Research Article

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Limiting profile of positive solutions to heterogeneous elliptic BVPs with nonlinear flux decaying to negative infinity on a portion of the boundary

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Abstract: This paper ascertains the limiting profile of the positive solutions of heterogeneous logistic elliptic boundary value problems under nonlinear mixed boundary conditions. Specifically, the study considers cases when the nonlinear flux on certain regions of the boundary decays to negative infinity, while vanishing on the complementary regions. The main result establishes that the limiting profile of these solutions is a positive function that satisfies the logistic equation, vanishes on the regions where the nonlinear flux decays to negative infinity, and exhibits zero flux on the complementary boundary pieces. The mathematical analysis carried out in this work employs functional and monotonicity techniques as key tools.

Keywords: positive solutions; heterogeneous logistic elliptic BVPs; nonlinear mixed boundary conditions; glued Dirichlet-Neumann boundary conditions; nonlinear flux on the boundary; spatial heterogeneities

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1 Introduction and main results

This work focuses on analyzing the asymptotic behavior of positive solutions to the following heterogeneous logistic elliptic boundary value problem with nonlinear mixed boundary conditions as $\gamma \uparrow \infty$:

$$\begin{cases} -\Delta u_\gamma = \lambda u_\gamma - a(x)u_\gamma^p & \text{in } \Omega, \quad p > 1, \\ u_\gamma = 0 & \text{on } \Gamma_0, \\ \partial u_\gamma = -\gamma b(x) u_\gamma^q & \text{on } \Gamma_1, \ q > 1. \end{cases} \tag{1.1}$$

The analysis is conducted under the following assumptions:

i) Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$ of class C^2, with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are two disjoint components of $\partial\Omega$ and $\Gamma_1 = \Gamma_1^{\mathfrak{D}} \cup \Gamma_1^{\mathcal{N}}$, being $\Gamma_1^{\mathfrak{D}}$ and $\Gamma_1^{\mathcal{N}}$ two connected pieces, open and closed respectively as $N-1$ dimensional manifolds, such that $\partial\Gamma_1^{\mathcal{D}} = \partial\Gamma_1^{\mathcal{N}} \subset \Gamma_1^{\mathcal{N}}$.

ii) $-\Delta$ stands for the minus Laplacian operator in $\mathbb{R}^N$ and $\lambda \in \mathbb{R}$.

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iii) The potential $a \in C(\bar{\Omega})$, with $a \gneq 0$, measures the spatial heterogeneities in Ω and satisfies that

$$\Omega_0 := \operatorname{int}\{x \in \Omega : a(x) = 0\} \neq \emptyset, \qquad \Omega_0 \in C^2, \tag{1.2}$$

$$\partial\Omega_0 = \Gamma_1 \cup \Gamma_0^0, \qquad \Gamma_0^0 := \partial\Omega_0 \cap \Omega, \qquad \operatorname{dist}\left(\Gamma_0^0, \Gamma_1\right) > 0. \tag{1.3}$$

Set $\Omega^+ := \Omega \backslash \bar{\Omega}_0$.

iv) $\partial u_\gamma = \nabla u_\gamma \circ \bar{n}$, where $\bar{n}$ is the outward normal vector field to $\partial\Omega$.

v) $b \in C(\Gamma_1)$ is a positive potential satisfying

$$\Gamma_1^{\mathcal{N}} = b^{-1}(0) \qquad \text{and} \qquad \Gamma_1^{\mathfrak{D}} = b^{-1}(0, \|b\|_{L_\infty(\Gamma_1)}] \tag{1.4}$$

and $\gamma > 0$.

Figure 1 shows a possible configuration of the domain Ω, its boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1^{\mathfrak{D}} \cup \Gamma_1^{\mathcal{N}}$ and the boundary conditions in each piece of the boundary.

The existence and asymptotic behavior of positive solutions to elliptic boundary value problems with a bifurcation-continuation parameter in the boundary conditions has been extensively studied in previous works, such as [1–4]. In this paper, we analyze the limiting profile of positive solutions to (1.1) as γ tends to infinity. Equation (1.1) models a logistic elliptic boundary value problem with nonlinear mixed boundary conditions, arising in the context of coastal fishery harvesting under spatially heterogeneous conditions (cf. [5]). Additionally, taking into account that the nonnegative solutions of (1.1) correspond to the steady states of positive solutions in the associated parabolic problem, (1.1) plays a key role in population dynamics with spatial heterogeneities. This is particularly relevant in scenarios where, due to the heterogeneous distribution of natural resources, some regions of the habitat boundary exhibit zero population flux, while others experience a nonlinear population flux.

To analyze the limiting behavior of the positive solutions to (1.1) as γ tends to infinity, we focus on the positive weak solutions of the following heterogeneous logistic elliptic boundary value problem, which involves mixed and glued Dirichlet-Neumann boundary conditions:

$$\begin{cases} -\Delta u = \lambda u - a(x)u^p & \text{in } \Omega, \quad p > 1, \\ u = 0 & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma_1^{\mathfrak{D}}, \\ \partial u = 0 & \text{on } \Gamma_1^{\mathcal{N}}. \end{cases} \tag{1.5}$$

These weak solutions will play a crucial role in our analysis.

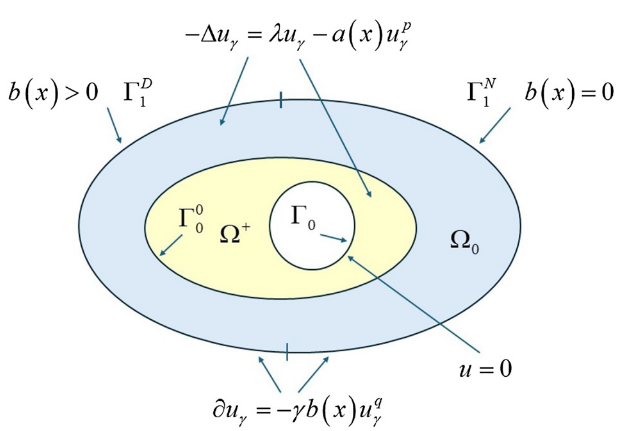

Figure 1: Configuration of Ω and $\partial\Omega = \Gamma_0 \cup \Gamma_1^{\mathfrak{D}} \cup \Gamma_1^{\mathcal{N}}$.

The main result of this work (Theorem 1.1) states that if the parameter λ belongs to a suitable interval, to specify later, the limiting behavior of the positive solutions to (1.1) in $H^1(\Omega)$ as γ tends to infinity coincides with the unique positive weak solution of (1.5).

