

Research Article

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The hybrid power mean involving the Kloosterman sums and Dedekind sums

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Abstract: Kloosterman sums and Dedekind sums are two important sums in analytic number theory, the study of their various properties is a very interesting subject. The primary objective of this article is used the analytical methods and the properties of L -functions to examine and interpret the calculating problems of a particular type hybrid power mean involving the classical Kloosterman sums and Dedekind sums, and provide several new and intrinsically intriguing mean value formulae for them.

Keywords: Kloosterman sums; Dedekind sums; Dirichlet L -functions; hybrid power mean; calculating formula

MSC 2020: 11F20; 11M20

1 Introduction

To delineate the findings of the present study, it is first necessary to give the definitions of the classical Kloosterman sums $K(s, t; q)$ (see [1]) and the Dedekind sums $S(r, q)$ (see [2]). For any integer $q > 1$, we define the classical Kloosterman sums $K(s, r; q)$ as

$$K(s, r; q) = \sum_{x=1}^q e\left(\frac{rx + s\bar{x}}{q}\right),$$

where $\sum_{x=1}^q$ denotes the summation over all $1 \leq x \leq q$ such that $(x, q) = 1$, r and s are any integers, $x \cdot \bar{x} \equiv 1 \pmod{q}$, $e(z) = e^{2\pi iz}$ and $i^2 = -1$.

This sum plays a very crucial role in the study of analytic number theory, as many number theory problems within the field are intimately connected with it. For instance, Zhang Yitang's important work [3] is the case, he gave a strong upper bound estimate for $K(r, s; p)$ on some special sets, as a result, substantial progress has been made on the major problem of twin primes. About the arithmetical properties of the Kloosterman sums, many scholars have conducted extensive research on this topic, yielding numerous significant findings. For example, Li Jianhua and Liu Yanni [4] used the properties of Gauss sums and the analytic method to studied the hybrid mean value problem involving between the Gauss sums and the general Kloosterman sums, and to gave several interesting identities of it. Li Xiaoxue and Hu Jiayuan [5] used the analytic method and the properties of the classical Gauss sums to studied the computational problem of one kind fourth hybrid power mean of the quartic Gauss sums and Kloosterman sums, and gave an exact computational formula for it.

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Perhaps the most basic and the most important work in this area is the upper bound estimation for $K(r, s; q)$ (see Chowla [6] or Estermann [7]). That is,

$$\sum_{x=1}^q e\left(\frac{rx + s\bar{x}}{q}\right) \ll (r, s, q)^{\frac{1}{2}} \cdot d(q) \cdot q^{\frac{1}{2}},$$

where $d(q)$ denotes the Dirichlet divisor function, (r, s, q) denotes the greatest common factor of r, s and q .

In addition, Salié [8] and Iwaniec [9] proved that for any prime p , an individual possesses a distinct identity

$$\sum_{n=0}^{p-1} \left| \sum_{x=1}^{p-1} e\left(\frac{nx + \bar{x}}{p}\right) \right|^4 = 2p^3 - 3p^2 - 3p.$$

Let $p \geq 3$ be a prime. For any integer n , the Legendre's symbol $\left(\frac{n}{p}\right)$ modulo p is defined as follows:

$$\left(\frac{n}{p}\right) = \begin{cases} 1, & \text{if } (n, p) = 1 \text{ and } n \text{ is a quadratic residue modulo } p; \\ -1, & \text{if } (n, p) = 1 \text{ and } n \text{ is a quadratic nonresidue modulo } p; \\ 0, & \text{if } p \mid n. \end{cases}$$

This arithmetical function occupies a very important position in the elementary number theory and analytic number theory, and many classical number theory problems are closely related to it.

Zhang [10] used the elementary methods to prove a generalized result. That is, for any positive integer R and integer n with $(u, R) = 1$, one has the identity

$$\sum_{h=1}^R \left| \sum_{x=1}^R e\left(\frac{hx + u\bar{x}}{R}\right) \right|^4 = 3^{\omega(R)} \cdot R^2 \cdot \phi(R) \cdot \prod_{p \parallel R} \left(\frac{2}{3} - \frac{1}{3p} - \frac{4}{3p(p-1)} \right),$$

where $\phi(R)$ is the Euler function, $\prod_{p \parallel R}$ denotes the product over all prime divisors of R with $p \mid R$ and $p^2 \nmid R$, and $\omega(R)$ denotes the number of all different prime divisors of R .

