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Research Article

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Growth theorems and coefficient bounds for g-starlike mappings of complex order λ

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Abstract: Let f be a g-starlike mapping of complex order λ such that $x = 0$ is a zero of order $k + 1$ of $f(x) - x$. By utilizing the geometric properties of f, we characterize its growth theorems and coefficient bounds. The established results yield a unified representation for the growth theorems and coefficient bounds of the subfamilies of normalized biholomorphic mappings with distinct geometric interpretations, respectively. In particular, the estimates are sharp when $\lambda \le 0$.

Keywords: g-starlike mappings of complex order λ; growth theorems; quasi-convex mappings; coefficient bounds

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1 Introduction and preliminaries

Let $\mathbb{C}$ be the complex plane. The unit disk is denoted by $\mathbb{D} = \{\zeta \in \mathbb{C}: |\zeta| < 1\}$. Let $\mathbb{C}^n = \left\{z = (z_1, \dots, z_n)^T: z_k \in \mathbb{C}, k = 1, \dots, n\right\}$ denote the space of n complex variables equipped with Euclidean norm $\|z\| = \sqrt{\sum_{k=1}^n |z_k|^2}$. Let X be a complex Banach space with respect to the norm $\|\cdot\|_X$. Let $\mathscr{B} = \{x \in X: \|x\|_X < 1\}$ be the open unit ball in X. Let $\Omega \subseteq X$ be a domain that contains the origin. The set of holomorphic mappings from Ω into X is denoted by $H(\Omega)$.

If $f \in H(\Omega)$, then

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} D^{(n)} f(x)((y-x)^n)$$

for all y in some neighborhood of $x \in \Omega$, where $D^{(n)}f(x)$ is the n th Fréchet derivative of f at x, and

$$D^{(n)} f(x)((y-x)^n) = D^{(n)} f(x)(y-x, \dots, y-x)$$

for $n \ge 1$. In particular, the 1st Fréchet derivative $D^{(1)}f(x) = Df(x)$. We note that $D^{(n)}f(x)$ is a bounded symmetric n-linear mapping from $\prod_{j=1}^n X$ into X.

A mapping $f \in H(\Omega)$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$, where I is the identity operator on X. A mapping $f \in H(\Omega)$ is said to be biholomorphic if the inverse f^{-1} exists and it is holomorphic on the open

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set $f(\Omega)$. A mapping $f \in H(\Omega)$ is said to be locally biholomorphic if each $x \in \Omega$ has a neighberhood V such that $f|_V$ is biholomorphic. We denote by $\mathscr{L}(\Omega)$ the family of normalized locally biholomorphic mappings on Ω.

Let $T: X \to \mathbb{C}$ be a continuous linear functional. Then

$$\|T\| = \sup\{|Tx|: x \in \partial\mathscr{B}\}.$$

For each $x \in X\backslash\{0\}$, we define $T(x) = \{l_x \in X^*: \|l_x\| = 1, l_x(x) = \|x\|\}$, where X^* denotes the dual space of X. According to the Hahn–Banach theorem, $T(x)$ is nonempty. For any fixed $x \in X, \zeta \in \mathbb{C}\backslash\{0\}$, we have $l_{\zeta x} = \frac{|\zeta|}{\zeta} l_x$.

If for any $x \in \Omega, t \in [0,1]$, $(1-t)x \in \Omega$ holds, then Ω is said to be starlike (with respect to the origin). A domain $\Omega \subseteq X$ is said to be convex if given $x_1, x_2 \in \Omega$, $tx_1 + (1-t)x_2 \in \Omega$, for all $t \in [0,1]$.

Let $f \in H(\Omega)$ be biholomorphic mapping with $0 \in f(\Omega)$. If $f(\Omega)$ is starlike (with respect to the origin), then f is said to be starlike. If $f(\Omega)$ is convex, then f is said to be convex.

If $f, g \in H(\mathscr{B})$, and there exists a holomorphic mapping $v: \mathscr{B} \to \mathscr{B}$ with $v(0) = 0$ such that $f = g \circ v$, then we say that f is subordinate to g, denoted by $f \prec g$. If g is biholomorphic on $\mathscr{B}$, then $f \prec g$ is equivalent to requiring that $f(0) = g(0)$ and $f(\mathscr{B}) \subseteq g(\mathscr{B})$.

The growth theorem is one of the central research topics in the geometric function theory of several complex variables. In 1991, Barnard, Fitzgerald, and Gong [1] using the analytical characterization of the normalized biholomorphic starlike mappings showed that both the growth theorem and the Koebe 1/4-Theorem hold, and the same results were demonstrated by Kubicka and Poreda [2] using the method of Loewner chains.

Theorem A. [1, 2] *Let f be a normalized biholomorphic starlike mapping on $\mathbb{B}_n = \{z \in \mathbb{C}^n: \|z\| < 1\}$. Then*

$$\frac{\|z\|}{(1+\|z\|)^2} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|)^2}, \quad z \in \mathbb{B}_n.$$

Moreover, $f(\mathbb{B}_n)$ contains a ball of radius $1/4$, and the above estimates are sharp.

If f is k-fold symmetry, that is, $\exp\left(-\frac{2\pi i}{k}\right) f\left(e^{\frac{2\pi i}{k}} x\right) = f(x)$, then Theorem A can be strengthened:

Theorem B. [1] *If $f: \mathbb{B}_n \to \mathbb{C}^n$ is normalized biholomorphic and is starlike with a k-fold symmetric image for $k \ge 1$, then*

$$\frac{\|z\|}{(1+\|z\|^k)^{2/k}} \le \|f(z)\| \le \frac{\|z\|}{(1-\|z\|^k)^{2/k}}, \quad z \in \mathbb{B}_n.$$

Moreover, $f(\mathbb{B}_n)$ contains a ball of radius $2^{-2/k}$, and these estimates are sharp.

Let f be a convex mapping. Then the following growth theorem and Koebe's type theorem are due to Honda [3] using the analytical characterization of convex mappings.

Theorem C. *Let $f: \mathscr{B} \to X$ be a k-fold symmetric normalized biholomorphic convex mapping. Then, for any point $x \in \mathscr{B}$, we have*

$$\frac{\|x\|}{(1+\|x\|^k)^{1/k}} \le \|f(x)\| \le \frac{\|x\|}{(1-\|x\|^k)^{1/k}}.$$

And $f(\mathscr{B})$ contains a ball of radius $2^{-1/k}$ centered at the origin.

