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Research Article

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Fixed point approaches to the stability of Jensen's functional equation

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Abstract: We will investigate the stability problem of the Jensen's functional equation using Brzdęk fixed point theorem and fixed point alternative method on non-Archimedean fuzzy normed spaces.

Keywords: Jensen's functional equation; Hyers-Ulam-Rassias stability; Brzdęk fixed point theorem; fixed point alternative method; non-Archimedean fuzzy normed spaces

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1 Introduction

In 1940, Ulam [1] introduced the stability problem for group homomorphisms, which can be framed as: under what conditions does an object that approximately satisfies a given property necessarily approximate an object that satisfies the property exactly? In 1941, Hyers [2] provided a partial affirmative answer to Ulam's question for additive functions (Cauchy functions) in Banach spaces, known as the Hyers stability theorem. This result was subsequently extended by Aoki [3], who generalized the stability concept to functions involving p-powers of the norm. Later, in 1978, Rassias [4] further expanded Hyers' theorem to allow for an unbounded Cauchy difference.

The study of stability problems in functional equations has seen substantial progress through the contributions of numerous mathematicians. This advancement can be primarily attributed to the exploration of three distinct perspectives: the theoretical analysis of functional equations, the study of normed spaces, and the development and application of various methodological approaches.

Over the past few decades, the stability of various functional equations has been the subject of extensive study and generalization by many mathematicians (see [5–9]). These investigations have significantly advanced the understanding and application of stability results for functional equations. In particular, a classic contribution to the field was made by Alsina [10], who investigated the Hyers-Ulam stability problem for functional equations in the generalized context of probabilistic normed spaces. A notable methodological approach in the study of stability problems is the fixed point method. Baker [11] was the first to introduce Ulam's type stability using this method, which we refer to as the *fixed point alternative method*. This approach has since been applied extensively in a range of studies (see [12–18]). Fixed point theory, as a powerful tool, plays a crucial role in the research, study, and application of nonlinear functional analysis, optimization theory, and variational inequalities (see [19–22]). Numerous authors have contributed to this field by introducing new types of fixed point theorems across various directions. In particular, Brzdęk and Ciepliński [23] introduced a fixed point

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existence theorem for nonlinear operators in metric spaces and utilized their results to address stability problems in Non-Archimedean metric spaces. This approach is referred to as *Brzdęk's fixed point method*. Recently, Brzdęk's fixed point method has been applied to various functional equations and normed spaces (see [24–28]). For instance, Benzarouala, Brzdęk, and Oubbi [29] applied a fixed point theorem to establish the Ulam stability for a general linear functional equation, which includes the Jensen equation as a particular case, within the framework of random normed spaces.

The theory of fuzzy sets, first introduced by Zadeh [30] in 1965, provided a new framework for modeling uncertainty. This foundation was soon extended to topological concepts. A crucial step was generalizing the triangle inequality, which required a binary operation known as the continuous triangular norm (t-norm). The theory of t-norms was extensively developed by Schweizer and Sklar [31], whose work, along with that of Alsina [32], established their fundamental role in constructing the triangle inequality for probabilistic metric spaces. This concept became central to the axiomatic definition of a fuzzy metric space first proposed by Kramosil and Michálek [33]. A different and widely used definition of a fuzzy metric space was later introduced by George and Veeramani [34], which has provided the basis for many subsequent developments. In a parallel development, these topological ideas were combined with algebraic properties to create fuzzy normed spaces, with influential early formulations by pioneers such as Katsaras [35] and Felbin [36].

With these fuzzy structures established, researchers began to investigate classical problems within them. Mirmostafaee and Moslehian made significant contributions by applying Hyers-Ulam stability theory first to general fuzzy normed spaces [37] and then more specifically to the non-Archimedean setting [38]. The study of stability for functional equations within these frameworks has since become an active and fruitful area of research.

This paper focuses on investigating the stability of Jensen's functional equations in non-Archimedean fuzzy normed spaces. For completeness, we begin by recalling the fundamental definitions that form the basis of our work.

Definition 1.1 (Continuous triangular norm(t-norm) [31]). A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a *continuous triangular norm* (or simply a continuous t-norm) if it satisfies the following properties for all $a, b, c, d \in [0,1]$:
1. $a * b = b * a, a * (b * c) = (a * b) * c$;
2. If $a \leq c$ and $b \leq d$, then $a * b \leq c * d$;
3. $a * 1 = a$;
4. $*$ is continuous.

Well-known examples of continuous t-norms include the *product t-norm* $a * b = a \cdot b$, and the *minimum t-norm* $a * b = \min\{a, b\}$.

Definition 1.2 (Fuzzy Normed Space [37]). Let X be a real linear space and let $*$ be a continuous t-norm. A function $N: X \times (0, \infty) \to [0,1]$ is said to be a fuzzy norm on X if it satisfies the following conditions for all $x, y \in X$:
(FN1) $N(x, t) = 1$ for all $t > 0$ if and only if $x = 0$;
(FN2) $N(kx, t) = N(x, \frac{t}{|k|})$ for all $t > 0$ and $k \in \mathbb{R}\backslash\{0\}$;
(FN3) $N(x + y, t + s) \geq N(x, t) * N(y, s)$ for all $s, t > 0$;
(FN4) The function $N_x(t) := N(x, t)$ is continuous on $(0, \infty)$ and $\lim_{t\to\infty} N(x, t) = 1$.

