

Research Article

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A kind of univariate improved Shepard-Euler operators

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Abstract: In this paper, a kind of univariate Shepard-Euler operators is studied by combining the known Shepard operator with the generalized Taylor polynomial as the expansion in the Euler polynomials. For practical purposes, another kind of improved Shepard-Euler operators without any derivative of the approximated function f is given by using divided differences to approximate the derivatives. Some error bounds and convergence rates of the combined operators are studied. Finally, some numerical experiments are shown to compare the approximation capacity of our operators with that of Caira-Dell'Accio's scheme. Furthermore, there is no demand for the derivatives of f in the proposed operator, so it does not increase the orders of smoothness of f .

Keywords: Shepard operator; Euler polynomials; convergence rates; the generalized Taylor polynomial

MSC 2020: 41A05; 65D05; 65D15

1 Introduction

The classical Shepard operator, first introduced in [1], is a well suited operator for two-dimensional interpolation of very large scattered data sets. Let f be a real valued function defined on $X \subset \mathbb{R}$. Let $X = \{x_i\}_{i=1}^N$ be a set of some distinct points. The Shepard operator in the univariate case is defined by

$$S_{N,\mu}[f](x) = \sum_{i=1}^N A_{\mu,i}(x) f(x_i), \quad \mu > 0, \quad (1)$$

where

$$A_{\mu,i}(x) = \frac{|x - x_i|^{-\mu}}{\sum_{k=1}^N |x - x_k|^{-\mu}} \quad (2)$$

and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}$. It is easy to check that

$$A_{\mu,i}(x_v) = \delta_{i,v}, \quad i, v = 1, 2, \dots, N, \quad (3)$$

and

$$\sum_{i=1}^N A_{\mu,i}(x) = 1. \quad (4)$$

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Because $S_{N,\mu}[f](x)$ reproduces only constant functions, so Shepard has suggested to apply $S_{N,\mu}[f](x)$ not directly to the $f(x_i)$, but to the Taylor polynomial of f of degree 1 at x_i . In this case the combined operator has the degree of exactness 1 in [1].

To increase the degree of exactness of the Shepard operator, several combined operators have been introduced and studied on Taylor [1–5], Lagrange [6], Hermite [7], Birkhoff [8], and Bernoulli [9]. The Shepard method can also refer to recent developments on the subject, see [10–16] for details.

Based on the idea in [9], we first combine the Shepard operator $S_{N,\mu}[f](x)$ in [1] with the generalized Taylor polynomial, the Euler-based expansion as one instance of two-point generalized Taylor polynomials introduced in [17] to obtain a kind of Shepard-Euler operators. The proposed Shepard-Euler operator $\tilde{S}_{E_m}$ possesses good reproduction qualities and high accuracy just like the Shepard-Bernoulli operator [9]. However, they involve the derivatives of f at every node. For practical purposes, applying the divided difference formula in [18] to the proposed operator $\tilde{S}_{E_m}$, we present another kind of Shepard-Euler operators S_{E_m} which do not require values of the derivatives at nodes. We show that the new operators S_{E_m} and $\tilde{S}_{E_m}$ could reproduce all polynomials of degree $\leq m$, and give the convergence rate of $\mathcal{O}(h^{m+1})$. Further, the constructed operator S_{E_m} could provide the desired smoothness and precision in the practical applications.

The organization of the remainder of this paper is as follows. In Section 2, we recall the definition of univariate Euler polynomials and give three useful theorems for the error of approximation that will be used later in the paper. In Section 3, we apply the previous results to derive a kind of Shepard-Euler operators with derivatives, and prove their convergence rates. In Section 4, another kind of improved Shepard-Euler operators without derivatives is provided. In Section 5, numerical examples are shown to demonstrate the accuracy of the proposed combination in some special situations. In Section 6, we give the main conclusions.

2 Some remarks about the generalized Taylor polynomial

The generalized Taylor polynomial is an expansion in the Euler polynomials $E_n(x)$, i.e., the polynomials of the sequence defined recursively by means of the following relations, see [19]

$$\begin{cases} E_0(x) = 1, \\ E'_n(x) = nE_{n-1}(x), & n \geq 1, \\ E_n(x+1) + E_n(x) = 2x^n, & n \geq 1. \end{cases} \quad (5)$$