For further results on the growth theorem and Koebe's type theorem for convex mappings, we refer to [4–7].

Let $f \in H(\mathscr{B})$. If $f(0) = 0, Df(0) = \ldots = D^{(k-1)}f(0) = 0$, but $D^{(k)}f(0) \ne 0$, we say that $x = 0$ is the zero of order k of $f(x)$, where $k = 1, 2, \ldots$. It is easy to deduce that $x = 0$ is a zero of order m of $f(x) - x$ for some m with $m \ge k+1$ if f is k-fold symmetric and $f(x) \ne x$.

Let $g: \mathbb{D} \to \mathbb{C}$ be a holomorphic univalent function such that $g(0) = 1$ and $\Re g(\zeta) > 0$. Suppose that $g(\overline{\zeta}) = \overline{g(\zeta)}$ and satisfies the conditions

$$\begin{cases}\min\limits_{|\zeta|=r}\Re g(\zeta)=\min\{g(r),g(-r)\};\\ \max\limits_{|\zeta|=r}\Re g(\zeta)=\max\{g(r),g(-r)\}.\end{cases}$$

We denote by $G(\mathbb{D})$ the set of all functions g defined as above. It is well known that all functions which are convex in the direction of the imaginary axis and symmetric about the real axis are contained in $G(\mathbb{D})$ (see [8]).

Let $g\in G(\mathbb{D})$. And let

$$\mathscr{M}_g(\mathscr{B})=\left\{h\in H(\mathscr{B})\colon h(0)=0,Dh(0)=I,\frac{1}{\|x\|_X}l_x\{h(x)\}\in g(\mathbb{D}),l_x\in T(x),x\in\mathscr{B}\backslash\{0\}\right\}.$$

If $g(\zeta)=\frac{1-\zeta}{1+\zeta},\zeta\in\mathbb{D}$, then the Carathéodory family $\mathscr{M}(\mathscr{B})=\mathscr{M}_g(\mathscr{B})$ on the unit ball $\mathscr{B}$ can be obtained, i.e.

$$\mathscr{M}(\mathscr{B})=\{h\in H(\mathscr{B})\colon h(0)=0,Dh(0)=I,\Re\{l_x\{h(x)\}\}>0,l_x\in T(x),x\in\mathscr{B}\backslash\{0\}\},$$

which plays a crucial role in the function theory of several complex variables.

Using $\mathscr{M}_g(\mathscr{B})$ and $\mathscr{M}(\mathscr{B})$, we can also construct some unified representation for subfamilies of normalized biholomorphic mappings. We denote by

$$\mathscr{S}_g^*(\mathscr{B})=\{f\in\mathscr{L}(\mathscr{B})\colon(Df(x))^{-1}f(x)\in\mathscr{M}_g(\mathscr{B})\}$$

and

$$\mathscr{S}_\lambda^*(\mathscr{B})=\{f\in\mathscr{L}(\mathscr{B})\colon(1-\lambda)(Df(x))^{-1}f(x)+\lambda x\in\mathscr{M}(\mathscr{B})\}$$

the families of g-starlike mappings and almost starlike mappings of complex order λ, respectivelly, for $g\in G(\mathbb{D})$, $\lambda\in\mathbb{C}$ with $\Re\lambda\le 0$. We note that the family $\mathscr{S}_g^*(\mathscr{B})$ introduced by Hamada and Kohr [9] provided a unified representation for the subfamily of spirallike mappings, the family $\mathscr{S}_\lambda^*(\mathscr{B})$ introduced by Bălăeţi and Nechita [10] established a unified formulation for the subfamilies of spirallike mappings. Based on the definition of $\mathscr{S}_g^*(\mathscr{B})$, Hamada and Honda [9] used the parametric representation method to obtain the sharp growth and covering theorems, as well as the sharp coefficient bounds for $f\in\mathscr{S}_g^*(\mathscr{B})$, where $x=0$ is the zero of order $k+1$ of $f(x)-x$. Moreover, in 2020, Graham, Hamada, and Kohr [11] established some coefficient bounds for g-starlike mappings on the unit ball of a complex Hilbert space as an application of the estimation of $\|Df(z_0)\|$ on the holomorphic tangent space for homogeneous polynomial mappings f between Hilbert balls.

The main purpose of this paper is to establish unified formulations for both growth theorems and coefficient limits concerning fundamental subfamilies of normalized biholomorphic mappings, respectively. Building upon these unified representations, we can derive numerous classical results on growth theorems and coefficient bounds for normalized biholomorphic mappings that are well-established in the literature. To develop these results, we shall recall the family of g-starlike mappings of complex order λ, denoted by $\mathscr{S}_{g,\lambda}^*(\mathscr{B})$, originally introduced by the first author of this paper in [12], where

$$\mathscr{S}_{g,\lambda}^*(\mathscr{B})=\{f\in\mathscr{L}(\mathscr{B})\colon(1-\lambda)(Df(x))^{-1}f(x)+\lambda x\in\mathscr{M}_g(\mathscr{B})\},$$

$g\in G(\mathbb{D})$, $\lambda\in\mathbb{C}$ with $\Re\lambda\le 0$. Clearly, when $\lambda=0$, we have $\mathscr{S}_{g,\lambda}^*(\mathscr{B})=\mathscr{S}_g^*(\mathscr{B})$; and when $g(\zeta)=\frac{1-\zeta}{1+\zeta},\zeta\in\mathbb{D}$, we have $\mathscr{S}_{g,\lambda}^*(\mathscr{B})=\mathscr{S}_\lambda^*(\mathscr{B})$. Furthermore, as established in [9,12,13], we derive the analytic characterization for: (i) the family of starlike mappings of order α when $\lambda=0$ and $g(\zeta)=\frac{1-\zeta}{1+(1-2\alpha)\zeta},\zeta\in\mathbb{D},0<\alpha<1$; (ii) the family of strongly starlike mappings of order α when $\lambda=0$ and $g(\zeta)=\frac{(1-\zeta)^\alpha}{(1+\zeta)^\alpha},\zeta\in\mathbb{D},0<\alpha\le 1$, $(1-\zeta)^\alpha|_{\zeta=0}=(1+\zeta)^\alpha|_{\zeta=0}=1$; (iii) the family of almost starlike mappings of order α when $\lambda=0$ and $g(\zeta)=(1-\alpha)\frac{1-\zeta}{1+\zeta}+\alpha,\zeta\in\mathbb{D},0\le\alpha<1$; (iv) the family of Janowski-starlike mappings of complex order λ when $g(\zeta)=\frac{1+A\zeta}{1+B\zeta},\zeta\in\mathbb{D},-1\le B<A\le 1$; (v) the family of spirallike mappings of type $\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$ when $g(\zeta)=\frac{1-\zeta}{1+\zeta},\zeta\in\mathbb{D}$ and $\lambda=i\tan\beta$.