The triple $(X, N, *)$ is called a fuzzy normed space.

Definition 1.3 (Non-Archimedean Fuzzy Normed Space [38]). A fuzzy normed space $(X, N, *)$ is called a non-Archimedean fuzzy norm if the triangle inequality (FN3) is replaced by the following stronger condition for all $x, y \in X$ and $s, t > 0$:

(NA-FN3) $N(x + y,\ \max\{s, t\}) \geq N(x, s) * N(y, t)$.

Definition 1.4 (Fuzzy Metric Space [34]). Let X be an arbitrary set and let $*$ be a continuous t-norm. A function $M: X \times X \times (0, \infty) \to [0, 1]$ is called a fuzzy metric on X if it satisfies the following conditions for all $x, y, z \in X$ and $s, t > 0$:
(FM1) $M(x, y, t) = 1$ if and only if $x = y$;
(FM2) $M(x, y, t) = M(y, x, t)$;
(FM3) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(FM4) The function $M_{xy}(t) := M(x, y, t)$ is continuous on $(0, \infty)$.

The triple $(X, M, *)$ is called a fuzzy metric space.

Definition 1.5 (Non-Archimedean Fuzzy Metric Space [39]). A fuzzy metric space $(X, M, *)$ is called a non-Archimedean fuzzy metric space if the triangle inequality (FM3) is replaced by the following stronger condition for all $x, y, z \in X$ and $s, t > 0$:
(NA-FM3) $M(x, z, \max\{s, t\}) \geq M(x, y, s) * M(y, z, t)$.

Example 1.6. This example illustrates a construction of both a non-Archimedean fuzzy normed space and its associated non-Archimedean fuzzy metric space using the minimum t-norm, $a * b = \min\{a, b\}$.

Let the underlying vector space be the set of rational numbers $X = \mathbb{Q}$, equipped with the p-adic norm $\|\cdot\|_p$, which is a well-known ultranorm satisfying

$$\|x + y\|_p \leq \max\{\|x\|_p, \|y\|_p\}, \quad \text{for all } x, y \in \mathbb{Q}.$$

Using this p-adic norm, we define the fuzzy norm $N: \mathbb{Q} \times (0, \infty) \to [0, 1]$ by

$$N(x, t) = e^{-\|x\|_p / t}, \quad \text{for all } x \in \mathbb{Q}, \ t > 0.$$

Then the triple $(\mathbb{Q}, N, *)$ forms a non-Archimedean fuzzy normed space.

Next, we define the induced fuzzy metric $M: \mathbb{Q} \times \mathbb{Q} \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) := N(x - y, t) = e^{-\|x - y\|_p / t}, \quad \text{for all } x, y \in \mathbb{Q}, \ t > 0.$$

Since the fuzzy norm N satisfies the non-Archimedean inequality

$$N(x + y, \max\{s, t\}) \geq \min\{N(x, s), N(y, t)\},$$

the induced fuzzy metric M inherits the same property. Therefore, the triple $(\mathbb{Q}, M, *)$ is a non-Archimedean fuzzy metric space.

In particular, Moslehian and Rassias [40] studied the stability problem of functional equations in non-Archimedean spaces. For a comprehensive overview of recent developments in this area, we refer the reader to the very recent survey by Ciepliński [41]. Let f be a function satisfying $f(n^2 + m^2) = f(n^2) + f(m^2)$ for all positive integers m and n, and assume that f is multiplicative or completely multiplicative. Such a function can be regarded as a solution to an arithmetic functional equation. Koh [28] investigated the stability problem of this arithmetic functional equation by applying Brzdęk's fixed point method within the framework of a non-Archimedean fuzzy normed space.

In this paper, we focus on the following Jensen's functional equation:

$$2f\left(\frac{x + y}{2}\right) = f(x) + f(y). \tag{1.1}$$

Kominek [42] was the first to obtain results concerning the stability of Jensen's equation, and the Hyers-Ulam-Rassias stability of this equation was subsequently explored by Jung [43]. Furthermore, Moslehian [44] examined Jensen's functional equation in the setting of non-Archimedean normed spaces, while Cădariu and Radu [45] addressed its stability via the fixed point alternative method.

One of the main objectives of the present work is to investigate the applicability of two fixed point techniques – namely, Brzdęk's fixed point method and the fixed point alternative method – to a broad class of functional equations, including Jensen's functional equation (1.1), in various settings such as non-Archimedean fuzzy normed spaces. Additionally, we aim to examine and compare the distinctive features and advantages associated with each method.