Before stating our main findings, we introduce some notations and previous results. Let us denote

$$W^2(\Omega) := \bigcap_{p>1} W^2_p(\Omega),$$

$$C^\infty_{\Gamma_0\cup\Gamma_1^{\mathfrak{D}}}(\Omega) := \left\{ \phi : \bar\Omega \to \mathbb{R} : \phi \in C^\infty(\Omega)\cap C(\bar\Omega) \wedge supp\,\phi \subset \bar\Omega \backslash \left(\Gamma_0 \cup \Gamma_1^{\mathfrak{D}}\right)\right\},$$

and let $H^1_*(\Omega)$ be the closure in $H^1(\Omega)$ of the set of functions $C^\infty_{\Gamma_0\cup\Gamma_1^{\mathcal{D}}}(\Omega)$, that is

$$H^1_*(\Omega) = \overline{C^\infty_{\Gamma_0\cup\Gamma_1^{\mathcal{D}}}(\Omega)}^{H^1(\Omega)}.$$

By construction if $u \in H^1_*(\Omega)$, then $u = 0$ on $\Gamma_0 \cup \Gamma_1^{\mathfrak{D}}$.

By a *positive weak solution* of (1.5) we mean any function $\varphi \in H^1_*(\Omega)$ satisfying

$$\varphi > 0, \qquad \int_{\Omega^+} a(x)\varphi^{p+1} < \infty,$$

and such that for each $\xi \in C^\infty_{\Gamma_0\cup\Gamma_1^{\mathfrak{D}}}(\Omega)$, or $\xi \in H^1_*(\Omega)$, the following holds

$$\int_\Omega \nabla\varphi\,\nabla\xi + \int_\Omega a(x)\varphi^p\xi = \lambda\int_\Omega \varphi\xi.$$

In particular, taking $\xi = \varphi \in H^1_*(\Omega)$ we have that

$$\int_\Omega |\nabla\varphi|^2 + \int_\Omega a(x)\varphi^{p+1} = \lambda\int_\Omega \varphi^2.$$

Hereafter we denote $\mathfrak{B}^{\mathcal{N}}$, $\mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)$ and $\mathfrak{B}^*_0\left(\Gamma_1^{\mathcal{N}}\right)$ the boundary operators defined by

$$\mathfrak{B}^{\mathcal{N}} u := \begin{cases} u & \text{on } \Gamma_0, \\ \partial u & \text{on } \Gamma_1, \end{cases} \qquad \mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)u := \begin{cases} u & \text{on } \Gamma_0, \\ \partial u & \text{on } \Gamma_1^{\mathcal{N}}, \\ u & \text{on } \Gamma_1^{\mathfrak{D}}, \end{cases} \qquad \mathfrak{B}^*_0\left(\Gamma_1^{\mathcal{N}}\right)u := \begin{cases} u & \text{on } \Gamma_0^0, \\ \partial u & \text{on } \Gamma_1^{\mathcal{N}}, \\ u & \text{on } \Gamma_1^{\mathfrak{D}}, \end{cases}$$

and by $\mathfrak{D}$ the Dirichlet boundary operator on $\partial\Omega$.

In the sequel we will say that a function $u \in W^2_p(\Omega)$, $p > N$ is *strongly positive* in Ω, and we will denote it by $u \gg 0$, if $u(x) > 0$ for each $x \in \Omega \cup \Gamma_1$ and $\partial u(x) < 0$ for each $x \in \Gamma_0$ such that $u(x) = 0$.

Let us consider the eigenvalue problem

$$\begin{cases} -\Delta\varphi = \sigma\varphi \text{ in } \Omega, \\ \mathfrak{B}^{\mathcal{N}}\varphi = \bar 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

By a *principal eigenvalue* of (1.6) we mean any eigenvalue of it which possesses a one-signed eigenfunction and in particular a positive eigenfunction. Owing to the results in [6, Theorem 12.1] it is known that (1.6) possesses a unique principal eigenvalue, denoted in the sequel by $\sigma_1^\Omega[\mathfrak{B}^{\mathcal{N}}]$, which is simple and the least eigenvalue of (1.6). Moreover, the positive eigenfunction $\varphi_1^{\mathcal{N}}$ associated to it, unique up to a multiplicative constant, satisfies

$$\varphi_1^{\mathcal{N}} \gg 0 \quad \text{in} \quad \Omega,$$

and in addition

$$\varphi_1^{\mathcal{N}} \in W^2(\Omega) \subset C^{1+\alpha}(\bar\Omega) \qquad \text{for all} \quad \alpha \in (0,1).$$

Also, hereafter we denote $\sigma_1^{\Omega}[\mathfrak{D}]$ the principal eigenvalue of $-\Delta$ in Ω subject to homogeneous Dirichlet boundary conditions.

A function $\varphi \in W_p^2(\Omega)$, $p > N$ is said to be a positive strict supersolution of the problem $(-\Delta, \Omega, \mathfrak{B}^{\mathcal{N}})$ if $\varphi > 0$ in Ω and the following holds

$$\begin{cases} -\Delta\varphi \geq 0 \text{ in } \Omega, \\ \mathfrak{B}^{\mathcal{N}}\varphi \geq 0 \text{ on } \partial\Omega, \end{cases}$$

with some of the inequalities strict.

Now, let us consider the eigenvalue problem with mixed and glued Dirichlet-Neumann boundary conditions on Γ_1 given by

$$\begin{cases} -\Delta\varphi = \mu\varphi \qquad \text{in } \Omega, \\ \mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)\varphi = \bar{0} \text{ on } \partial\Omega. \end{cases} \tag{1.7}$$

A function φ is said to be a weak solution of (1.7) if $\varphi \in H_*^1(\Omega)$ and for each $\xi \in H_*^1(\Omega)$ the following holds

$$\int_\Omega \nabla\varphi\,\nabla\xi = \mu \int_\Omega \varphi\,\xi.$$

The value μ is an eigenvalue of (1.7), if there exists a weak solution $\varphi \neq 0$ of (1.7) associated to μ. In that case, it is said that φ is a weak eigenfunction of (1.7) associated to the eigenvalue μ. By a *principal eigenvalue* of (1.7) we mean any eigenvalue of it which possesses a one-signed eigenfunction and in particular a positive eigenfunction.