Another motivation for studying the family $\mathscr{S}_{g,\lambda}^*(\mathscr{B})$ stems from their intrinsic relationship with quasi-convex mappings. As a natural generalization to the higher dimensions of convex functions in the plane, Roper and Suffridge [14] introduced the family of quasiconvex mappings of type A. Let f be a normalized local biholomorphic mapping on $\mathscr{B}$, and $u \in X$ with $\|u\| = 1$. If

$$\Re\left\{\frac{2\alpha}{l_u\left[\left(Df(\alpha u)\right)^{-1}\left(f(\alpha u) - f(\beta u)\right)\right]} - \frac{\alpha+\beta}{\alpha-\beta}\right\} > 0, \quad |\beta| \le |\alpha| < 1. \tag{1.1}$$

Then f is called a quasi-convex mapping of type A. The set of those mappings is denoted by $Q_A(\mathscr{B})$. Hamada and Kohr [15] not only showed that convex mappings were also quasi-convex mappings of type A, but also proved that the growth result for the family of convex mappings was also valid for the family of quasi-convex mappings of type A on the unit ball $\mathscr{B}$ in complex Banach spaces. At the same time, Zhang and Liu [16] introduced the family of quasi-convex mappings on the unit ball $\mathscr{B}$ in complex Banach spaces, which is denoted by $Q(\mathscr{B})$, i.e.

$$Q(\mathscr{B}) = \left\{f \in \mathscr{L}(\mathscr{B}) : \Re\left\{l_x\left[(Df(x))^{-1}(f(x) - f(\zeta x))\right]\right\} \ge 0, x \in \mathscr{B}, \zeta \in \overline{\mathbb{D}}\right\}.$$

And they proved the inclusion relation $K(\mathscr{B}) \subsetneq Q_A(\mathscr{B}) = Q(\mathscr{B})$, where $K(\mathscr{B})$ denotes the set of convex mappings on $\mathscr{B}$. Furthermore, if $X = \mathbb{C}^n$, the "quasi-convex mapping" is exactly the "quasi-convex mapping of type A" introduced by Roper and Suffridge in [14].

If $\beta = 0$ in (1.1), then the follwing relation holds

$$\Re\left\{\frac{\|x\|_X}{l_x\{(Df(x))^{-1}f(x)\}}\right\} > \frac{1}{2}, \quad x \in \mathscr{B}\backslash\{0\},$$

which is equivalent to

$$\left|\frac{1}{\|x\|_X} l_x\{(Df(x))^{-1}f(x)\} - 1\right| < 1, \, x \in \mathscr{B}\backslash\{0\}.$$

Therefore, we have the following inclusions:

$$K(\mathscr{B}) \subsetneq Q_A(\mathscr{B}) = Q(\mathscr{B}) \subseteq \mathscr{S}^*_{g,\lambda}(\mathscr{B}), \quad g(\zeta) = 1 - \zeta, \quad \lambda = 0.$$

It is precisely these inclusion relations among the family of convex mapping, quasi-convex mapping and the subfamilies of starlike mappings that motivate our investigation into the family of $S^*_{g,\lambda}(\mathscr{B})$.

Below we first construct a general normalized biholomorphic mapping $f \in S^*_{g,\lambda}(\mathbb{B}_n)$.

Example 1.1. Let $a, \ \lambda \in \mathbb{C}, \ \Re\lambda \le 0, \ g \in G(\mathbb{D})$. Assume that $f(z) = (z_1 + az_1z_2, z_2, \dots, z_n)^T$ is a holomorphic mapping on the unit ball $\mathbb{B}_n$. If $|a| \le \frac{l}{1+l}$, where $l = \frac{\operatorname{dist}(1, \partial g(\mathbb{D}))}{\frac{2}{3\sqrt{3}}|1-\lambda|}$, then $f \in S^*_{g,\lambda}(\mathbb{B}_n)$.

Proof. By direct calculation, we obtain

$$(Df(z))^{-1} = \begin{pmatrix} \frac{1}{1+az_2} & -\frac{az_1}{1+az_2} & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Thus

$$\frac{1}{\|z\|^2}\langle(1-\lambda)(Df(z))^{-1}f(z) + \lambda z, z\rangle = 1 - \frac{a(1-\lambda)|z_1|^2 z_2}{\left(|z_1|^2 + |z_2|^2 + \dots + |z_n|^2\right)(1+az_2)}.$$

Since $|a| \le \frac{l}{1+l}$ and the function $\frac{u}{1-u}$ is increasing on $[0,1)$, it yields that

$$\left|\frac{a(1-\lambda)|z_1|^2 z_2}{\left(|z_1|^2+|z_2|^2+\cdots+|z_n|^2\right)(1+az_2)}\right| \le \left|\frac{a(1-\lambda)|z_1|^2 z_2}{\left(|z_1|^2+|z_2|^2\right)(1+az_2)}\right| < \frac{2}{3\sqrt{3}}|1-\lambda|\frac{|a|}{1-|a|} \le \operatorname{dist}(1,\partial g(\mathbb{D})).$$

This implies

$$1-\frac{a(1-\lambda)|z_1|^2 z_2}{\left(|z_1|^2+|z_2|^2+\cdots+|z_n|^2\right)(1+az_2)} \in g(\mathbb{D}).$$

Thus $f \in S^*_{g,\lambda}(\mathbb{B}_n)$. □

In this paper, we will consider growth theorems and coefficient bounds for $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ such that $x=0$ is a zero of order $k+1$ of $f(x)-x$. In particular, we recover the results in [9,12,13,15–19].