2 Brzdęk's fixed point method

We will first reproduce the Brzdęk's fixed point method. Brzdęk and Ciepliński [23] introduced the existence theorem of the fixed point for nonlinear operators in metric spaces:

Theorem 2.1 ([23]). *Let X be a non-empty set, (Y, d) be a complete metric space and $\Lambda\colon \mathbb{R}_+^X \to \mathbb{R}_+^X$ be a non-decreasing operator satisfying the hypothesis*

$$\lim_{n\to\infty} \Lambda\delta_n = 0 \textit{ for every sequence } \{\delta_n\}_{n\in\mathbb{N}} \text{ in } \mathbb{R}_+^X \textit{ with } \lim_{n\to\infty}\delta_n = 0.$$

Suppose that $\mathcal{T}\colon Y^X \to Y^X$ is an operator satisfying the inequality

$$d(\mathcal{T}\xi(x), \mathcal{T}\mu(x)) \le \Lambda(\triangle(\xi, \mu))(x), \quad \textit{for all } \xi, \mu \in Y^X \textit{ and } x \in X$$

where $\triangle\colon (Y^X)^2 \to \mathbb{R}_+^X$ is a mapping which is defined by

$$\triangle(\xi, \mu)(x) := d(\xi(x), \mu(x)), \quad \textit{for all } \xi, \mu \in Y^X \textit{ and } x \in X.$$

If there exist functions $\varepsilon\colon X \to \mathbb{R}_+$ and $\phi\colon X \to Y$ such that

$$d(\mathcal{T}\phi(x), \phi(x)) \le \varepsilon(x)$$

and

$$\varepsilon^*(x) := \sum_{n\in\mathbb{N}_0} (\Lambda^n \varepsilon)(x) < \infty$$

for all $x \in X$, then the limit

$$\lim_{n\to\infty} (\mathcal{T}^n\phi)(x)$$

exists for each $x \in X$. Moreover, the function $\phi \in Y^X$ defined by

$$\psi(x) := \lim_{n\to\infty} (\mathcal{T}^n\phi)(x)$$

is a fixed point of $\mathcal{T}$ with

$$d(\phi(x), \psi(x)) \le \varepsilon^*(x)$$

for all $x \in X$.

Brzdęk and Ciepliński [23] used this result to prove the stability problem of functional equations in non-Archimedean metric spaces and obtained the fixed point results in arbitrary metric spaces. In particular, Brzdęk's fixed point method was also obtained from Theorem 2.1 (see [27]).

Theorem 2.2 ([27]). *Let X be a non-empty set, (Y, d) be a complete metric space and $f_1, f_2, f_3 : X \to X$ be given mappings. Suppose that $\mathcal{T} : Y^X \to Y^X$ and $\Lambda : \mathbb{R}_+^X \to \mathbb{R}_+^X$ are two operators satisfying the following conditions*

$$d(\mathcal{T}\xi(x),\ \mathcal{T}\mu(x)) \le \sum_{k=1}^{3} d(\xi(f_k(x)), \mu(f_k(x))) \tag{2.1}$$

and

$$\Lambda\delta(x) := \sum_{k=1}^{3} \delta(f_k(x)) \tag{2.2}$$

for all $\xi,\ \mu \in Y^X$, $\delta \in \mathbb{R}_+^X$ and $x \in X$. If there exist $\varepsilon : X \to \mathbb{R}_+$ and $\phi : X \to Y$ such that

$$d(\mathcal{T}\phi(x),\ \phi(x)) \le \varepsilon(x) \text{ and } \varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) < \infty \tag{2.3}$$

for all $x \in X$, then the limit $\lim_{n\to\infty}(\mathcal{T}^n\phi)(x)$ exists for each $x \in X$. Moreover, the function $\psi(x) := \lim_{n\to\infty}(\mathcal{T}^n\phi)(x)$ is a fixed point of $\mathcal{T}$ with

$$d(\phi(x),\ \psi(x)) \le \varepsilon^*(x)$$

for all $x \in X$.

Now, we will investigate the stability problem for the Jensen's functional equation (1.1) by using Brzdęk fixed point method on an non-Archimedean fuzzy metric space and non-Archimedean fuzzy normed space. We consider the space $\mathbb{R}_+$ equipped with a non-Archimedean fuzzy metric M and the minimum t-norm $*$, denoted by $(\mathbb{R}_+, M, *)$. We assume that M is generated from an underlying ultrametric on $\mathbb{R}_+$ that makes it *invariant*, that is,

$$M(x+z, y+z, t) = M(x, y, t) \quad \text{for all } x, y, z \in \mathbb{R}_+ \text{ and } t > 0.$$

Furthermore, let $(\mathbb{R}, N, *)$ be a complete non-Archimedean fuzzy normed space, also with the minimum t-norm $*$. These structures provide the setting for studying the stability of Jensen's functional equation under non-Archimedean fuzzy norms.

