B_N(x,r,s)$. Therefore, the net $(f_d)_{d\in D}$ is $\mathcal{I}$-semi-α convergent to f at x, and then it is $\mathcal{I}$-semi-α convergent to f.

(2) ⇒ (3) First we prove that f is continuous. For each $\varepsilon \in (0,1)$, there is $\varepsilon_1 \in (0,\varepsilon)$ such that $(1-\varepsilon_1)*(1-\varepsilon_1) > 1-\varepsilon$. For each $x \in X$ and $t > 0$, since the net $(f_d)_{d\in D}$ satisfies the $\mathcal{I}$-semi-α property at x with respect to f, there exists $r_1 \in (0,1)$ and $s_1 > 0$ such that for every $A \in \mathcal{F}(\mathcal{I})$ there is $m \in A$ such that

$$M\Big(f_m(y), f(x), \frac{t}{2}\Big) > 1-\varepsilon_1$$

for each $y \in B_N(x,r_1,s_1)$. Since $(f_d)_{d\in D}$ pointwise $\mathcal{I}$-converges to f, there exists $A_y \in \mathcal{F}(\mathcal{I})$ such that

$$M\Big(f_d(y), f(y), \frac{t}{2}\Big) > 1-\varepsilon_1$$

for every $d \in A_y$. Let $A \in \mathcal{F}(\mathcal{I})$. Then, for each $y \in B_N(x,r_1,s_1)$, there is $m_1 \in A \cap A_y$ such that

$$M(f(y), f(x), t) \geq M\Big(f_{m_1}(y), f(x), \frac{t}{2}\Big) * M\Big(f_{m_1}(y), f(y), \frac{t}{2}\Big)$$

$$\geq (1-\varepsilon_1)*(1-\varepsilon_1) > 1-\varepsilon.$$

It follows that f is continuous at x, then f is continuous.

It remains to prove that the net $(f_d)_{d\in D}$ is $\mathcal{I}$-semi uniformly convergent to f. Clearly, (2) implies that $(f_d)\xrightarrow{\mathcal{I}} f$. For each $x \in X$ and $t > 0$, since the net $(f_d)_{d\in D}$ satisfies the $\mathcal{I}$-semi-α property at x with respect to f, there exists $r_1 \in (0,1)$ and $s_1 > 0$ such that for every $A \in \mathcal{F}(\mathcal{I})$ there is $m \in A$ such that

$$M\Big(f_m(y), f(x), \frac{t}{2}\Big) > 1-\varepsilon_1$$

for all $y \in B_N(x,r_1,s_1)$. By the continuity of f, there exists $r_2 \in (0,1)$ and $s_2 > 0$ such that

$$M\Big(f(y), f(x), \frac{t}{2}\Big) > 1-\varepsilon_1$$

for each $y \in B_N(x,r_2,s_2)$. Let $r = \min\{r_1, r_2\}$ and $s = \min\{s_1, s_2\}$. Then for each $y \in B_N(x,r,s)$ we have

$$M(f_m(y), f(y), t) \geq M\Big(f_m(y), f(x), \frac{t}{2}\Big) * M\Big(f(y), f(x), \frac{t}{2}\Big)$$

$$\geq (1-\varepsilon_1)*(1-\varepsilon_1) > 1-\varepsilon.$$

It follows that the net is $\mathcal{I}$-semi uniformly convergent to f at x, then it is $\mathcal{I}$-semi uniformly convergent to f.

(3) ⇒ (1) It is clear that (3) implies that $(f_d) \xrightarrow{\mathcal{I}} f$. For each $\varepsilon \in (0, 1)$, there is $\varepsilon_1 \in (0, \varepsilon)$ such that $(1 - \varepsilon_1) * (1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon$. For each $x \in X$ and $t > 0$, since $(f_d)_{d \in D}$ is $\mathcal{I}$-semi uniformly convergent to f at x, there exists $r_1 \in (0, 1)$ and $s_1 > 0$ such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that

$$M\Big(f_m(y), f(y), \frac{t}{3}\Big) > 1 - \varepsilon_1$$

for all $y \in B_N(x, r_1, s_1)$. By the continuity of f, there exists $r_2 \in (0, 1)$ and $s_2 > 0$ such that

$$M\Big(f(y), f(x), \frac{t}{3}\Big) > 1 - \varepsilon_1$$

for all $y \in B_N(x, r_2, s_2)$. Let $r = \min\{r_1, r_2\}$ and $s = \min\{s_1, s_2\}$. Then for each $y \in B_N(x, r, s)$ we have

$$M(f_m(y), f_m(x), t) \geq M\Big(f_m(y), f(y), \frac{t}{3}\Big) * M\Big(f(y), f(x), \frac{t}{3}\Big) * M\Big(f_m(x), f(x), \frac{t}{3}\Big)$$

$$\geq (1 - \varepsilon_1) * (1 - \varepsilon_1) * (1 - \varepsilon_1) > 1 - \varepsilon.$$

Thus the net is $\mathcal{I}$-semi-exhaustive at x, then it is $\mathcal{I}$-semi-exhaustive. □

Remark 3.17. By Theorem 3.16, Example 3.4 also shows that there exists a directed set D such that $\mathcal{I}$-semi uniform convergence and semi uniform convergence are different whenever $\mathcal{I} \neq \mathcal{I}_0$.