On the other hand, it is also imperative to introduce the concept of Dedekind sums $S(r, q)$. For any integers $q \geq 2$ and r , the classical Dedekind sums $S(r, q)$ is defined as follows (see [2]):

$$S(r, q) = \sum_{b=1}^q \left(\left(\frac{b}{q} \right) \right) \left(\left(\frac{br}{q} \right) \right),$$

where as usual, $((z))$ is defined as

$$((z)) = \begin{cases} z - [z] - \frac{1}{2}, & \text{if } z \text{ is not an integer;} \\ 0, & \text{if } z \text{ is an integer.} \end{cases}$$

It is clear that $S(r, q)$ describes the behaviour of the logarithm of the η -function (see [11] and [12]) under modular transformations. Certain arithmetical properties of $S(r, q)$ are documented within various references [13–16]. In order to avoid the tedious, we do not want to list them here. However, it should be noted that perhaps the most critical attribute of $S(h, q)$ is its adherence to the principle of reciprocity (see [2] and [13]). That is, for all positive integers u and v with $(u, v) = 1$, one has the identity

$$S(u, v) + S(v, u) = \frac{u^2 + v^2 + 1}{12uv} - \frac{1}{4}. \quad (1)$$

Rademacher and Grosswald [12] also obtained a three-term formula similar to (1).

The main purpose of this paper is to study the calculating problems of one kind hybrid mean values involving $K(m, 1; p)$ and $S(r, p)$. That is,

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} K^2(m, 1; p) \cdot K^2(n, 1; p) \cdot S^k(m \cdot \bar{n}, p), \quad (2)$$

where p is an odd prime.

About the contents, Liu and Zhang [17] proved the following conclusion:

For any odd square-full number H , one has the identity

$$\sum_{m=1}^H \sum_{n=1}^H K(m, 1; H) \cdot K(n, 1; H) \cdot S(m \cdot \bar{n}, H) = \frac{1}{12} \cdot \phi^2(H) \cdot H \cdot \prod_{p|H} \left(1 + \frac{1}{p}\right).$$

But there seems to be no research on (2), at least we have not seen such a result in the existing literature. In the present manuscript, we employ analytical methodologies coupled with the intrinsic characteristics of Dirichlet L -functions to investigate the computational intricacies associated with (2), thereby establishing several intriguing findings. That is to say, we have arrived at the following three inferences:

Theorem 1. Let p be an odd prime with $4 \mid (p-1)$. Then for any integer $k \geq 0$, we have the identity

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} K^2(m, 1; p) \cdot K^2(n, 1; p) \cdot S^{2k+1}(m \cdot \bar{n}, p) = \frac{p^3 \cdot (p-1)^{2k+1}}{(12)^{2k+1}} \cdot \left(1 - \frac{2}{p}\right)^{2k+1}.$$

Theorem 2. Let p be an odd prime with $4 \mid (p-3)$, then we have the identity

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} K^2(m, 1; p) \cdot K^2(n, 1; p) \cdot S(m \cdot \bar{n}, p) = \frac{p^2 \cdot (p-1) \cdot (p-2)}{12} - p^2 \cdot d_p^2,$$

where d_p denotes the class number of the imaginary quadratic field $Q(\sqrt{-p})$.

Theorem 3. For any odd prime p , we have the asymptotic formula

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} K^2(m, 1; p) \cdot K^2(n, 1; p) \cdot S^2(m \cdot \bar{n}, p) = \frac{1}{24} \cdot p^5 + O\left(p^4 \cdot \exp\left(\frac{4 \ln p}{\ln \ln p}\right)\right),$$

where $\exp(y) = e^y$.