Theorem 2.3. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying*

$$L_0 := \left\{ m \in \mathbb{N} \mid s(m) + 2s\left(\frac{1+m}{2}\right) < 1 \right\} \ne \emptyset, \tag{2.4}$$

where

$$s(m) := \inf\left\{ t \in \mathbb{R}_+ \mid h(mx) \le t\, h(x) \quad \text{for all } x \in \mathbb{R}_+ \right\}.$$

Assume that for all $x, y \in \mathbb{R}_+$, $n, m \in \mathbb{N}$, and $s > 0$, the following inequality holds:

$$N\big(h(nx) + h(my),\ s\big) \ge N\big(s(n)\, h(x) + s(m)\, h(y),\ s\big). \tag{2.5}$$

Suppose $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$M\left(2f\left(\frac{x+y}{2}\right),\ f(x) + f(y),\ t\right) \ge N\big(h(x) + h(y),\ t\big) \tag{2.6}$$

for all $x, y \in \mathbb{R}_+$ and $t > 0$. Then there exists a unique function $T : \mathbb{R}_+ \to \mathbb{R}$ satisfying Jensen's functional equation (1.1)*, such that*

$$M\big(f(x),\ T(x),\ t\big) \ge N\big(s_0\, h(x),\ t\big) \tag{2.7}$$

for all $x \in \mathbb{R}_+$ and $t > 0$, where

$$s_0 := \inf_{m \in L_0} \left\{ \frac{1 + s(m)}{1 - s(m) - 2s\left(\frac{1+m}{2}\right)} \right\}. \tag{2.8}$$

Proof. Let $m \in L_0$. By letting $y = mx$ in the inequality (2.6), we have

$$M\Big(2f\Big(\frac{1+m}{2}x\Big),\ f(x)+f(mx),\ t\Big) \geq N(h(x)+h(mx),\ t), \quad t>0,\ x \in \mathbb{R}_+. \tag{2.9}$$

From inequality (2.5), it follows that

$$M\Big(2f\Big(\frac{1+m}{2}x\Big),\ f(x)+f(mx),\ t\Big) \geq N(c_m(x),\ t), \tag{2.10}$$

where

$$c_m(x) := (1+s(m))h(x), \quad t>0,\ x \in \mathbb{R}_+.$$

To utilize the Brzdęk fixed-point method, we must first define two operators as outlined in Theorem 2.2:

1. Define $\mathcal{T}_m : \mathbb{R}^{\mathbb{R}_+} \to \mathbb{R}^{\mathbb{R}_+}$ by

$$\mathcal{T}_m\xi(x) := \xi\Big(\frac{1+m}{2}x\Big) + \xi\Big(\frac{1+m}{2}x\Big) - \xi(mx) = 2\xi\Big(\frac{1+m}{2}x\Big) - \xi(mx). \tag{2.11}$$

2. Define $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_+} \to \mathbb{R}_+^{\mathbb{R}_+}$ by

$$\Lambda_m\mu(x) := \mu\Big(\frac{1+m}{2}x\Big) + \mu\Big(\frac{1+m}{2}x\Big) + \mu(mx) = 2\mu\Big(\frac{1+m}{2}x\Big) + \mu(mx) \tag{2.12}$$

for all $x \in \mathbb{R}_+$ and $\xi \in \mathbb{R}^{\mathbb{R}_+}$, $\mu \in \mathbb{R}_+^{\mathbb{R}_+}$. Clearly, Λ_m satisfies the condition (2.1) of Theorem 2.2. Now, we will check the condition (2.2) in Theorem 2.2.

For $\xi, \mu \in \mathbb{R}^{\mathbb{R}_+}$ and $t>0$, we have

$$\begin{aligned}
&M\big(\mathcal{T}_m\xi(x), \mathcal{T}_m\mu(x), t\big)\\
&= M\Big(2\xi\Big(\frac{1+m}{2}x\Big) - \xi(mx),\ 2\mu\Big(\frac{1+m}{2}x\Big) - \mu(mx),\ t\Big)\\
&\geq \min\Big\{M\Big(2\xi\Big(\frac{1+m}{2}x\Big) - \xi(mx),\ \mu\Big(\frac{1+m}{2}x\Big) + \xi\Big(\frac{1+m}{2}x\Big) - \xi(mx), t\Big),\\
&\quad M\Big(\mu\Big(\frac{1+m}{2}x\Big) + \xi\Big(\frac{1+m}{2}x\Big) - \xi(mx),\ 2\mu\Big(\frac{1+m}{2}x\Big) - \mu(mx), t\Big)\Big\}\\
&= \min\{M(\xi(f_1(x)), \mu(f_1(x)), t),\ M(\xi(f_2(x)), \mu(f_2(x)), t)\},
\end{aligned}$$

where $f_1(x) = \frac{1+m}{2}x$, $f_2(x) = mx$.

From (2.10), we have

$$M\big(\mathcal{T}_m f(x), f(x), t\big) = M\Big(2f\Big(\frac{1+m}{2}x\Big), f(x)+f(mx), t\Big) \geq N\big((1+s(m))h(x), t\big).$$

Also, note

$$\Lambda_m c_m(x) \leq (1+s(m))\Big[2s\Big(\frac{1+m}{2}\Big) + s(m)\Big]h(x), \quad x \in \mathbb{R}_+.$$

By induction on $k \in \mathbb{N}$, we obtain

$$\Lambda_m^k c_m(x) = (1+s(m))\Big[2s\Big(\frac{1+m}{2}\Big) + s(m)\Big]^k h(x), \quad x \in \mathbb{R}_+.$$

Hence, summing over k gives

$$c_m^*(x) := \sum_{j=0}^{\infty} \Lambda_m^j c_m(x) = \frac{1+s(m)}{1-s(m)-2s\big(\frac{1+m}{2}\big)}h(x), \quad \Lambda_m^0 c_m(x) = c_m(x).$$