Owing to the results in [7, Theorem 1.1] it is known that (1.7) possesses a unique principal eigenvalue, denoted in the sequel by $\sigma_1^{\Omega}\left[\mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)\right]$, which is simple and the smallest eigenvalue of all eigenvalues of (1.7). Moreover, the positive eigenfunction φ^* associated to it, unique up to a multiplicative constant, satisfies that $\varphi^* \in H_*^1(\Omega)$ and

$$\varphi^*(x) > 0 \qquad \text{a.e. in} \quad \Omega.$$

Moreover, $\sigma_1^{\Omega}\left[\mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)\right]$ comes characterized by

$$\sigma_1^{\Omega}\left[\mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)\right] = \inf_{\varphi \in H_*^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2} = \frac{\int_\Omega |\nabla\varphi^*|^2}{\int_\Omega (\varphi^*)^2} \tag{1.8}$$

(cf. [7, (2.27)]). In the same way, substituting in (1.7) Ω by Ω_0 and $\mathfrak{B}^*\left(\Gamma_1^{N}\right)$ by $\mathfrak{B}_0^*\left(\Gamma_1^{N}\right)$, owing to [7, Theorem 1.1] we obtain the following variational characterization for $\sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right]$

$$\sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right] = \inf_{\varphi \in H_*^1(\Omega_0)\setminus\{0\}} \frac{\int_{\Omega_0} |\nabla\varphi|^2}{\int_{\Omega_0} \varphi^2} = \frac{\int_{\Omega_0} |\nabla\varphi_0^*|^2}{\int_{\Omega_0} \left(\varphi_0^*\right)^2}, \tag{1.9}$$

where φ_0^* stands for the positive principal eigenfunction associated to the principal eigenvalue $\sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right]$, unique up to a multiplicative constant. Taking into account the variational characterizations (1.8) and (1.9), it is clear that

$$\sigma_1^{\Omega}\left[\mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)\right] < \sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right].$$

Moreover, owing to [7, Corollary 3.5] and [8, Proposition 3.2] it is known that

$$\sigma_1^{\Omega}[\mathfrak{B}^{\mathcal{N}}] < \sigma_1^{\Omega}\left[\mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)\right] < \sigma_1^{\Omega}[\mathfrak{D}] < \sigma_1^{\Omega_0}[\mathfrak{D}], \tag{1.10}$$

and

$$\sigma_1^{\Omega}[\mathfrak{B}^{\mathcal{N}}] < \sigma_1^{\Omega_0}[\mathfrak{B}^{\mathcal{N}}] < \sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right] < \sigma_1^{\Omega_0}[\mathfrak{D}], \tag{1.11}$$

but no clear monotonicity relationship exists between $\sigma_1^{\Omega}[\mathfrak{D}]$ and $\sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right]$, since the relative position of both depends on the sizes of Ω_0 with respect to Ω and of $\Gamma_1^{\mathcal{N}}$ with respect to Γ_1.

The problem of ascertaining the limiting profile of the positive solutions of (1.1) when γ tends to infinity was already analyzed in [3], in the particular case when the potential b is a positive potential bounded away from zero on Γ_1, that is, $b(x) \geq \underline{b} > 0$ on Γ_1 and in addition, either $\Omega = \Omega_0$, i.e. $a = 0$ in Ω, or $\bar{\Omega}_0 \subset \Omega$ (Theorem 1.1 and Theorem 1.2-ii) therein, respectively). In both cases it was proved (adapting the notation therein to our framework) that if $\lambda \in \left(\sigma_1^{\Omega}[\mathfrak{B}^{\mathcal{N}}], \sigma_1^{\Omega}[\mathfrak{D}]\right)$, then

$$\lim_{\gamma\uparrow\infty} \| u_\gamma \|_{L_\infty} = 0. \tag{1.12}$$

Owing to the fact that under assumptions of [3, Th.1.1 and Th.1.2] for each fixed $\lambda \in \left(\sigma_1^{\Omega}[\mathfrak{B}^{\mathcal{N}}], \sigma_1^{\Omega}[\mathfrak{D}]\right)$ and for each $\gamma > 0$ there exists a unique positive solution u_γ of (1.1), considering γ as the bifurcation parameter, we conclude from (1.12) that (1.1) exhibits bifurcation from the trivial branch $(\gamma, u) = (\gamma, 0)$ when γ tends to infinity. In this work we extend the previous analysis about the limiting profile of the positive solutions of (1.1) when γ tends to infinity, to cover the more complicated case when $\Omega_0 \subset \Omega$ satisfying (1.2) and (1.3), and in addition, either the potential b vanishes on some regions of Γ_1 (cf. Theorem 1.1), or b is bounded away from zero on Γ_1 (cf. Theorem 1.2).

The following is the main result of this work

Theorem 1.1. *Under the general assumptions* (1.2), (1.3) *and* (1.4), *assume in addition that*

$$\sigma_1^{\Omega}[\mathfrak{D}] < \sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right] \tag{1.13}$$

and

$$\sigma_1^{\Omega}[\mathfrak{D}] < \lambda < \sigma_1^{\Omega_0}\left[\mathcal{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right]. \tag{1.14}$$

Then,

$$\mathit{lim}_{\gamma\uparrow\infty} \| u_\gamma - u^* \|_{H^1(\Omega)} = 0, \tag{1.15}$$

where u_γ and u^* stand for the unique positive solution of (1.1) *and* (1.5), *respectively.*

Figure 2 shows the behavior of the limiting profile u^* of the positive solution u_γ of (1.1) when γ tends to infinity versus the profile of the potential $b(x)$ on Γ_1.

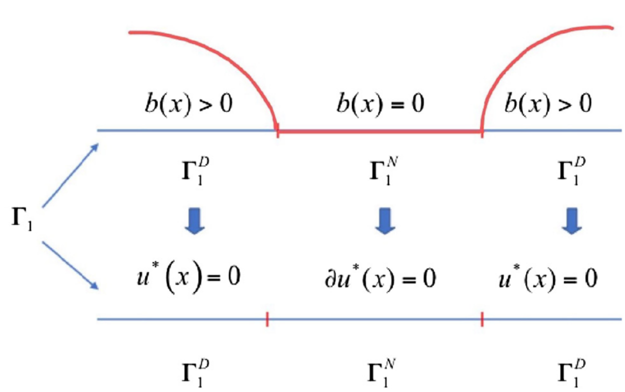

Figure 2: Behavior of u^* on Γ_1 versus profile of $b(x)$

Now, Theorem 1.1 asserts that if (1.2), (1.3) and (1.4) hold, then, the contrary to the cases analyzed in [3], the bifurcation of (1.1) to positive solutions from the trivial branch $(\gamma, u) = (\gamma, 0)$ when γ tends to infinity fails, since (1.15) holds.

Remark 1.1. Owing to (1.10) and (1.11) we have that

$$\sigma_1^{\Omega}[\mathfrak{D}] < \sigma_1^{\Omega_0}[\mathfrak{D}] \qquad \text{and} \qquad \sigma_1^{\Omega_0}[\mathfrak{B}^{\mathcal{N}}] < \sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right] < \sigma_1^{\Omega_0}[\mathfrak{D}].$$

Moreover, it is known that when $\Gamma_1^{\mathcal{N}}$ is very small versus $\Gamma_1^{\mathcal{D}}$, that is, when $\Gamma_1^{\mathcal{D}}$ is almost Γ_1, then, $\sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right]$ approaches to $\sigma_1^{\Omega_0}[\mathfrak{D}]$. Then, in this situation, condition (1.13) will be satisfied.