2 Growth theorems for normalized biholomorphic mapping $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$

In the following subsection, we shall characterize the growth theorems for normalized biholomorphic mapping $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ such that $f(x)-x$ has a zero of order $k+1$ at $x=0$. To this end, several preparatory lemmas are required.

2.1 Several lemmas

Lemma 2.1. [20] *Let f be a spirallike mapping with respect to $(1-\lambda)I$ on $\mathscr{B}$, where $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$. Suppose that $x(t) = f^{-1}\left(\exp\left(-(1-\lambda)t\right)f(x)\right)$, $t \in [0,+\infty)$, then*

(i) $\|x(t)\|$ is strictly decreasing on $[0,+\infty)$ with respect to t;

(ii) $\lim_{t\to+\infty}\frac{\|f(x(t))\|}{\|x(t)\|} = 1$, and

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = -(1-\lambda)\left[Df(x(t))\right]^{-1}f(x(t)), \quad \forall t \in (0,+\infty);$$

(iii) $\frac{\mathrm{d}\|f(x(t))\|}{\mathrm{d}t} = -(1-\Re\lambda)\|f(x(t))\|, \forall t \in (0,+\infty)$.

Lemma 2.2. [21] *Let $x\colon [0,+\infty) \to X$ be differentiable at the point $s \in (0,+\infty)$. If $\|x(t)\|$ is differentiable at the point s with respect to t, then*

$$\Re\left\{T_{x(s)}\left[\frac{\mathrm{d}x}{\mathrm{d}t}(s)\right]\right\} = \frac{\mathrm{d}\|x(s)\|}{\mathrm{d}t}, \quad s \in [0,+\infty).$$

Lemma 2.3. [22] *If $f \in H(\mathbb{D})$, h is a biholomorphic function on $\mathbb{D}$, k is a positive integer,*

$$f(0) = h(0), f'(0) = \ldots = f^{(k-1)}(0) = 0,$$

and $f \prec h$. Then

$$f(r\mathbb{D}) \subseteq h(r^k\mathbb{D}), r \in (0,1), r\mathbb{D} = \{\zeta \in \mathbb{C}\colon |\ \zeta\ | < r\}.$$

We now establish a Harnack inequality for $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$.

Lemma 2.4. *Let $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ be a normalized biholomorphic mapping such that $x = 0$ is a zero of order $k+1$ of $f(x) - x$. Then*

$$-\|x\|\Re\lambda + \|x\| \min\{g(\|x\|^k), g(-\|x\|^k)\} \le \Re\{(1-\lambda)l_x[(Df(x))^{-1}f(x)]\}$$
$$\le -\|x\|\Re\lambda + \|x\| \max\{g(\|x\|^k), g(-\|x\|^k)\}.$$

Proof. Fixing $x \in \mathscr{B}\backslash\{0\}$, let $x_0 = \frac{x}{\|x\|}$. Then

$$q(\zeta) = \begin{cases} (1-\lambda)\frac{1}{\zeta} l_{x_0}[(Df(\zeta x_0))^{-1} f(\zeta x_0)] + \lambda, & \zeta \in \mathbb{D}\backslash\{0\}, \\ 1, & \zeta = 0 \end{cases}$$

is a holomorphic function on $\mathbb{D}$.

Observe that

$$q(\zeta) = (1-\lambda)\frac{1}{|\zeta|} l_{\zeta x_0}[(Df(\zeta x_0))^{-1} f(\zeta x_0)] + \lambda, \quad \zeta \ne 0,$$

and $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$. We conclude that $q(0) = g(0) = 1, q(\mathbb{D}) \subseteq g(\mathbb{D})$, i.e. $q \prec g$. Furthermore, $q'(0) = \ldots = q^{(k-1)}(0) = 0$.

Lemma 2.3 immediately implies that $q(r\mathbb{D}) \subseteq g(r^k\mathbb{D}), r \in (0,1)$. Hence

$$\min\{g(r^k), g(-r^k)\} \le \Re q(\zeta) \le \max\{g(r^k), g(-r^k)\}.$$

Let $\zeta = \|x\|$. Then

$$-\|x\|\Re\lambda + \|x\| \min\{g(\|x\|^k), g(-\|x\|^k)\} \le \Re\{(1-\lambda)l_x[(Df(x))^{-1}f(x)]\}$$
$$\le -\|x\|\Re\lambda + \|x\| \max\{g(\|x\|^k), g(-\|x\|^k)\}.$$

□

2.2 Growth theorems for $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$

By utilizing the geometric properties of the family of g-starlike mappings of complex order λ, we establish growth theorems for normalized biholomorphic mapping $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ on the unit ball $\mathscr{B}$ such that $f(x) - x$ has a zero of order $k+1$ at $x = 0$. Specifically, sharp results are still obtainable when λ is restricted to real numbers. This result generalized the results in [9,12,13,15–17,19].

Theorem 2.5. *Let $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ be a normalized biholomorphic mapping such that $x = 0$ is a zero of order $k+1$ of $f(x) - x$. Then*

$$\|x\| \exp\left(\int_0^{\|x\|} \left[\frac{1-\Re\lambda}{\max\{g(y^k), g(-y^k)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right)$$
$$\le \|f(x)\|$$
$$\le \|x\| \exp\left(\int_0^{\|x\|} \left[\frac{1-\Re\lambda}{\min\{g(y^k), g(-y^k)\} - \Re\lambda} - 1\right]\frac{dy}{y}\right).$$

Proof. Note that f is also a spirallike mapping with respect to $(1-\lambda)I$, since f is a g-starlike mapping of complex order λ on $\mathscr{B}$. Fix $x \in \mathscr{B}\backslash\{0\}$, let $x(t) = f^{-1}(\exp(-(1-\lambda)t) f(x)), t \in [0, +\infty)$. It follows from Lemma 2.1 that $x(t)$ is differentiable on $[0, +\infty)$ and

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = -(1-\lambda)\big[Df(x(t))\big]^{-1} f(x(t)).$$