By the Brzdęk fixed-point theorem (Theorem 2.2), the limit

$$T_m(x) := \lim_{k\to\infty} \mathcal{T}_m^k f(x)$$

exists for each $m \in L_0$ and $x \in \mathbb{R}_+$, with

$$M(f(x), T_m(x), t) \geq N(c_m^*(x), t), \quad t > 0.$$

Using induction on $k \in \mathbb{N}_0$, we verify

$$M\left(\mathcal{T}_m^k\left(2f\left(\frac{x+y}{2}\right)\right), \mathcal{T}_m^k f(x) + \mathcal{T}_m^k f(y), t\right) \geq N\left(\left[2s\left(\frac{1+m}{2}\right) + s(m)\right]^k (h(x)+h(y)), t\right), \quad x, y \in \mathbb{R}_+.$$

The base case $k = 0$ holds from (2.10). The inductive step uses the non-Archimedean property of M:

$$\begin{aligned}
&M\left(\mathcal{T}_m^{k+1}\left(2f\left(\frac{x+y}{2}\right)\right), \mathcal{T}_m^{k+1} f(x) + \mathcal{T}_m^{k+1} f(y), t\right) \\
&= M\Big(2\mathcal{T}_m^k\left(2f\left(\frac{1+m}{2}\frac{x+y}{2}\right)\right) - 2\mathcal{T}_m^k f\left(m\frac{x+y}{2}\right), \\
&\quad \mathcal{T}_m^k\left(2f\left(\frac{1+m}{2}x\right)\right) - \mathcal{T}_m^k f(mx) + \mathcal{T}_m^k\left(2f\left(\frac{1+m}{2}y\right)\right) - \mathcal{T}_m^k f(my), t\Big) \\
&\geq \min\Big\{M\left(\mathcal{T}_m^k\left(2f\left(\frac{1+m}{2}\frac{x+y}{2}\right)\right), \mathcal{T}_m^k f\left(\frac{1+m}{2}x\right) + \mathcal{T}_m^k f\left(\frac{1+m}{2}y\right), t\right), \\
&\quad M\left(\mathcal{T}_m^k\left(2f\left(m\frac{x+y}{2}\right)\right), \mathcal{T}_m^k f(mx) + \mathcal{T}_m^k f(my), t\right)\Big\} \\
&\geq \min\Big\{N\left(\left[2s\left(\frac{1+m}{2}\right) + s(m)\right]^k \left(h\left(\frac{1+m}{2}x\right) + h\left(\frac{1+m}{2}y\right)\right), t\right), \\
&\quad N\left(\left[2s\left(\frac{1+m}{2}\right) + s(m)\right]^k (h(mx) + h(my)), t\right)\Big\} \\
&\geq \min\left\{N\left(\left[2s\left(\frac{1+m}{2}\right) + s(m)\right]^k (h(x)+h(y)), \frac{t}{s\left(\frac{1+m}{2}\right)}\right),\right. \\
&\quad \left. N\left(\left[2s\left(\frac{1+m}{2}\right) + s(m)\right]^k (h(x)+h(y)), \frac{t}{s(m)}\right)\right\}.
\end{aligned}$$

We note that

$$0 < s(m) < 2s\left(\frac{1+m}{2}\right) + s(m) < 1 \text{ and } 0 < s\left(\frac{1+m}{2}\right) < 2s\left(\frac{1+m}{2}\right) + s(m) < 1.$$

Since $N(x, \cdot)$ is a non-decreasing, we have

$$\begin{aligned}
&M\left(\mathcal{T}_m^{k+1}\left(2f\left(\frac{x+y}{2}\right)\right), \mathcal{T}_m^{k+1} f(x) + \mathcal{T}_m^{k+1} f(y), t\right) \\
&\quad \geq N\left(\left[2s\left(\frac{1+m}{2}\right) + s(m)\right]^k (h(x)+h(y)), \frac{t}{\left[2s\left(\frac{1+m}{2}\right) + s(m)\right]}\right) \\
&\quad = N\left((h(x)+h(y)), \frac{t}{\left[2s\left(\frac{1+m}{2}\right) + s(m)\right]^{k+1}}\right)
\end{aligned}$$

for $t > 0$. Passing to the limit $k \to \infty$, we deduce

$$N\left((h(x)+h(y)), \frac{t}{\left[2s\left(\frac{1+m}{2}\right)+s(m)\right]^{k+1}}\right) \to 1,$$

where $0 < 2s(\frac{1+m}{2}) + s(m) < 1$.

Hence we have

$$2T_m\left(\frac{x+y}{2}\right) = T_m(x) + T_m(y), \quad x, y \in \mathbb{R}_+, \tag{2.13}$$

so T_m satisfies the functional equation (1.1) uniquely.

Assume $T\colon \mathbb{R}_+ \to \mathbb{R}$ satisfies the equation (1.1) and

$$M(f(x), T(x), t) \geq N(Lh(x), t)$$

for $L > 0$ constant.