In the particular case when the potential $b \in \mathcal{C}(\Gamma_1)$ is positive and bounded away from zero on Γ_1, that is, when $\Gamma_1^{\mathfrak{D}} = \Gamma_1$ and $\Gamma_1^{\mathcal{N}} = \emptyset$, the following result holds, which is the second main result of this paper.

Theorem 1.2. *Under the general conditions* (1.2) *and* (1.3), *assume in addition that* $b \in \mathcal{C}(\Gamma_1)$ *is positive and bounded away from zero on* Γ_1 *and*

$$\sigma_1^{\Omega}[\mathfrak{D}] < \lambda < \sigma_1^{\Omega_0}[\mathfrak{D}]. \tag{1.16}$$

Then,

$$\lim_{\gamma\uparrow\infty} \| u_\gamma - u_{\mathfrak{D}} \|_{H^1(\Omega)} = 0, \tag{1.17}$$

where u_γ *stands for the unique positive solution of* (1.1) *and* $u_{\mathfrak{D}}$ *denotes the unique positive solution of the problem*

$$\begin{cases} -\Delta u = \lambda u - a(x)u^p \text{ in } \Omega, \quad p > 1, \\ u = 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{1.18}$$

Then, the results obtained in [3, Th.1.1, Th.1.2] together with Theorem 1.1 and Theorem 1.2 show that the profile of the positive potential b on the boundary condition plays a crucial role in the shape of the limiting profile u^* of the positive solutions of (1.1) when γ tends to infinity.

The main technical tools used to carry out the mathematical analysis of this work are functional and monotonicity techniques.

The structure of this paper is as follows. Section 2 collects some previous results that are going to be used throughout this work, and Section 3 contains the proofs of Theorem 1.1 and Theorem 1.2.

2 Preliminaries, notations and previous results

Let us denote by Λ_γ, Λ^* and $\Lambda_{\mathfrak{D}}$ the range of values of the parameter λ for which (1.1), (1.5) and (1.18) possess positive solution, respectively. It is known that

$$\Lambda_{\mathfrak{D}} = \left(\sigma_1^{\Omega}[\mathfrak{D}], \sigma_1^{\Omega_0}[\mathfrak{D}]\right), \tag{2.1}$$

and for each $\lambda \in \Lambda_{\mathfrak{D}}$ the positive solution of (1.18) is unique and strongly positive in Ω (cf. [9, Lemma 3.1, Theorem 3.5]).

Let $b \in \mathcal{C}(\Gamma_1)$ be the positive continuous potential appearing on the boundary conditions of (1.1) satisfying (1.4) and $\gamma > 0$. Then, by construction we have that

$$\gamma b \in \mathcal{C}(\Gamma_1), \qquad \gamma b \gneq 0, \tag{2.2}$$

and

$$\Gamma_1^{\mathcal{N}} = (\gamma b)^{-1}(0) = b^{-1}(0), \qquad \Gamma_1^{\mathcal{D}} = (\gamma b)^{-1}(0, \|\gamma b\|_{L_\infty(\Gamma_1)}] = b^{-1}(0, \|b\|_{L_\infty(\Gamma_1)}]. \tag{2.3}$$

Owing to (2.2) and (2.3) next result follows from [10, Theorem 1.1-i)].

Proposition 2.1. *For each* $\gamma > 0$, (1.1) *possesses a positive solution if, and only if*

$$\sigma_1^{\Omega}[\mathcal{B}^{\mathcal{N}}] < \lambda < \sigma_1^{\Omega_0}\left[\mathcal{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right], \tag{2.4}$$

that is

$$\Lambda_\gamma = \left(\sigma_1^{\Omega}[\mathcal{B}^{\mathcal{N}}], \sigma_1^{\Omega_0}\left[\mathcal{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right]\right). \tag{2.5}$$

Moreover, for each $\lambda \in \Lambda_\gamma$, *the positive solution of* (1.1) *is unique and strongly positive in* Ω. *In the sequel we will denote it by* u_γ. *Furthermore,*

$$u_\gamma \in W^2(\Omega) \subset \mathcal{C}^{1+\alpha}(\bar{\Omega}) \quad \forall\, \alpha \in (0,1).$$

Next result provides us with a comparison method and it is proved following similar arguments to those used in the proof of [11, Proposition 3.2].

Proposition 2.2. *Assume* (2.4) *and let* $\Theta_\lambda \in W_p^2(\Omega)$, $p > N$ *be a positive strict supersolution (subsolution) of* (1.1). *Then,*

$$\Theta_\lambda > u_\gamma \;\; (\Theta_\lambda < u_\gamma).$$

As for the existence and uniqueness of positive solution of (1.5), next result follows adapting to our framework the arguments given in [12, Theorem 3].

Proposition 2.3. *Problem* (1.5) *admits a positive weak solution* $u^* \in H_*^1(\Omega) \cap L_\infty(\Omega)$ *if, and only if*

$$\sigma_1^{\Omega}\left[\mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)\right] < \lambda < \sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right],$$

that is,

$$\Lambda_* = \left(\sigma_1^{\Omega}\left[\mathfrak{B}^*\left(\Gamma_1^{\mathcal{N}}\right)\right], \sigma_1^{\Omega_0}\left[\mathfrak{B}_0^*\left(\Gamma_1^{\mathcal{N}}\right)\right]\right). \tag{2.6}$$

In this case, the solution u^* *is unique.*

3 Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1: Pick λ satisfying (1.14). Since any positive constant is a positive strict supersolution of the problem $(-\Delta, \Omega, \mathcal{D})$, it follows from the Characterization of the strong maximum principle [13, Theorem 2.5] that $\sigma_1^{\Omega}[\mathfrak{D}] > 0$ and hence, (1.14) implies

$$0 < \sigma_1^{\Omega}[\mathfrak{D}] < \lambda.$$

To prove the result we will show that (1.15) holds for every sequence of real numbers $\{\gamma_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \gamma_n = \infty. \tag{3.1}$$

Subsequently, we fix a sequence satisfying (3.1) and set

$$u_n := u_{\gamma_n}, \qquad \Lambda_n := \Lambda_{\gamma_n}, \qquad n \geq 1.$$

Since (3.1) holds, we can assume without loss of generality that

$$\gamma_n > \gamma_1 > 0, \qquad n > 1. \tag{3.2}$$

Also, due to (1.10), (1.11), (1.13), (1.14), (2.5) and (2.6), we have that

$$\lambda \in \left(\sigma_1^{\Omega}[\mathcal{D}], \sigma_1^{\Omega_0}\left[\mathcal{B}^*\left(\Gamma_1^{\mathcal{N}}\right)\right]\right) = \Lambda_n \cap \Lambda_* \cap \Lambda_{\mathcal{D}}, \qquad n \geq 1. \tag{3.3}$$

Then, it follows from Proposition 2.1 and (2.1), the existence for each $n \geq 1$ of a unique positive solution of (1.1) and (1.18), u_n and $u_{\mathfrak{D}}$, respectively, which are strongly positive in Ω and

$$u_n,\ u_{\mathfrak{D}} \in W^2(\Omega) \subset H^2(\Omega).$$

In particular,

$$\partial u_{\mathcal{D}} < 0 \qquad \text{on } \partial\Omega. \tag{3.4}$$

Also, owing to (3.3), it follows from Proposition 2.3 the existence of a unique positive weak solution $u^* \in H^1_*(\Omega) \cap L_\infty(\Omega)$ of (1.5).