Let $D = \mathbb{N}$ and $\mathcal{I} \neq \mathcal{I}_0$. The sequences of functions $(f_n)_{n \in \mathbb{N}}$ defined in Example 3.8 is semi-exhaustive and pointwise $\mathcal{I}$-convergent to $f(x) = \begin{cases} 1, & x < 0; \\ 0, & x \geq 0. \end{cases}$ Clearly, f is not continuous at $x_0 = 0$. This shows that semi-exhaustiveness is too weak to preserve the continuity of the pointwise $\mathcal{I}$-limit function. However, by Theorems 3.16, we have the following corollaries.

Corollary 3.18. *If a net of functions $(f_d)_{d \in D}$ is $\mathcal{I}$-semi-exhaustive and $(f_d) \xrightarrow{\mathcal{I}} f \in Y^X$, then f is continuous.*

Corollary 3.19. *If a net of functions $(f_d)_{d \in D}$ $\mathcal{I}$-semi-α converges to $f \in Y^X$, then f is continuous.*

4 Conclusions

Recently, N. Papanastassiou [39] introduced the concepts of semi-α convergence, semi-exhaustiveness and semi uniform convergence of sequences of functions between metric spaces. Since fuzzy metric spaces can be regarded as a generalization of metric spaces in a certain sense, it is natural to consider whether these new concepts can be extended to fuzzy metric spaces and which properties of them remain invariant. Note that $\mathcal{I}$-convergence of nets implies convergence of sequences, when the directed set is $\mathbb{N}$ and $\mathcal{I} = \mathcal{I}_0$. This motivates us to define appropriate ideal versions of semi-α convergence, semi-exhaustiveness and semi uniform convergence for nets of functions such that they imply the classical one when the directed set is $\mathbb{N}$ and $\mathcal{I} = \mathcal{I}_0$, and to study them in the modified way.

Thus, in this paper, we define and study ideal versions of semi-α convergence, semi-exhaustiveness and semi uniform convergence of nets of functions in fuzzy metric spaces. Our work extends some corresponding results in [39]. We also prove that when $\mathcal{I}$ is a maximal D-admissible ideal, then a net of functions is $\mathcal{I}$-exhaustive if and only if it is $\mathcal{I}$-semi-exhaustive. However, ideal versions of α-convergence and almost uniform convergence of nets of functions in fuzzy metric spaces have not been considered in this paper. Thus, one can define ideal versions of α-convergence and almost uniform convergence of nets of functions in fuzzy metric spaces, and study their properties or their relations with the corresponding concepts.

Concluding this section, we briefly mention another related study on $\mathcal{I}$-convergence. In 2024, the first and third authors studied the properties of $\mathcal{I}$-convergence in cone metric spaces [42]. The paper discusses $\mathcal{I}$-sequential compactness, $\mathcal{I}$-sequential countable compactness, and $\mathcal{I}$-completeness in cone metric spaces, and provides a negative answer to an open problem posed by P. Das [24, Open Problem 2.3]. Both reference [42] and this paper concern $\mathcal{I}$-convergence, but they focus on different specific aspects. As a direction for future research, we can study $\mathcal{I}$-sequential compactness, sequential countable compactness, and completeness in fuzzy metric spaces.

Acknowledgments: The authors sincerely thank the reviewers for their careful reading and valuable suggestions in improving this manuscript.
Research ethics: Not applicable.
Informed consent: Not applicable.
Author contributions: All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results and approved the final version of the manuscript. All authors contributed equally to this work.
Use of Large Language Models, AI and Machine Learning Tools: None declared.
Conflict of interest: The authors state no conflicts of interest.
Research funding: This research was supported by National Natural Science Foundation of China (No.∼11901274), the Fujian Provincial Natural Science Foundation of China (Nos.∼2025J01360,∼2024J02022, ∼2024J01804), the Natural Science Foundation of Jiangsu Province (No. ∼BK20200834), the Institute of Fujian Key Laboratory of Granular Computing and Applications, the Institute of Meteorological Big Data-Digital Fujian, Fujian Key Laboratory of Data Science and Statistics, and President's fund of Minnan Normal University (KJ18007).
Data availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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