Let

$$2T\left(\frac{x+y}{2}\right) = T(x) + T(y) \tag{2.14}$$

for all $x,\ y \in \mathbb{R}_+$. Then we will show that $T = T_m$ for each $m \in L_0$. By letting $y = mx$ in the inequality (2.14), we have

$$T(x) = 2T\left(\frac{(1+m)x}{2}\right) - T(mx),$$

for $x \in \mathbb{R}_+$ and $t > 0$. Let $m_0 \in L_0$. Then

$$\begin{aligned} M(T(x), T_{m_0}(x), t) &\geq \min\{M(T(x), f(x), t), M(f(x), T_{m_0}(x), t)\} \\ &\geq \min\left\{N(L\,h(x), t), N\left(\frac{1+s(m_0)}{1-s(m_0)-2s\left(\frac{1+m_0}{2}\right)}h(x), t\right)\right\}. \end{aligned}$$

By letting $S_0 = ((1+s(m_0)) + (1-s(m_0)-2s(\frac{1+m_0}{2}))L)$, we have

$$\frac{1+s(m_0)}{1-s(m_0)-2s\left(\frac{1+m_0}{2}\right)} + L = S_0 \sum_{k=0}^{\infty}\left[s(m_0)+2s\left(\frac{1+m_0}{2}\right)\right]^k.$$

Since $N(s,\cdot)$ is a non-decreasing, we note

$$\begin{aligned} &\min\left\{N(L\,h(x), t), N\left(\frac{1+s(m_0)}{1-s(m_0)-2s\left(\frac{1+m_0}{2}\right)}h(x), t\right)\right\} \\ &\geq N\left(h(x), \frac{1}{\frac{1+s(m_0)}{1-s(1+m_0^2)-s(m_0^2)}+L}t\right) \\ &= N\left(S_0\sum_{k=0}^{\infty}\left[s(m_0)+2s\left(\frac{1+m_0}{2}\right)\right]^k h(x), t\right). \end{aligned}$$

Hence we get

$$M(T(x), T_{m_0}(x), t) \geq N\left(S_0\sum_{k=0}^{\infty}\left[s(m_0)+2s\left(\frac{1+m_0}{2}\right)\right]^k h(x),\ t\right), \tag{2.15}$$

for $x \in \mathbb{R}_+$ and $t > 0$. For $l \in \mathbb{N}_0$, assume that

$$M(T(x), T_{m_0}(x), t) \geq N\left(S_0 \sum_{k=l}^{\infty} \left[s(m_0) + 2s\left(\frac{1+m_0}{2}\right)\right]^k h(x), t \right),$$

for $x \in \mathbb{R}_+$ and $t > 0$. We will check it by using mathematical induction on l. If $l = 0$, it follows from the inequality (2.15). Then

$$\begin{aligned}
&M(T(x), T_{m_0}(x), t) \\
&\quad = M\left(2T\left(\frac{(1+m_0)x}{2}\right) - T(m_0 x),\ 2T_{m_0}\left(\frac{(1+m_0)x}{2}\right) - T_{m_0}(m_0 x), t \right) \\
&\quad \geq \min\left\{ M\left(T\left(\frac{(1+m_0)x}{2}\right), T_{m_0}\left(\frac{(1+m_0)x}{2}\right), t \right), M(T(m_0 x), T_{m_0}(m_0 x), t) \right\} \\
&\quad \geq \min\left\{ N\left(2s\left(\frac{1+m_0}{2}\right) S_0 \sum_{k=l}^{\infty} \left[s(m_0) + 2s\left(\frac{1+m_0}{2}\right)\right]^k h(x), t \right), \right. \\
&\qquad \left. N\left(s(m_0) S_0 \sum_{k=l}^{\infty} \left[s(m_0) + 2s\left(\frac{1+m_0}{2}\right)\right]^k h(x), t \right) \right\} \\
&\quad \geq N\left(S_0 \sum_{k=l+1}^{\infty} \left[s(m_0) + 2s\left(\frac{1+m_0}{2}\right)\right]^k h(x), t \right),
\end{aligned}$$

for $x \in \mathbb{R}_+$ and $t > 0$, where S_0 is a positive constant depending on m_0 and L. Hence it holds whenever $l \in \mathbb{N}_0$. Letting the summation index $l \to \infty$, we obtain

$$N\left(S_0 \sum_{k=l+1}^{\infty} \left[s(m_0) + 2s\left(\frac{1+m_0}{2}\right)\right]^k h(x), t \right) \to 1,$$

implying $T = T_{m_0}$ for all $m_0 \in L_0$. □

Corollary 2.4. *Let $h: \mathbb{R}_+ \to (0, \infty)$ be a mapping such that*

$$\liminf_{n\to\infty} \sup_{x \in \mathbb{R}_+} \frac{h\left(\frac{1+n}{2}x\right) + h(nx)}{h(x)} = 0. \tag{2.16}$$

Suppose that $f: \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$M\left(2f\left(\frac{x+y}{2}\right), f(x) + f(y), t\right) \geq N\big(h(x) + h(y), t\big) \tag{2.17}$$

for all $x, y \in \mathbb{R}_+$ and $t > 0$. Then there exists a unique additive function $T: \mathbb{R}_+ \to \mathbb{R}$ such that

$$M\big(f(x), T(x), t\big) \geq N\big(h(x), t\big) \tag{2.18}$$

for all $x \in \mathbb{R}_+$ and $t > 0$.