Furthermore, using Lemmas 2.2 and 2.4, we have

$$\begin{aligned}
&\|x(t)\|\Re\lambda - \|x(t)\| \max\{g(\|x(t)\|^k), g(-\|x(t)\|^k)\} \\
&\quad\le -\Re T_{x(t)}\big[(1-\lambda)(Df(x(t)))^{-1} f(x(t))\big] \\
&\quad= \Re T_{x(t)}\left[\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right] = \frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t} \\
&\quad\le \|x(t)\|\Re\lambda - \|x(t)\| \min\{g(\|x(t)\|^k), g(-\|x(t)\|^k)\},
\end{aligned}$$

i.e.

$$\begin{aligned}
&-\big(1-\Re\lambda\big)\|x(t)\|\left[1 - \frac{1-\max\{g(\|x(t)\|^k), g(-\|x(t)\|^k)\}}{1-\Re\lambda}\right] \\
&\quad\le \frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t} \\
&\quad\le -\big(1-\Re\lambda\big)\|x(t)\|\left[1 - \frac{1-\min\{g(\|x(t)\|^k), g(-\|x(t)\|^k)\}}{1-\Re\lambda}\right].
\end{aligned}$$

Note that

$$\frac{1}{\|f(x(t))\|}\frac{\mathrm{d}\|f(x(t))\|}{\mathrm{d}t}\|x(t)\|\left[1 - \frac{1-\max\{g(\|x(t)\|^k), g(-\|x(t)\|^k)\}}{1-\Re\lambda}\right] \tag{2.1}$$

$$\le \frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t}$$

$$\le \frac{1}{\|f(x(t))\|}\frac{\mathrm{d}\|f(x(t))\|}{\mathrm{d}t}\|x(t)\|\left[1 - \frac{1-\min\{g(\|x(t)\|^k), g(-\|x(t)\|^k)\}}{1-\Re\lambda}\right], \tag{2.2}$$

since $\frac{\mathrm{d}\|f(x(t))\|}{\mathrm{d}t} = -(1-\Re\lambda)\|f(x(t))\|$.

By integrating inequality (2.1) on both sides over $t \in [0,\tau]$, we obtain

$$\begin{aligned}
&\int_0^\tau \frac{1}{\|f(x(t))\|}\frac{\mathrm{d}\|f(x(t))\|}{\mathrm{d}t}\,\mathrm{d}t \\
&\quad\le \int_0^\tau \frac{1-\Re\lambda}{\max\{g(\|x(t)\|^k), g(-\|x(t)\|^k)\} - \Re\lambda}\frac{1}{\|x(t)\|}\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t}\,\mathrm{d}t.
\end{aligned}$$

Hence

$$\begin{aligned}
&\log\|f(x(\tau))\| - \log\|f(x)\| \\
&\quad\le \int_0^\tau \left[\frac{1-\Re\lambda}{\max\{g(\|x(t)\|^k), g(-\|x(t)\|^k)\} - \Re\lambda} - 1\right]\frac{1}{\|x(t)\|}\frac{\mathrm{d}\|x(t)\|}{\mathrm{d}t}\,\mathrm{d}t + \log\|x(\tau)\| - \log\|x\|,
\end{aligned}$$

i.e.

$$\log \frac{\|f(x(\tau))\|}{\|x(\tau)\|} \leq \int_{\|x\|}^{\|x(\tau)\|} \left[\frac{1-\Re\lambda}{\max\{g(y^k), g(-y^k)\} - \Re\lambda} - 1\right]\frac{1}{y}\mathrm{d}y + \log\frac{\|f(x)\|}{\|x\|}.$$

Let $\tau \to +\infty$. Then

$$\|f(x)\| \geq \|x\| \exp\left(\int_0^{\|x\|} \left[\frac{1-\Re\lambda}{\max\{g(y^k), g(-y^k)\} - \Re\lambda} - 1\right]\frac{1}{y}\mathrm{d}y\right).$$

Applying analogous arguments to inequality (2.2) yields conclusion

$$\|f(x)\| \leq \|x\| \exp\left(\int_0^{\|x\|} \left[\frac{1-\Re\lambda}{\min\{g(y^k), g(-y^k)\} - \Re\lambda} - 1\right]\frac{\mathrm{d}y}{y}\right).$$

□

Remark 2.6. Let $k = 1$ in Theorem 2.5. We immediately deduce Theorem 3.6 in [12]. The proof of Theorem 2.5 strongly relies on the fact that $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ possesses the geometric properties of spirallike mappings with respect to $(1-\lambda)I$. By utilizing these geometric properties, it is possible to avoid dependence on the reflexivity of Banach spaces, thereby allowing the discussion of related issues in generalized Banach spaces. In a sense, Theorem 2.5 generalizes the scope of Theorem 3.6 in [12].

Remark 2.7. Building upon the method introduced in [9], we establish the sharpness of Theorem 2.5 in the case of $\lambda \leq 0$. More precisely, by constructing particular normalized biholomorphic mappings, we demonstrate that equalities hold in Theorem 2.5. We organize the proof as follows:

(Step 1) Let $b \in \mathscr{S}^*_{g,\lambda}(\mathbb{D})$ satisfy $b(0) = b'(0) - 1 = 0$, and

$$(1-\lambda)\frac{b(\zeta)}{\zeta b'(\zeta)} + \lambda = g(\zeta), \quad \zeta \in \mathbb{D}.$$

For a positive integer k, let $b_k(\zeta) = \zeta\left(\varphi(\zeta^k)\right)^{\frac{1}{k}}$, where $\varphi(\zeta) = \frac{b(\zeta)}{\zeta}$ and $\left(\varphi(\zeta^k)\right)^{\frac{1}{k}}|_{\zeta=0} = 1$. Then $b_k(\zeta) \in \mathscr{S}^*_{g,\lambda}(\mathbb{D})$ such that $\zeta = 0$ is a zero of order $k+1$ of $b_k(\zeta) - \zeta$.