Moreover, thanks to (3.4) it is clear that the function $u_{\mathfrak{D}}$ is a positive strict subsolution of (1.1) for each $n \geq 1$, and therefore, it follows from Proposition 2.2 that

$$u_{\mathcal{D}} < u_n, \qquad n \geq 1. \tag{3.5}$$

Also, thanks to (3.2), it is easy to prove that u_1 is a positive strict supersolution of (1.1) for each $n \geq 2$ and hence, it follows from Proposition 2.2 that

$$u_n < u_1, \qquad n \geq 2. \tag{3.6}$$

Thus, (3.5) and (3.6) imply that

$$0 < u_{\mathcal{D}} < u_n \leq u_1, \qquad n \geq 1. \tag{3.7}$$

On the other hand, multiplying (1.1) by u_n and integrating by parts it becomes apparent that

$$0 < \int_\Omega |\nabla u_n|^2 = \lambda \int_\Omega u_n^2 - \int_\Omega a(x) u_n^{p+1} - \gamma_n \int_{\Gamma_1^{\mathfrak{D}}} b\, u_n^{q+1}, \tag{3.8}$$

and hence, since u_n is strongly positive in Ω, $a \gneq 0$, $b \gneq 0$ and $\gamma_n > 0$, it follows from (3.7) and (3.8) that

$$0 < \int_\Omega |\nabla u_n|^2 < \lambda \int_\Omega u_n^2 \leq \lambda \int_\Omega u_1^2. \tag{3.9}$$

Now, owing to the fact that $u_1 \in W^2(\Omega) \subset L_\infty(\Omega)$, it follows from (3.7) and (3.9) the existence of a constant $M > 0$ such that

$$\|u_n\|_{H^1(\Omega)} \leq M, \quad n \geq 1. \tag{3.10}$$

Moreover, owing to (3.7) and (3.10), it is apparent that along some subsequence, again labeled by n,

$$0 < L := \lim_{n\to\infty} \| u_n \|_{H^1(\Omega)}. \tag{3.11}$$

In the sequel we will restrict ourselves to dealing with functions of this subsequence.

Owing to (3.7) and (3.8) we have that

$$\gamma_n \int_{\Gamma_1^{\mathfrak{D}}} b\, u_n^{q+1} < \lambda \int_\Omega u_n^2 \leq \lambda \int_\Omega u_1^2, \qquad n \geq 1. \tag{3.12}$$

Thus, since $u_1 \in L_\infty(\Omega)$, it follows from (3.12) that there exists a constant $C > 0$ such that

$$0 < \gamma_n \int_{\Gamma_1^{\mathfrak{D}}} b\, u_n^{q+1} \leq C, \qquad n \geq 1, \tag{3.13}$$

and hence, (3.13) and (3.1) imply that along some subsequence, again labeled by n,

$$\lim_{n\to\infty}\int_{\Gamma_1^{\mathfrak{D}}} bu_n^{q+1} = 0. \tag{3.14}$$

In particular, since $b(x) > 0$ for all $x \in \Gamma_1^{\mathfrak{D}}$, it follows from (3.14) that

$$\lim_{n\to\infty} u_n(x) = 0 \qquad \text{a.e.} \quad x \in \Gamma_1^{\mathfrak{D}}.$$

On the other hand, since the injection operator $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact, it follows from (3.10) the existence of $u \in L_2(\Omega)$ and a subsequence of u_n, $n \geq 1$, again labeled by n, such that

$$\lim_{n\to\infty} \| u_n - u\|_{L_2(\Omega)} = 0. \tag{3.15}$$

To complete the rest of the proof it suffices to prove that (3.11) and (3.15) imply that $u = u^*$ and

$$\lim_{n\to\infty} \| u_n - u\|_{H^1(\Omega)} = 0,$$

since this argument can be repeated along any subsequence of the original sequence. To prove it, set

$$v_n := \frac{u_n}{\|u_n\|_{H^1(\Omega)}}, \qquad n \geq 1.$$

By construction,

$$\|v_n\|_{H^1(\Omega)} = 1, \qquad n \geq 1, \tag{3.16}$$

and owing to (3.7) the following holds

$$\|v_n\|_{L_\infty(\Omega)} = \|u_n\|_{L_\infty(\Omega)}\|u_n\|^{-1}_{H^1(\Omega)} \leq \|u_1\|_{L_\infty(\Omega)}\|u_n\|^{-1}_{L_2(\Omega)} \leq \|u_1\|_{L_\infty(\Omega)}\|u_D\|^{-1}_{L_2(\Omega)} := \tilde{M}. \tag{3.17}$$

Also, owing to (3.16), it follows from the continuity of the trace operator on Γ_1, $t_1 \in \mathcal{L}(H^1(\Omega), W_2^{\frac{1}{2}}(\Gamma_1))$ and of the injection operator $j\colon W_2^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L_2(\Gamma_1)$, the existence of a constant $C_1 > 0$ such that

$$\|v_n|_{\Gamma_1}\|_{W_2^{\frac{1}{2}}(\Gamma_1)} \leq C_1, \qquad \|v_n|_{\Gamma_1}\|_{L_2(\Gamma_1)} \leq C_1, \qquad n \geq 1. \tag{3.18}$$

Now, since by construction v_n provides us with a positive solution of the problem

$$\begin{cases} -\Delta v_n = \lambda v_n - a(x)u_n^{p-1}v_n & \text{in } \Omega, \\ v_n = 0 & \text{on } \Gamma_0, \\ \partial v_n = -\gamma_n b\, u_n^{q-1} v_n & \text{on } \Gamma_1, \end{cases} \tag{3.19}$$