Proof. For each $n \in \mathbb{N}$, define

$$a_n = \sup_{x \in \mathbb{R}_+} \frac{h\left(\frac{1+n}{2}x\right) + h(nx)}{h(x)}.$$

By the definition of $s(n)$ as in Theorem 2.3, we observe that

$$s\left(\frac{1+n}{2}\right) = \sup_{x \in \mathbb{R}_+} \frac{h\left(\frac{1+n}{2}x\right)}{h(x)} \leq a_n,$$

$$s(n) = \sup_{x \in \mathbb{R}_+} \frac{h(nx)}{h(x)} \leq a_n.$$

Therefore, we get

$$s\left(\frac{1+n}{2}\right) + s(n) \leq 2a_n. \tag{2.19}$$

By assumption, the sequence $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ such that

$$\lim_{k\to\infty} \sup_{x \in \mathbb{R}_+} \frac{h\left(\frac{1+n_k}{2}x\right) + h(n_k x)}{h(x)} = 0. \tag{2.20}$$

Then from (2.19) and (2.20), we have

$$\lim_{k\to\infty}\left[s\left(\frac{1+n_k}{2}\right) + s(n_k)\right] = 0,$$

which implies

$$\lim_{k\to\infty} s\left(\frac{1+n_k}{2}\right) = 0, \quad \lim_{k\to\infty} s(n_k) = 0.$$

Thus,

$$\lim_{k\to\infty} \frac{1 + s(n_k)}{1 - s(n_k) - 2s\left(\frac{1+n_k}{2}\right)} = 1.$$

Now, letting $s_0 = 1$ as in Theorem 2.3, the inequality (2.18) follows directly from the stability result (2.7). □

3 Fixed point alternative method

We will first present the theorems of the fixed point alternative in a generalized metric space. Subsequently, we will analyze stability using the fixed point alternative method.

Definition 3.1. Let X be a set. A function $d: X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

It is important to note that the principal distinction between a generalized metric and a traditional metric is that the range of a generalized metric can include infinity. We will now present one of the fundamental results in fixed point theory. For the proof, refer to [46].

Theorem 3.2 (The alternative of fixed point [46], [47]). *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T: X \to X$ with Lipschitz constant $0 < L < 1$. Then for each given $x \in X$, either*

$$d(T^n x,\ T^{n+1} x) = \infty \quad \textit{for all } n \geq 0,$$

or there exists a natural number n_0 such that
1. $d(T^n x, T^{n+1} x) < \infty$ *for all* $n \geq n_0$;
2. *The sequence* $\{T^n x\}$ *is convergent to a fixed point* y^* *of* J;
3. y^* *is the unique fixed point of* T *in the set*

$$Y = \{y \in X \mid d(T^{n_0} x, y) < \infty\};$$

4. $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in Y$.

To investigate the stability of the Jensen's functional equation, we define an operator $Df(x, y)$ which measures how much a function f deviates from satisfying the equation. For a mapping $f: \mathbb{R}_+ \to \mathbb{R}$, the operator is defined as follows:

$$Df(x, y) = 2f\left(\frac{x+y}{2}\right) - f(x) - f(y), \quad x, y \in \mathbb{R}_+.$$

The following theorem, which is proved by applying the fixed point alternative method, is the main result of our analysis.

Theorem 3.3. *Let $(\mathbb{R}, N, *)$ be a non-Archimedean fuzzy normed space with the minimum t-norm $*$. Let $\phi: \mathbb{R}_+^2 \to [0, \infty)$ be a function for which there exists a constant L with $0 < L < 1$ such that*

$$N\left(\frac{1}{2}\phi(2x, 2y), t\right) \geq N(L\phi(x, y), t),$$

for all $x, y \in \mathbb{R}_+$ and $t > 0$. Let $f: \mathbb{R}_+ \to \mathbb{R}$ be a function satisfying $f(0) = 0$ and the inequality

$$N(Df(x, y), t) \geq N(\phi(x, y), t) \tag{3.1}$$

for all $x, y \in \mathbb{R}_+$ and $t > 0$. Then there exists a unique Jensen's function $R: \mathbb{R}_+ \to \mathbb{R}$ defined by $R(x) = N-\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ such that

$$N(f(x) - R(x), t) \geq N\left(\frac{L}{1-L}\phi(x, 0), t\right) \tag{3.2}$$

for all $x \in \mathbb{R}_+$.

Proof. Consider the set

$$\Omega = \{g: \mathbb{R}_+ \to \mathbb{R} \mid g(0) = 0\}$$

and define the generalized metric on Ω by

$$d(g, h) = \inf\left\{c \in (0, \infty) \mid N(g(x) - h(x), t) \geq N(c\phi(x, 0), t) \quad \text{for all } x \in \mathbb{R}_+,\ t > 0\right\}.$$

It is easy to show that (Ω, d) is a complete metric space. Now define a function $T: \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{2}g(2x), \quad g \in \Omega,$$

for all $x \in \mathbb{R}_+$.