Taking $k = 0$ in Theorem 1, we can derive the following inferences:

Corollary. Let p be a prime with $p \equiv 1 \pmod{4}$, then we have

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} K^2(m, 1; p) \cdot K^2(n, 1; p) \cdot S(m \cdot \bar{n}, p) = \frac{p^2 \cdot (p-1) \cdot (p-2)}{12}.$$

Some notes: If k is an even number, then we only consider the case $k = 2$ in (2). Because when the even number $k > 2$, the situation is considerably more complex, hence, we will defer its consideration. If $p \equiv 3 \pmod{4}$ and $k = 2n + 1 \geq 3$ is an odd number, then for $2n + 1$ odd characters $\chi_i(-1) = -1$ ($i = 1, 2, \dots, 2n + 1$), it will definitely appear $\chi_1 \chi_2 \dots \chi_{2n+1} = \left(\frac{*}{p}\right)$. As a result, the mean theorem of Dirichlet L -functions are difficult to deal with, so we do not consider it in Theorem 2 when $k = 2n + 1 \geq 3$. Obviously, in our all theorems, the modulo is a prime p , and it is an interesting open problem whether these conclusions can be generalized to the general composite number q .

2 Several lemmas

In this section, we present three elementary lemmas that are essential for the demonstration of our theorems. Subsequently, it is imperative to incorporate certain principles of analytic number theory, including concepts Gauss sums and Dedekind sums, the characteristics of which are extensively documented in references [18], [19] and [20], thereby obviating the need for redundant explication. Initially, we present the following lemmas:

Lemma 1. *Let $p > 2$ be a prime. Then for any Dirichlet character χ modulo p , we have the identity*

$$\left| \sum_{m=1}^{p-1} \chi(m) \left(\sum_{a=1}^{p-1} e\left(\frac{ma + \bar{a}}{p}\right) \right)^2 \right| = \begin{cases} p^2 - p - 1, & \text{if } \chi = \chi_0 \text{ is the principal character modulo } p; \\ p, & \text{if } \chi = \chi_2 \text{ is the Legendre's symbol modulo } p; \\ p^{\frac{3}{2}}, & \text{if } \chi^2 \neq \chi_0. \end{cases}$$

Proof. If χ^2 is not the principal character modulo p , then from the definition and properties of the classical Gauss sums we have

$$\begin{aligned} \sum_{m=1}^{p-1} \chi(m) \left(\sum_{a=1}^{p-1} e\left(\frac{ma + \bar{a}}{p}\right) \right)^2 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=1}^{p-1} \chi(m) e\left(\frac{m(a+b) + \bar{a} + \bar{b}}{p}\right) \\ &= \tau(\chi) \sum_{a=1}^{p-1} \bar{\chi}(a+1) \sum_{b=1}^{p-1} \bar{\chi}(b) e\left(\frac{\bar{b}(\bar{a}+1)}{p}\right) \\ &= \tau^2(\chi) \cdot \sum_{a=1}^{p-1} \bar{\chi}(a+1) \bar{\chi}(\bar{a}+1) = \tau^2(\chi) \cdot \sum_{a=1}^{p-1} \chi(a) \bar{\chi}^2(a+1) \\ &= \frac{\tau^2(\chi)}{\tau(\chi^2)} \cdot \sum_{b=1}^{p-1} \chi^2(b) \sum_{a=1}^{p-1} \chi(a) e\left(\frac{b(a+1)}{p}\right) \\ &= \frac{\tau^3(\chi)}{\tau(\chi^2)} \cdot \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b}{p}\right) = \frac{\tau^4(\chi)}{\tau(\chi^2)}. \end{aligned} \quad (3)$$

Note that $|\tau(\chi)| = |\tau(\chi^2)| = \sqrt{p}$, from (3) we have

$$\left| \sum_{m=1}^{p-1} \chi(m) \left(\sum_{a=1}^{p-1} e\left(\frac{ma + \bar{a}}{p}\right) \right)^2 \right| = p^{\frac{3}{2}}. \quad (4)$$

If $\chi^2 = \chi_0$ is the principal character modulo p and $\chi \neq \chi_0$, then from (3) we have

$$\begin{aligned} \sum_{m=1}^{p-1} \chi(m) \left(\sum_{a=1}^{p-1} e\left(\frac{ma + \bar{a}}{p}\right) \right)^2 &= \tau^2(\chi) \cdot \sum_{a=1}^{p-1} \chi(a) \bar{\chi}^2(a+1) \\ &= \tau^2(\chi) \cdot \sum_{a=1}^{p-2} \chi(a) = -\chi(-1) \cdot \tau^2(\chi). \end{aligned} \quad (5)$$