(3.7), (3.16) and (3.19) imply that

$$\| -\Delta v_n\|_{L_2(\Omega)} = \|\lambda v_n - a(x)u_n^{p-1}v_n\|_{L_2(\Omega)} \leq C_2\|v_n\|_{L_2(\Omega)} \leq C_2, \tag{3.20}$$

for

$$C_2 := \lambda + \|a\|_{L_\infty(\Omega)}\|u_1\|^{p-1}_{L_\infty(\Omega)}.$$

Then, owing to (3.18) and (3.20), it follows from the L_p-elliptic estimates of Agmon, Douglis and Nirenberg [14] the existence of a constant $C_3 > 0$ such that

$$\|v_n\|_{H^2(\Omega)} \leq C_3, \qquad n \geq 1. \tag{3.21}$$

Moreover, taking into account the continuity of the trace operator on Γ_1, $t_1 \in \mathcal{L}(H^1(\Omega), W_2^{\frac{1}{2}}(\Gamma_1))$ and of the injection operator $j\colon W_2^{\frac{1}{2}}(\Gamma_1) \hookrightarrow L_2(\Gamma_1)$, it follows from (3.21) the existence of a constant $C_4 > 0$ such that

$$\|\nabla v_n|_{\Gamma_1}\|_{L_2(\Gamma_1)} \leq C_4, \qquad n \geq 1. \tag{3.22}$$

Since $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$, it follows from (3.16) the existence of $v \in L_2(\Omega)$ and a subsequence of v_n, again labeled by n, such that

$$\lim_{n\to\infty} \| v_n - v\|_{L_2(\Omega)} = 0. \tag{3.23}$$

In particular,

$$\lim_{n\to\infty} v_n(x) = v(x) \qquad \text{a.e.} \quad x \in \Omega,$$

and since $v_n > 0$, $n \geq 1$, we obtain that

$$v \geq 0 \qquad \text{in } \Omega.$$

In addition, due to the compactness of the injection operator from $W_2^{\frac{1}{2}}(\Gamma_1)$ to $L_2(\Gamma_1)$, it follows from (3.18) the existence of $\tilde{v} \in L_2(\Gamma_1)$ and a subsequence of v_n, again labeled by n, such that

$$\lim_{n\to\infty} \| v_n - \tilde{v}\|_{L_2(\Gamma_1)} = 0. \tag{3.24}$$

Now, let K be any compact subset of $\Gamma_1^{\mathfrak{D}}$. Since $b(x) > 0$ for all $x \in \Gamma_1^{\mathfrak{D}}$ and $b \in C(\Gamma_1)$, set

$$b_K := \min_{x\in K} \{ b(x)\} > 0.$$

Then, owing to (3.19) the following holds on K

$$(\partial v_n(x))^2 = \gamma_n^2 b^2(x) v_n^2(x) u_n^{2(q-1)}(x) \geq \gamma_n^2 b_K^2 u_n^{2q}(x) \|u_n\|_{H^1(\Omega)}^{-2}. \tag{3.25}$$

Also, since (3.11) holds, there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n\|_{H^1(\Omega)} \leq 2L, \qquad n \geq n_0. \tag{3.26}$$

Now, (3.25) and (3.26) imply that

$$(\partial v_n(x))^2 \geq \left(\frac{\gamma_n b_K u_n^q(x)}{2L} \right)^2 \qquad \text{for each} \quad x \in K, \quad n \geq n_0,$$

and hence,

$$u_n^{2q}(x) \leq \left(\frac{2L\, \partial v_n(x)}{\gamma_n\, b_K} \right)^2 \qquad \text{for each} \quad x \in K, \quad n \geq n_0. \tag{3.27}$$

Then, (3.27) and (3.22) imply that for $n \geq n_0$ the following holds

$$\|u_n\|_{L_{2q}(K)}^{2q} \leq \left(\frac{2L}{\gamma_n b_K} \right)^2 \int_K (\partial v_n(x))^2 \leq \left(\frac{2L}{\gamma_n b_K} \right)^2 \|\nabla v_n\|_{L_2(\Gamma_1)}^2 \leq \left(\frac{2LC_4}{\gamma_n b_K} \right)^2. \tag{3.28}$$

Now, owing to (3.1), letting $n \to \infty$ in (3.28) gives

$$\lim_{n\to\infty} \| u_n\|_{L_{2q}(K)} = 0,$$

and since $L_{2q}(K) \subset L_2(K)$ we have that

$$\lim_{n\to\infty} \| u_n\|_{L_2(K)} = 0. \tag{3.29}$$

Thus, (3.11) and (3.29) imply that

$$\lim_{n\to\infty} \| v_n\|_{L_2(K)} = 0 \tag{3.30}$$

in any compact subset $K \subset \Gamma_1^{\mathfrak{D}}$. In particular,

$$\lim_{n\to\infty} v_n(x) = 0 \qquad \text{a.e.} \quad x \in K \subset \Gamma_1^{\mathfrak{D}}.$$

Now we are going to prove that since (3.30) holds in any compact subset $K \subset \Gamma_1^{\mathfrak{D}}$ and (3.17) holds, then

$$\lim_{n\to\infty} \| v_n\|_{L_2\left(\Gamma_1^{\mathfrak{D}}\right)} = 0. \tag{3.31}$$

In particular

$$\lim_{n\to\infty} v_n(x) = 0 \quad \text{a.e. } x \in \Gamma_1^{\mathfrak{D}}.$$

Indeed, given any $\varepsilon > 0$, take a compact subet K contained in $\Gamma_1^{\mathcal{D}}$ such that

$$|\Gamma_1^{\mathcal{D}} \backslash K|_{\text{Ł}_2(\Gamma_1)} < \frac{\varepsilon^2}{2\tilde{M}^2}, \tag{3.32}$$

where $|\cdot|$ stands for the Lebesgue measure in $L_2(\Gamma_1)$ and $\tilde{M}$ the constant defined in (3.17). On the other hand, taking into account that (3.30) holds, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that

$$\|v_n\|_{L_2(K)} < \frac{\varepsilon}{\sqrt{2}}, \qquad \forall\, n > n_0 := n_0(\varepsilon). \tag{3.33}$$

Now, owing to (3.17), (3.32) and (3.33), it is apparent that

$$\|v_n\|^2_{L_2(\Gamma_1^{\mathcal{D}})} = \int\limits_{L_2(\Gamma_1^{\mathcal{D}})} v_n^2 = \int\limits_K v_n^2 + \int\limits_{\Gamma_1^{\mathcal{D}}\backslash K} v_n^2 < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2, \qquad n > n_0.$$

Thus, for any $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $\|v_n\|_{L_2(\Gamma_1^{\mathcal{D}})} < \varepsilon$ for any $n > n_0$, which concludes the proof of (3.31).