Let $g, h \in \Omega$ and suppose $d(g, h) \leq c$ for some $c \in (0, \infty)$. Then

$$N(g(x) - h(x), 2t) \geq N(c\phi(x, 0), 2t)$$

for all $x \in \mathbb{R}_+$ and $t > 0$. By replacing x with $2x$, we obtain

$$N\left(\frac{1}{2}g(2x) - \frac{1}{2}h(2x), t\right) \geq N\left(\frac{1}{2}\phi(2x, 0), \frac{t}{c}\right) \geq N(Lc\phi(x, 0), t),$$

where the last inequality follows from the assumption on ϕ and L. Hence,

$$d(Tg, Th) \leq L\, d(g, h),$$

which shows that T is a strictly contractive mapping on Ω with Lipschitz constant L.

Now, by setting $x = 2x$, $y = 0$, and $t = 2t$ in inequality (3.1), we obtain

$$N\left(f(x) - \frac{1}{2}f(2x), t\right) \geq N(L\phi(x, 0), t) = N\left(\phi(x, 0), \frac{t}{L}\right)$$

for all $x \in \mathbb{R}_+$ and $t > 0$, which implies

$$d(Tf, f) \leq L < \infty.$$

By induction, we obtain

$$N\Big(f(x)-\frac{1}{2^n}f(2^nx),t\Big)\geq N\Big(\phi(x,0),\frac{t}{L^n}\Big)$$

for all $x\in\mathbb{R}_+$ and $t>0$. As $n\to\infty$, we have $N\Big(\phi(x,0),\frac{t}{L^n}\Big)\to 1$, since $0<L<1$.

Hence, by the alternative fixed point theorem (Theorem 3.2), there exists a fixed point R of T in Ω such that

$$R(x)=N-\lim_{n\to\infty}\frac{1}{2^n}f(2^nx) \tag{3.3}$$

for all $x\in\mathbb{R}_+$.

Now, let $x=2^nx$, $y=2^ny$, and $t=2^nt$ in equation (3.1). Then,

$$N(Df(2^nx,2^ny),2^nt)\geq N(\phi(2^nx,2^ny),2^nt),$$

which implies

$$N\Big(\frac{1}{2^n}Df(2^nx,2^ny),t\Big)\geq N\Big(\frac{1}{2^n}\phi(2^nx,2^ny),t\Big)\geq N\Big(\phi(x,y),\frac{t}{L^n}\Big)$$

for all $x,y\in\mathbb{R}_+$ and $t>0$.

As $n\to\infty$, we have $N\Big(\phi(x,y),\frac{t}{L^n}\Big)\to 1$, so R satisfies the Jensen's functional equation (1.1). Therefore, R is a solution to the Jensen equation. The uniqueness of R also follows from the fixed point theorem.

Furthermore, we obtain

$$d(f,R)\leq\frac{1}{1-L}\,d(Tf,f),$$

and thus

$$N(f(x)-R(x),t)\geq N\Big(\frac{L}{1-L}\,\phi(x,0),t\Big)$$

for all $x\in\mathbb{R}_+$ and $t>0$. This proves the desired inequality (3.2). □

Corollary 3.4. *Let θ and L be positive real numbers with $0<L<1$, and let $f\colon\mathbb{R}_+\to\mathbb{R}$ be a mapping satisfying $f(0)=0$ and the inequality*

$$N(Df(x,y),t)\geq N\big(\theta(\|x\|+\|y\|),t\big) \tag{3.4}$$

for all $x,y\in\mathbb{R}_+$ and $t>0$. Then there exists a unique solution $R\colon\mathbb{R}_+\to\mathbb{R}$ such that

$$N(f(x)-R(x),t)\geq N\bigg(\frac{L\theta}{1-L}\|x\|,t\bigg) \tag{3.5}$$

for all $x\in\mathbb{R}_+$ and $t>0$.

Proof. Let $\phi(x,y)=\theta(\|x\|+\|y\|)$ for all $x,y\in\mathbb{R}_+$. Then the inequality (3.4) becomes

$$N(Df(x,y),t)\geq N(\phi(x,y),t),$$

which satisfies the condition of Theorem 3.3. Therefore, by applying Theorem 3.3, we obtain

$$N(f(x)-R(x),t)\geq N\Big(\frac{L}{1-L}\phi(x,0),t\Big)=N\bigg(\frac{L\theta}{1-L}\|x\|,t\bigg)$$

for all $x\in\mathbb{R}_+$ and $t>0$. □

Remark 3.5. One of the primary aims of this paper is to assess whether Brzdęk's fixed point method can be effectively applied in various spaces, including non-Archimedean fuzzy normed spaces. The primary results on stability estimates within Brzdęk's fixed point method indicate that this approach necessitates the non-Archimedean fuzzy metric to possess the invariant property. This invariance property is not a requirement in other stability methods. Additionally, it is important to highlight the use of the relation $y = mx$ in the Brzdęk's fixed point method. This linear relationship between the variables x and y makes it possible to prove the results and contributes to a highly effective stability approach. On the other hand, the fixed point alternative method typically necessitates strictly contractive mappings and scaling processes. To investigate the stability problem related to Jensen's functional equation (1.1) using the fixed point alternative method, it is essential that one of the variables, x or y, should be zero. We also propose the following open problems for further investigation: Can Brzdęk's fixed point method be extended to other types of functional euations and normed spaces?

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