If $\chi = \chi_0$ is the principal character modulo p , then we obtain

$$\begin{aligned} \sum_{m=1}^{p-1} \left(\sum_{a=1}^{p-1} e\left(\frac{ma + \bar{a}}{p}\right) \right)^2 &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=1}^p e\left(\frac{mb(a+1) + \bar{b}(\bar{a}+1)}{p}\right) - 1 \\ &= p \cdot (p-1) - 1 = p^2 - p - 1. \end{aligned} \quad (6)$$

Combining (4), (5) and (6),

$$\begin{aligned} &\left| \sum_{m=1}^{p-1} \chi(m) \left(\sum_{a=1}^{p-1} e\left(\frac{ma + \bar{a}}{p}\right) \right)^2 \right| \\ &= \begin{cases} p^2 - p - 1, & \text{if } \chi = \chi_0 \text{ is the principal character modulo } p; \\ p, & \text{if } \chi = \chi_2 \text{ is the Legendre's symbol modulo } p; \\ p^{\frac{3}{2}}, & \text{if } \chi^2 \neq \chi_0. \end{cases} \end{aligned}$$

This proves Lemma 1.

Lemma 2. Let $q > 2$ be an integer. Then for any integer v with $(v, q) = 1$, we have

$$S(v, q) = \frac{1}{\pi^2 q} \sum_{r|q} \frac{r^2}{\phi(r)} \sum_{\substack{\chi \bmod r \\ \chi(-1)=-1}} \chi(v) |L(1, \chi)|^2,$$

where $L(s, \chi)$ is L -function corresponding to $\chi \bmod h$, $\phi(r)$ is the Euler function.

Proof. This is a result established by Zhang, see Lemma 2 in [14].

Lemma 3. Let $p > 2$ be a prime. Then for the fourth power mean of Dirichlet L -functions, we have the asymptotic formula

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^4 = \frac{5 \cdot \pi^4}{144} \cdot p + O\left(\exp\left(\frac{4 \ln p}{\ln \ln p}\right)\right).$$

Proof. See reference [21].

3 Proofs of the theorems

In this section, we shall employ the three fundamental lemmas introduced in Section 2 to establish our main results. First if q is an odd prime p , then from Lemma 2 we obtain

$$S(r, p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(r) \cdot |L(1, \chi)|^2. \quad (7)$$

From (7) we have the identity

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 = \frac{\pi^2}{12} \cdot \frac{(p-1)^2(p-2)}{p^2}. \quad (8)$$

If $p \equiv 1 \pmod{4}$, $k = 2h + 1$ is a positive odd number, then for all Dirichlet characters χ_i , satisfying $\chi_i(-1) = -1$ and $\chi_1 \chi_2 \dots \chi_k(-1) = -1$, it follows from (7) and Lemma 1 that the product $\chi_1 \dots \chi_k$ is not the Legendre's symbol modulo p . Moreover, since the Kloosterman sums is a real number, we obtain the following identity

$$\begin{aligned} & \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \left(\sum_{x=1}^{p-1} e\left(\frac{mx + \bar{x}}{p}\right) \right)^2 \left(\sum_{y=1}^{p-1} e\left(\frac{ny + \bar{y}}{p}\right) \right)^2 S^k(m \cdot \bar{n}, p) \\ &= \frac{p^k}{\pi^{2k}(p-1)^k} \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} K^2(m, 1; p) \cdot K^2(n, 1; p) \cdot \left(\sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \chi(m\bar{n}) \cdot |L(1, \chi)|^2 \right)^k \\ &= \frac{p^k}{\pi^{2k}(p-1)^k} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1(-1)=-1}} \dots \sum_{\substack{\chi_k \pmod{p} \\ \chi_k(-1)=-1}} \left| \frac{\tau^4(\chi_1 \chi_2 \dots \chi_k)}{\tau((\chi_1 \chi_2 \dots \chi_k)^2)} \right|^2 \cdot |L(1, \chi_1)|^2 \dots |L(1, \chi_k)|^2 \\ &= \frac{p^{k+3}}{\pi^{2k}(p-1)^k} \cdot \left(\sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} |L(1, \chi)|^2 \right)^k = \frac{p^{k+3}}{\pi^{2k}(p-1)^k} \cdot \left(\frac{\pi^2}{12} \cdot \frac{(p-1)^2(p-2)}{p^2} \right)^k \\ &= \frac{p^3 \cdot (p-1)^k}{(12)^k} \cdot \left(1 - \frac{2}{p} \right)^k. \end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. If $p \equiv 3 \pmod{4}$, note that $\left(\frac{-1}{p}\right) = -1 = \chi_2(-1)$ and $|L(1, \chi_2)| = \pi \cdot d_p / \sqrt{p}$, from Lemma 1 and Lemma 2,