Then, since by construction $v_n|_{\Gamma_0} = 0$, $n \geq 1$, it follows from (3.31) that

$$\lim_{n\to\infty} \| v_n\|_{L_2\left(\Gamma_0 \cup \Gamma_1^{\mathfrak{D}}\right)} = 0.$$

We now show that v_n is a Cauchy sequence in $H^1(\Omega)$. Indeed, since (3.19) holds, it is apparent that

$$\begin{cases} -\Delta(v_m - v_k) = \lambda(v_m - v_k) - a(x)\left(v_m u_m^{p-1} - v_k u_k^{p-1}\right) & \text{in } \Omega, \\ v_m - v_k = 0 & \text{on } \Gamma_0, \\ \partial(v_m - v_k) = -b\left(\gamma_m u_m^{q-1} v_m - \gamma_k u_k^{q-1} v_k\right) & \text{on } \Gamma_1. \end{cases} \tag{3.34}$$

Then, multiplying the partial differential equation of (3.34) by $v_m - v_k$ and integrating by parts gives

$$\begin{aligned} \int\limits_\Omega |\nabla(v_m - v_k)|^2 = \lambda \int\limits_\Omega (v_m - v_k)^2 - \int\limits_\Omega a(x) v_m u_m^{p-1}(v_m - v_k) \\ + \int\limits_\Omega a(x) v_k u_k^{p-1}(v_m - v_k) + \int\limits_{\Gamma_1^{\mathfrak{D}}} \partial(v_m - v_k)(v_m - v_k). \end{aligned} \tag{3.35}$$

Now, thanks to (3.6), (3.16), (3.22) and applying the Holder's inequality, the following estimates hold:

$$\left| \int\limits_\Omega a(x) v_m u_m^{p-1}(v_m - v_k) \right| \leq \|a\|_{L_\infty(\Omega)} \|u_1\|^{p-1}_{L_\infty(\Omega)} \|v_m - v_k\|_{L_2(\Omega)}, \tag{3.36}$$

$$\left| \int\limits_\Omega a(x) v_k u_k^{p-1}(v_m - v_k) \right| \leq \|a\|_{L_\infty(\Omega)} \|u_1\|^{p-1}_{L_\infty(\Omega)} \|v_m - v_k\|_{L_2(\Omega)}, \tag{3.37}$$

$$\begin{aligned} \left| \int\limits_{\Gamma_1^{\mathfrak{D}}} \partial(v_m - v_k)(v_m - v_k) \right| &\leq \left(\|\nabla v_m\|_{L_2(\Gamma_1)} + \|\nabla v_k\|_{L_2(\Gamma_1)}\right) \|v_m - v_k\|_{L_2(\Gamma_1)} \\ &\leq 2C_4 \|v_m - v_k\|_{L_2(\Gamma_1)}. \end{aligned} \tag{3.38}$$

Finally, substituting (3.36), (3.37) and (3.38) in (3.35), it follows from (3.23) and (3.24) that for any $\varepsilon > 0$ there exists $\tilde{n}_0 = \tilde{n}_0(\varepsilon)$ such that for any $m,\ k \geq \tilde{n}_0$ the following holds

$$\|\nabla(v_k - v_m)\|_{L_2(\Omega)} \leq \varepsilon,$$

which proves that v_n, $n \geq 1$ is a Cauchy sequence in $H^1(\Omega)$. Now, combining this fact with (3.16) and (3.23) give

$$\lim_{n\to\infty} \| v_n - v\|_{H^1(\Omega)} = 0, \qquad \|v\|_{H^1(\Omega)} = 1 \tag{3.39}$$

and in particular, it shows that $v \in H^1(\Omega)$.

We now ascertain the behavior of v on $\partial\Omega$. We already know that $v_n - v \in H^1(\Omega)$. Let $i \in \mathcal{L}(W_2^{\frac{1}{2}}(\partial\Omega), L_2(\partial\Omega))$ be the injection operator $i\colon\ W_2^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L_2(\partial\Omega)$ and $t \in \mathcal{L}(H^1(\Omega), W_2^{\frac{1}{2}}(\partial\Omega))$ the trace operator on $\partial\Omega$. Owing to the continuity of i and t, there exists $\tilde{C} > 0$ such that

$$\|v_n - v\|_{L_2\left(\Gamma_0 \cup \Gamma_1^{\mathcal{D}}\right)} \leq \|v_n - v\|_{L_2(\partial\Omega)} \leq \tilde{C}\|v_n - v\|_{H^1(\Omega)}, \qquad n \geq 1,$$

and owing to (3.39) it is apparent that

$$\lim_{n\to\infty} \| v_n - v\|_{L_2\left(\Gamma_0 \cup \Gamma_1^{\mathcal{D}}\right)} = 0.$$

Now, since $v_n|_{\Gamma_0} = 0$, $n \geq 1$ and (3.31) holds, we have that

$$v = 0 \qquad \text{in } L_2\left(\Gamma_0 \cup \Gamma_1^{\mathcal{D}}\right),$$

and therefore, since $v \in H^1(\Omega)$, we obtain that

$$v \in H^1_*(\Omega). \tag{3.40}$$

Moreover, since $v_n \gneq 0$, $n \geq 1$, it follows from (3.39) that

$$v \gneq 0 \qquad \text{in } \Omega,$$

that is, $v(x) \geq 0$ almost everywhere in Ω, but $v \neq 0$. On the other hand, the following holds for each $n \geq 1$

$$\begin{aligned}\left\| v_n - \frac{u}{L}\right\|_{L_2(\Omega)} &= \left\| \frac{u_n}{\|u_n\|_{H^1(\Omega)}} - \frac{u}{L}\right\|_{L_2(\Omega)} \\ &\leq \frac{\|u_n - u\|_{L_2(\Omega)}}{\|u_n\|_{H^1(\Omega)}} + \left|\frac{1}{\|u_n\|_{H^1(\Omega)}} - \frac{1}{L}\right| \|u\|_{L_2(\Omega)},\end{aligned}$$

where $L > 0$ is the limit defined by (3.11). Then, it follows from (3.11) and (3.15) that

$$\lim_{n\to\infty} \| v_n - L^{-1}u\|_{L_2(\Omega)} = 0 \tag{3.41}$$

and therefore, (3.23) and (3.41) imply that

$$u = Lv \qquad \text{in } L_2(\Omega). \tag{3.42}$$

In particular, (3.40) and (3.42) imply that

$$u \in H^1_*(\Omega). \tag{3.43}$$

Now we show that u provides us with a weak solution of (1.5). We already now that $v \in H^1_*(\Omega)$. Now, pick $\xi \in H^1_*(\Omega)$ up. Then, multiplying the differential equation (3.19) by ξ and integrating by parts, taking into account that $\operatorname{supp}(\xi) \subset \Omega \cup \Gamma_1^{\mathcal{N}}$, the following holds

$$\int_\Omega \nabla v_n \nabla \xi = \lambda \int_\Omega v_n \xi - \int_\Omega a(x) v_n u_n^{p-1} \xi, \qquad n \geq 1. \tag{3.44}$$