(Step 2) For $u \in \partial\mathscr{B}$, let

$$F_u(x) = \frac{b_k(l_u(x))}{l_u(x)}x, \quad x \in \mathscr{B}. \tag{2.3}$$

Then $F_u(x) \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ such that $x = 0$ is a zero of order $k+1$ of $F_u(x) - x$. In indeed, let $f_u(x) = \frac{b_k(l_u(x))}{l_u(x)}$. Then $F_u(x) = f_u(x)x$. Hence $DF_u(x)\eta = (Df_u(x)\eta)x + f_u(x)\eta$, $\eta \in X$, and

$$(DF_u(x))^{-1}\eta = \frac{1}{f_u(x)}\left[\eta - \frac{(Df_u(x)\eta)x}{f_u(x) + Df_u(x)x}\right], \quad \eta \in X.$$

Elementary operations give

$$(DF_u(x))^{-1}F_u(x) = \frac{1}{f_u(x)}\left[x - \frac{(Df_u(x)x)x}{f_u(x) + Df_u(x)x}\right]f_u(x) = \frac{b_k(l_u(x))}{b_k'(l_u(x))l_u(x)}x.$$

It yields that

$$\frac{1}{\|x\|_X}T_x\left\{(1-\lambda)(DF_u(x))^{-1}F_u(x) + \lambda x\right\} = (1-\lambda)\frac{b_k(l_u(x))}{b_k'(l_u(x))l_u(x)} + \lambda \prec g.$$

(Step 3) Without loss of generality, we take

$$\min_{0\leq r<1}\{g(r), g(-r)\} = g(r), \quad \max_{0\leq r<1}\{g(r), g(-r)\} = g(-r).$$

Let $F_u(x) = \frac{b_k(l_u(x))}{l_u(x)} x$ be as in (2.3). Theorem 2.5 yields the following inequalities

$$\|x\| \exp\left(\int_0^{\|x\|} \left(\frac{y\widetilde{b_k}'(y)}{\widetilde{b_k}(y)} - 1\right)\frac{1}{y}\mathrm{d}y\right) \le \|F_u(x)\| \le \|x\| \exp\left(\int_0^{\|x\|} \left(\frac{yb_k'(y)}{b_k(y)} - 1\right)\frac{1}{y}\mathrm{d}y\right), x \in \mathscr{B},$$

where $\widetilde{b_k}(y) = e^{-\frac{\pi i}{k}} b_k(e^{\frac{\pi i}{k}} y)$. After simple computations, we deduce that

$$\|x\| \exp\left(\log \frac{\widetilde{b_k}(\|x\|)}{\|x\|}\right) \le \|F_u(x)\| \le \|x\| \exp\left(\log \frac{b_k(\|x\|)}{\|x\|}\right), \quad x \in \mathscr{B}.$$

This implies that

$$e^{-\frac{\pi i}{k}} b_k\left(e^{\frac{\pi i}{k}} \|x\|\right) \le \|F_u(x)\| \le b_k(\|x\|), \quad x \in \mathscr{B}. \tag{2.4}$$

With $x = ru$ or $x = e^{\frac{\pi i}{k}} ru$, it follows that $\|F_u(ru)\| = |b_k(r)|$, $\|F_u(e^{\frac{\pi i}{k}} ru)\| = |b_k(e^{\frac{\pi i}{k}} r)|$, we obtain the equalities in (2.4), as desired. This completes the proof.

Let $\lambda = 0$. Theorem 2.5 and Remark 2.7 reduce to the sharp growth theorem for g-starlike mapping f on the unit ball $\mathscr{B}$ such that $f(x) - x$ has a zero of order $k + 1$ at $x = 0$, it is due to Hamada and Honda [9].

Corollary 2.8. *Let $g \in G(\mathbb{D})$, and let $f: \mathscr{B} \to X$ be a g-starlike mapping such that $x = 0$ is a zero of order $k + 1$ of $f(x) - x$. Then*

$$\begin{aligned}
&\|x\| \exp\left(\int_0^{\|x\|} \left[\frac{1}{\max\{g(y^k), g(-y^k)\}} - 1\right]\frac{\mathrm{d}y}{y}\right) \\
&\quad \le \|f(x)\| \\
&\quad \le \|x\| \exp\left(\int_0^{\|x\|} \left[\frac{1}{\min\{g(y^k), g(-y^k)\}} - 1\right]\frac{\mathrm{d}y}{y}\right).
\end{aligned}$$

Moreover, the estimates are sharp.

Let $k = 1$, $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in \mathbb{D}$. Theorem 2.5 reduces to the growth theorems for almost starlike mappings of complex order λ on the unit ball $\mathscr{B}$ due to Zhang, Lu, and Li [13].

Corollary 2.9. *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \le 0$, and let $f: \mathscr{B} \to X$ be a almost starlike mapping of complex order λ. Then*

$$\begin{aligned}
\frac{\|x\|}{\left[1 + \frac{1+\Re\lambda}{1-\Re\lambda}\|x\|\right]^{\frac{2}{1+\Re\lambda}}} \le \ &\|f(x)\| \le \frac{\|x\|}{\left[1 - \frac{1+\Re\lambda}{1-\Re\lambda}\|x\|\right]^{\frac{2}{1+\Re\lambda}}}, && \Re\lambda \ne -1, \\
\|x\| \exp(-\|x\|) \le \ &\|f(x)\| \le \|x\| \exp(\|x\|), && \Re\lambda = -1.
\end{aligned}$$

Let $\lambda = 0$, $g(\zeta) = 1 - \zeta$. Theorem 2.5 and Remark 2.7 immediately yield the following two sharp growth theorems for normalized biholomorphic convex mapping or quasi-convex mapping f such that $x = 0$ is a zero of order $k+1$ of $f(x)-x$ (cf. [17,23]). Note that $x=0$ is a zero of order m of $f(x)-x$ for some m with $m \geq k+1$, since f is $k-$ fold symmetry. It follows naturally that Corollary 2.10 and Theorem C produce identical results.

Corollary 2.10. *Let $f\colon \mathscr{B} \to X$ be a normalized biholomorphic convex mapping such that $x=0$ is a zero of order $k+1$ of $f(x)-x$. Then, for any point $x \in \mathscr{B}$, we have*

$$\frac{\|x\|}{(1+\|x\|^k)^{1/k}} \leq \|f(x)\| \leq \frac{\|x\|}{(1-\|x\|^k)^{1/k}}.$$

Moreover, the estimates are sharp.