$$\begin{aligned} & \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} K^2(m, 1; p) \cdot K^2(n, 1; p) \cdot S(m \cdot \bar{n}, p) \\ &= \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \left| \sum_{m=1}^{p-1} \chi(m) \cdot K^2(m, 1; p) \right|^2 \cdot |L(1, \chi)|^2 \\ &= \frac{p^4}{\pi^2(p-1)} \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1 \\ \chi \neq \chi_2}} |L(1, \chi)|^2 + \frac{p^3}{\pi^2(p-1)} \cdot |L(1, \chi_2)|^2 \\ &= \frac{p^4}{\pi^2(p-1)} \cdot \frac{\pi^2 \cdot (p-1)^2(p-2)}{12 \cdot p^2} - \frac{p^3(p-1)}{\pi^2(p-1)} \cdot \frac{\pi^2 \cdot d_p^2}{p} \\ &= \frac{p^2 \cdot (p-1) \cdot (p-2)}{12} - p^2 \cdot d_p^2. \end{aligned}$$

This proves Theorem 2.

For any odd prime p and $k = 2$, note that $\chi \bar{\chi} = \chi_0$, from Lemma 1–3,

$$\begin{aligned} & \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \left(\sum_{x=1}^{p-1} e\left(\frac{mx + \bar{x}}{p}\right) \right)^2 \left(\sum_{y=1}^{p-1} e\left(\frac{ny + \bar{y}}{p}\right) \right)^2 S^2(m \cdot \bar{n}, p) \\ &= \frac{p^2}{\pi^4(p-1)^2} \sum_{\substack{\chi_1 \pmod{p} \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \pmod{p} \\ \chi_2(-1)=-1}} \left| \sum_{m=1}^{p-1} \chi_1 \chi_2(m) \cdot K^2(m, 1; p) \right|^2 \cdot |L(1, \chi_1)|^2 \cdot |L(1, \chi_2)|^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{p^5}{\pi^4(p-1)^2} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1 \\ \chi_1 \chi_2 \neq \chi_0}} |L(1, \chi_1)|^2 \cdot |L(1, \chi_2)|^2 + \frac{p^2 \cdot (p^2 - p - 1)^2}{\pi^4(p-1)^2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^4 \\
&= \frac{p^5}{\pi^4(p-1)^2} \left(\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 \right)^2 - \frac{p^5}{\pi^4(p-1)^2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^4 \\
&\quad + \frac{p^2 \cdot (p^2 - p - 1)^2}{\pi^4(p-1)^2} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^4 \\
&= \frac{p(p-1)^2(p-2)^2}{144} + \frac{p^2(p^2 - p - 1)^2 - p^5}{\pi^4(p-1)^2} \cdot \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^4 \\
&= \frac{p^5}{144} + \frac{5p^5}{144} + O\left(p^4 \cdot \exp\left(\frac{4 \ln p}{\ln \ln p}\right)\right) = \frac{1}{24} \cdot p^5 + O\left(p^4 \cdot \exp\left(\frac{4 \ln p}{\ln \ln p}\right)\right).
\end{aligned}$$

This completes the proof of Theorem 3.

4 Conclusions

This paper employs analytic techniques and properties of Dirichlet L -functions to study the computational problems of a specific hybrid power mean, which connects classical sums with Dedekind sums. As a result, we establish several novel and interesting identities. One of them is the identity

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} K^2(m, 1; p) \cdot K^2(n, 1; p) \cdot S^{2k+1}(m \cdot \bar{n}, p) = \frac{p^3 \cdot (p-1)^{2k+1}}{(12)^{2k+1}} \cdot \left(1 - \frac{2}{p}\right)^{2k+1},$$

where p denotes a prime with $p \equiv 1 \pmod{4}$ and $k \geq 0$ be any integer.

We believe that the methods used in this paper will contribute to further research in the relevant fields.

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