It should be noted that since

$$\xi = 0 \quad \text{on} \quad \Gamma_0 \cup \Gamma_1^{\mathfrak{D}} \qquad \text{and} \qquad \partial v_n = 0 \quad \text{on}\, \Gamma_1^{\mathcal{N}},$$

it is apparent that

$$\int_{\partial\Omega} \partial v_n \,\xi = \int_{\Gamma_0} \partial v_n \,\xi + \int_{\Gamma_1^{\mathfrak{D}}} \partial v_n \,\xi + \int_{\Gamma_1^{\mathcal{N}}} \partial v_n \,\xi = 0.$$

Then, taking into account (3.7), (3.11), (3.15) and (3.39), and letting $n \to \infty$ in (3.44) gives

$$\int_\Omega \nabla v \,\nabla \xi = \lambda \int_\Omega v\,\xi - \int_\Omega a(x) v\, u^{p-1} \xi. \tag{3.45}$$

Now, multiplying (3.45) by L and taking into account (3.42), it is apparent that for each $\xi \in H^1_*(\Omega)$ the following holds

$$\int_\Omega \nabla u \,\nabla \xi + \int_\Omega a(x) u^p \xi = \lambda \int_\Omega u\,\xi. \tag{3.46}$$

In particular, taking $\xi = u \in H^1_*(\Omega)$ in (3.46) we obtain that

$$\int_\Omega a(x) u^{p+1} = \lambda \int_\Omega u^2 - \int_\Omega |\nabla u|^2 < \infty, \tag{3.47}$$

and therefore, (3.43), (3.46) and (3.47) conclude that $u \in H^1_*(\Omega)$ provides us with a weak solution of (1.5). Hence, since u^* is the unique weak positive solution of (1.5) we have that

$$u = Lv = u^*, \tag{3.48}$$

and owing to (3.39) the following holds

$$\lim_{n\to\infty} \| v_n - L^{-1} u^* \|_{H^1(\Omega)} = 0.$$

Now, (3.10) and (3.48) imply that

$$\begin{aligned} \|u_n - u^*\|_{H^1(\Omega)} &\le \|u_n\|_{H^1(\Omega)} \left(\|v_n - v\|_{H^1(\Omega)} + \|u^*\|_{H^1(\Omega)} \left| \frac{1}{L} - \frac{1}{\|u_n\|_{H^1(\Omega)}} \right| \right) \\ &\le M \left(\|v_n - v\|_{H^1(\Omega)} + \|u^*\|_{H^1(\Omega)} \left| \frac{1}{L} - \frac{1}{\|u_n\|_{H^1(\Omega)}} \right| \right) \end{aligned} \tag{3.49}$$

and letting $n \to \infty$ in (3.49), it follows from (3.11) and (3.39) that (1.15) holds along some subsequence. Therefore, since the same argument works along any subsequence, the proof is completed.

□

Proof of Theorem 1.2: Assume that $b \in C(\Gamma_1)$ is positive and bounded away from zero on Γ_1, that is, there exists $\underline{b} > 0$ such that

$$b(x) \ge \underline{b} > 0 \qquad \text{for each } x \in \Gamma_1. \tag{3.50}$$

In this case we have that $\Gamma_1^{\mathfrak{D}} = \Gamma_1$ and $\Gamma_1^{\mathcal{N}} = \emptyset$. It follows from [10, Theorem 1.1] that for each $\gamma > 0$

$$\Lambda_\gamma = \left(\sigma_1^\Omega[\mathfrak{B}^{\mathcal{N}}], \sigma_1^{\Omega_0}[\mathfrak{D}] \right) \tag{3.51}$$

and for each fixed $\lambda \in \Lambda_\gamma$, (1.1) possesses a unique positive solution, which we denote by u_γ. Owing to (1.10), (2.1) and (3.51) we have that for each $\gamma > 0$ the following holds

$$\Lambda_\gamma \cap \Lambda_D = \left(\sigma_1^\Omega[\mathcal{D}], \sigma_1^{\Omega_0}[\mathcal{D}] \right).$$

Pick λ satisfying (1.16). In the same way as in the proof of Theorem 1.1, to prove (1.17) we will show that (1.17) holds for any sequence of real numbers $\{\gamma_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty}\gamma_n = \infty$. The proof of it readily follows adapting the arguments used in the proof of Theorem 1.1, substituting $\Gamma_1^{\mathcal{D}}$ by Γ_1, $\Gamma_1^{\mathcal{N}}$ by the empty set, the space $H^1_*(\Omega)$ by the Sobolev space $H^1_0(\Omega)$ and taking into account that Proposition 2.2 also works assuming (3.50) and (1.16). We omit the rest of the details of the proof by repetitive. As a remark, it should be pointed out that in this case, using the same notation as in the proof of Theorem 1.1, it is straight to prove that

$$\lim_{n\to\infty} \| v_n \|_{L_2(\Gamma_1)} = 0. \tag{3.52}$$

Indeed, since

$$\partial v_n = -\gamma_n\, b\, u_n^{q-1}\, v_n \qquad \text{on } \Gamma_1$$

(cf. (3.19)), taking into account (3.50) and the fact that $\|u_n\|_{H^1(\Omega)} \le 2L$ for some $n \ge n_0$ (cf. (3.26)), where L is defined by (3.11), it is apparent that

$$u_n^{2q}(x) \le \left(\frac{2L\partial v_n(x)}{\gamma_n \underline{b}} \right)^2 \qquad \text{for each} \qquad x \in \Gamma_1, \qquad n \ge n_0. \tag{3.53}$$

Then, owing to (3.22) it follows from (3.53) that

$$\|u_n\|^{2q}_{L_{2q}(\Gamma_1)} \le \left(\frac{2L}{\gamma_n \underline{b}} \right)^2 \int_{\Gamma_1} (\partial v_n(x))^2 \le \left(\frac{2L}{\gamma_n \underline{b}} \right)^2 \|\nabla v_n\|^2_{L_2(\Gamma_1)} \le \left(\frac{2LC_4}{\gamma_n \underline{b}} \right)^2, \tag{3.54}$$

for some constant $C_4 > 0$. Now, owing to the fact that $\lim_{n\to\infty}\gamma_n = \infty$, letting $n \to \infty$ in (3.54) gives

$$\lim_{n\to\infty} \| u_n \|_{L_{2q}(\Gamma_1)} = 0,$$

and since $L_{2q}(\Gamma_1) \subset L_2(\Gamma_1)$, it is apparent that

$$\lim_{n\to\infty} \| u_n \|_{L_2(\Gamma_1)} = 0. \tag{3.55}$$

Now, taking into account (3.11), it follows from (3.55) that (3.52) holds. This completes the proof.

□

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