Corollary 2.11. *Let $f\colon \mathscr{B} \to X$ be a normalized biholomorphic quasi-convex mapping such that $x=0$ is a zero of order $k+1$ of $f(x)-x$. Then, for any point $x \in \mathscr{B}$, we have*

$$\frac{\|x\|}{(1+\|x\|^k)^{1/k}} \leq \|f(x)\| \leq \frac{\|x\|}{(1-\|x\|^k)^{1/k}}.$$

Moreover, the estimates are sharp.

Remark 2.12. Based on the fact in Page 4 concerning the relationships among the family of g-starlike mappings of complex order λ, subfamilies of starlike mappings and subfamilies of spirallike mappings, taking $k=1$ in Theorem 2.5 leads to rigorous derivations of the classical results. For example, the growth theorems of strongly starlike mappings of order $\alpha \in (0,1]$, Janowski-starlike mappings of complex order λ, almost starlike mappings of order $\alpha \in [0,1)$, etc.

3 Coefficient bounds for $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$

As is well-known in geometric function theory, the Bieberbach conjecture holds for normalized biholomorphic functions on the unit disk $\mathbb{D}$, which states that:

Bieberbach Conjecture. Let $f(\zeta)=\zeta+\sum_{n=2}^{\infty} a_n \zeta^n$ be normalized biholomorphic function on $\mathbb{D}$. Then

$$|a_n| \leq n, \quad n=2,3,\dots.$$

Equality $|a_n|=n$ for a given $n \geq 2$ holds if and only if $f(\zeta)=\frac{\zeta}{(1-e^{i\theta}\zeta)^2}, \theta \in \mathbb{R}$.

However, the corresponding result fails to hold for normalized biholomorphic mappings in higher-dimensional complex spaces $\mathbb{C}^n, (n \geq 2)$. As a natural higher-dimensional generalization of the Bieberbach conjecture, theoretically important research developments have been made for normalized biholomorphic mappings under additional constraints, such as starlikeness or the existence of parametric representations. The specific achievements regarding coefficient bounds for subclasses of normalized biholomorphic mappings are well-documented in [9,11,18,23,24]. We now present a unified representation of known coefficient bounds by establishing the bounds for coefficients of $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ with $x=0$ is a zero of order $k+1$ of $f(x)-x$.

As preparation for our main results, we first record the following fundamental lemma in one complex variable.

Lemma 3.1. *Let $f \in H(\mathbb{D}), g$ be a biholomorphic function on $\mathbb{D}$ with $f(0)=g(0)$. If $f'(0)=\dots=f^{(k-1)}(0)=0, f^{(k)}(0) \neq 0$ and $f \prec g$, then*

$$\frac{|f^{(k)}(0)|}{k!} \leq |g'(0)|.$$

Proof. Let $f(0)=g(0)=a$. And let $f(\zeta)=a+\zeta^k p(\zeta), \zeta \in \mathbb{D}$. Where p is a holomorphic function on a neighborhood of 0 and $p(0) \neq 0$. Let $h(\zeta)=g^{-1}(f(\zeta))$. Since $f \prec g$, we know that $h\colon \mathbb{D} \to \mathbb{D}$ satisties $h(0)=0$. Since

$g^{-1}(a) = 0$, there exists a holomorphic function $G(w)$ on a neighborhood of a such that $g^{-1}(w) = (w-a)G(w)$ and $G(a) \neq 0$. Therefore, we obtain $h(\zeta) = g^{-1}(f(\zeta)) = \zeta^k p(\zeta) G(f(\zeta))$ on a neighborhood of 0. From the Schwarz lemma it follows that $|\frac{h^{(k)}(0)}{k!}| \leq 1$, i.e.

$$\frac{|f^{(k)}(0)|}{k!} \leq |g'(0)|.$$

□

We use in our paper the following result due to Rogosinski.

Lemma 3.2. [25] *Let $f(\zeta) = 1 + \sum_{j=1}^{\infty} b_j \zeta^j$, $g(\zeta) = 1 + \sum_{j=1}^{\infty} c_j \zeta^j$ be holomorphic functions on the unit disk $\mathbb{D}$, and let g is convex on $\mathbb{D}$. If $f \prec g$, then*

$$|b_j| \leq |c_1|, \quad j = 1, 2 \ldots .$$

Now, the bound for the $(k+1)$th order coefficients of $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ is established as follows.

Theorem 3.3. *Let $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ be a normalized biholomorphic mapping such that $x = 0$ is a zero of order $k+1$ of $f(x) - x$. Then*

$$\left|(1-\lambda)\frac{k}{(k+1)!} l_u\{D^{(k+1)} f(0)(u^{k+1})\}\right| \leq |g'(0)|, \quad u \in \partial\mathscr{B}.$$

Proof. Note that

$$f(x) = x + \frac{1}{(k+1)!} D^{(k+1)} f(0)(x^{k+1}) + \cdots, \quad x \in \mathscr{B},$$

since $x = 0$ is a zero of order $k+1$ of $f(x) - x$. Taking $h(x) = (Df(x))^{-1} f(x)$, Then

$$h(x) = x - \frac{k}{(k+1)!} D^{(k+1)} f(0)(x^{k+1}) + \cdots, \quad x \in \mathscr{B}.$$

Fix $u \in \partial\mathscr{B}$, $l_u \in T(u)$. Let

$$p(\zeta) = \begin{cases} (1-\lambda)\frac{1}{\zeta} l_u\{h(\zeta u)\} + \lambda, & \zeta \in \mathbb{D}\backslash\{0\}, \\ 1, & \zeta = 0. \end{cases}$$

Then $p \in H(\mathbb{D})$ and has the Taylor expansion

$$p(\zeta) = 1 - (1-\lambda)\frac{k}{(k+1)!} l_u\{D^{(k+1)} f(0)(u^{k+1})\}\zeta^k + \cdots, \quad \zeta \in \mathbb{D}.$$

Straightforward calculation shows that

$$p^{(k)}(0) = -(1-\lambda)\frac{k}{k+1} l_u\{D^{(k+1)} f(0)(u^{k+1})\},$$

Since $p(0) = g(0) = 1$ and $p \prec g$, using Lemma 3.1, it holds

$$\left|(1-\lambda)\frac{k}{(k+1)!} l_u\{D^{(k+1)} f(0)(u^{k+1})\}\right| \leq |g'(0)|.$$

□

Remark 3.4. Now, we establish the sharpness of Theorem 3.3 in the case of $\lambda \leq 0$. In fact, let b_k and F_u be as defined in Remark 2.7. A straightforward calculation shows that

$$b_k(\zeta) = \zeta - \frac{1}{1-\lambda}\frac{1}{k} g'(0)\zeta^{k+1} \ldots, \quad \zeta \in \mathbb{D},$$

and

$$F_u(\zeta u) = b_k(\zeta)u, \quad \zeta \in \mathbb{D}, \quad u \in \partial\mathscr{B}.$$

Hence,

$$\left|(1-\lambda)\frac{k}{(k+1)!}l_u\{D^{(k+1)}F_u(0)(u^{k+1})\}\right| = |g'(0)|.$$

This completes the proof.

The convexity condition on g leads to significant improvements in our estimates, as demonstrated by the following.

Theorem 3.5. *Let $g \in G(\mathbb{D})$ be a convex function, and let $f \in \mathscr{S}^*_{g,\lambda}(\mathscr{B})$ be a normalized biholomorphic mapping such that $x = 0$ is a zero of order $k+1$ of $f(x) - x$. Then*

$$\left|(1-\lambda)\frac{1}{m!}l_x\{D^{(m)}f(0)(x^m)\}\right| \le \frac{1}{m-1}|g'(0)|\|x\|^m,$$

for $m = k+1, \dots, 2k$.

Proof. Fix $w \in X \backslash \{0\}$, let $w_0 = \frac{w}{\|w\|} \in \partial\mathscr{B}$ and

$$p(\zeta) = \begin{cases} (1-\lambda)\frac{1}{\zeta}l_{w_0}\{(Df(\zeta w_0))^{-1}f(\zeta w_0)\} + \lambda, & \zeta \in \mathbb{D}\backslash\{0\}, \\ 1, & \zeta = 0. \end{cases}$$

Then $p(0) = g(0) = 1$ and $p \prec g$. Using the convexity of g and Lemma 3.2, we obtain

$$\left|\frac{p^{(m)}(0)}{m!}\right| \le |g'(0)|, \quad m \ge 1.$$

For a given $x \in \mathscr{B}$, let $h(x) = (Df(x))^{-1}f(x)$. Then $f(x) = Df(x)h(x)$. Since $f(x) - x$ has a zero of order $k+1$ at $x = 0$, we have

$$\begin{aligned} x &+ \frac{D^{(k+1)}f(0)(x^{k+1})}{(k+1)!} + \cdots + \frac{D^{(m)}f(0)(x^m)}{m!} + \cdots \\ &= \left(I + \frac{D^{(k+1)}f(0)(x^k,\cdot)}{k!} + \cdots + \frac{D^{(m)}f(0)(x^{m-1},\cdot)}{(m-1)!} + \cdots\right) \\ &\quad \times \left(Dh(0)x + \frac{D^{(2)}h(0)(x^2)}{2!} + \cdots + \frac{D^{(k+1)}h(0)(x^{k+1})}{(k+1)!} + \cdots + \frac{D^{(m)}h(0)(x^m)}{m!} + \cdots\right). \end{aligned}$$

Comparing the coefficients on both sides of the equation, we have

$$Dh(0)x = x, \quad D^{(l)}h(0)(x^l) = 0, \quad l = 2, \dots, k. \tag{3.1}$$

Thus

$$\frac{D^{(m)}f(0)(x^m)}{m!} = \frac{D^{(m)}h(0)(x^m)}{m!} + \frac{D^{(m)}f(0)(x^m)}{(m-1)!}, \quad m = k+1, \dots, 2k,$$

i.e.

$$\frac{D^{(m)}f(0)(x^m)}{m!} = -\frac{1}{m-1}\frac{D^{(m)}h(0)(x^m)}{m!}, \quad m = k+1, \dots, 2k.$$

Furthermore, by the defifiniton of p and equations (3.1), p has the Taylor expansion

$$\frac{1}{m!}p^{(m)}(0) = -(1-\lambda)\frac{m-1}{m!}l_{w_0}\{D^{(k+1)}f(0)(w_0^m)\}, \quad m = k+1, \dots, 2k,$$

i.e.

$$\frac{1}{m!}p^{(m)}(0) = -(1-\lambda)\frac{m-1}{m!}l_x\{D^{(k+1)}f(0)(x^m/\|x\|^m)\}, \quad m = k+1, \ldots, 2k.$$

It yields that

$$\left|(1-\lambda)\frac{1}{m!}l_x\{D^{(k+1)}f(0)(x^m)\}\right| \leq \frac{1}{m-1}\left|g'(0)\right|\|x\|^m, \quad m = k+1, \ldots, 2k.$$

□

Let $\lambda = 0$. We can get Theorem 4.2 in [9].

Corollary 3.6. *Let $g \in G(\mathbb{D})$ be a convex function, and let $f : \mathscr{B} \to X$ be a g-starlike mapping such that $x = 0$ is a zero of order $k + 1$ of $f(x) - x$. Then*

$$\left|\frac{1}{m!}l_x\{D^{(m)}f(0)(x^m)\}\right| \leq \frac{1}{m-1}\left|g'(0)\right|\|x\|^m,$$

for $m = k + 1, \ldots, 2k$. And the estimate is sharp when $m = k + 1$.

If $\lambda = 0, g(\zeta) = 1 - \zeta, \zeta \in \mathbb{D}$, then the following result for normalized biholomorphic quasi-convex mapping can be obtained.

Corollary 3.7. *Let $f : \mathscr{B} \to X$ be a quasi-convex mapping such that $x = 0$ is a zero of order $k + 1$ of $f(x) - x$. Then*

$$\left|\frac{1}{m!}l_x\{D^{(k+1)}f(0)(x^m)\}\right| \leq \frac{1}{m-1}\|x\|^m,$$

for $m = k + 1, \ldots, 2k$. And the estimate is sharp when $m = k + 1$.

Remark 3.8. Based on the fact in Page 4 concerning the relationships among the family of g-starlike mapping of complex order λ, subfamilies of starlike mappings and subfamilies of spirallike mappings, as an immediate consequence of Theorem 3.5, we recover the known results in [9,18].

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