For functions in the class $C^m([a, b])$, $a, b \in \mathbb{R}$, $a < b$, this expansion is realized by the following equation

$$f(x) = P_m^{<E>}[f; a, b](x) + R_m^{<E>}[f; a, b](x), \quad x \in [a, b], \quad (6)$$

where the polynomial expansion $P_m^{<E>}[f; a, b](x)$ in Euler polynomials is defined by

$$P_m^{<E>}(x) = \sum_{k=0}^m \frac{f^{(k)}(a) + f^{(k)}(b)}{2k!} h^k E_k\left(\frac{x-a}{h}\right) \quad (7)$$

and the remainder term $R_m^{<E>}[f; a, b](x)$ in its Peano's representation is given by:

$$R_m^{<E>}[f; a, b](x) = \frac{1}{(m-1)!} \int_a^b f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt, \quad (8)$$

where

$$K_{a,b}^{<E>}(x) = \begin{cases} -\frac{1}{2} \sum_{k=0}^m \binom{m-1}{k} h^k E_k \left(\frac{x-a}{h} \right) (b-t)^{m-1-k}, & x \leq t, \\ \frac{1}{2} \sum_{k=0}^m \binom{m-1}{k} h^k E_k \left(\frac{x-a}{h} \right) (a-t)^{m-1-k}, & x \geq t, \end{cases} \quad (9)$$

with $h = b - a$. The polynomial approximant $P_m^{<E>}[f; a, b](x)$ is derived from a nice property as follows:

$$\lim_{h \rightarrow 0} P_m^{<E>}[f; a, b](x) = T_m[f; a](x), \quad (10)$$

where $T_m[f; a](x)$ is the m th Taylor polynomial of f about a . Therefore, the expansion $P_m^{<E>}[f; a, b](x)$ in Euler polynomials is called the generalized Taylor polynomial.

To obtain bounds for the remainder $R_m^{<E>}[f; a, b](x)$ from the formula (8) even in points outside the interval $[a, b]$, we investigate the operator

$$f \rightarrow P_m^{<E>}[f; a, b],$$

where $f \in C^m[c, d]$ with $c < a$ and $b < d$. By using the Peano's kernel theorem [20], we provide the integral expression for the remainder (8) as follows.

Theorem 1. Let $f \in C^m[c, d]$ and $x \in [c, d]$, then for the remainder

$$R_m^{<E>}[f; a, b](x) = f(x) - P_m^{<E>}[f; a, b](x) \quad (11)$$

we have the following integral representations

$$R_m^{<E>}[f; a, b](x) = \begin{cases} \frac{1}{(m-1)!} \int_a^b f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt, & c \leq x \leq a, \\ \frac{1}{(m-1)!} \int_a^b f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt, & a \leq x \leq b, \\ \frac{1}{(m-1)!} \int_a^b f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt, & b \leq x \leq d, \end{cases} \quad (12)$$

where

$$K_{a,b}(x, t) = (x - t)_+^{m-1} - \sum_{k=0}^m \frac{(m-1)!}{2(m-1-k)!k!} [(a-t)_+^{m-1-k} + (b-t)_+^{m-1-k}] h^k E_k \left(\frac{x-a}{h} \right) \quad (13)$$

and $(\cdot)_+^k$ denotes the positive part of the k th power of the argument, i.e.,

$$(s)_+^k = \max\{s^k, 0\}. \quad (14)$$

Proof. On the one hand, in the polynomial approximation term (7) there are evaluations of derivatives of f up to the order m on points a and b of $[c, d]$; on the other hand, the exactness of the polynomial approximant (9) on the space $\mathbb{P}^m$ implies the exactness of the operator on the subspace $\mathbb{P}^{m-1}$. Peano's kernel theorem provides the following expression for the remainder (12)

$$R_m^{<E>}[f; a, b](x) = \frac{1}{(m-1)!} \int_c^d f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt, \quad (15)$$

where (13) is given by applying the linear functional $f \rightarrow R_m^{<E>}[f; a, b](x)$ to a function $(x - t)_+^{m-1}$ in x . If $x \in [c, a]$, then

$$\begin{aligned} R_m^{<E>}[f; a, b](x) &= \frac{1}{(m-1)!} \int_c^x f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt + \frac{1}{(m-1)!} \int_x^a f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt \\ &\quad + \frac{1}{(m-1)!} \int_a^b f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt + \frac{1}{(m-1)!} \int_b^d f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt. \end{aligned} \quad (16)$$

If $t \in [c, x]$, then

$$\begin{aligned} K_{a,b}^{<E>}(x, t) &= (x - t)^{m-1} \\ &\quad - \sum_{k=0}^m \frac{(m-1)!}{2(m-1-k)!k!} [(a-t)_+^{m-1-k} + (b-t)_+^{m-1-k}] h^k E_k\left(\frac{x-a}{h}\right) = 0, \end{aligned} \quad (17)$$

where $(x - t)^{m-1}$ is considered as a polynomial in x of degree $m - 1$.

If $t \in [b, d]$, then

$$K_{a,b}(x, t) = 0.$$

Thus, we now have proven the first case of (12). The remaining cases of (12) can be proved in an analogous manner. $\square$

By Theorem 1, we can obtain the following result.

Theorem 2. If $f \in C^m[c, d]$ and $x \in [c, d]$, then for the remainder we have

$$|R_m^{<E>}[f; a, b](x)| \leq \begin{cases} C^{<E>}(m) \|f^{(m)}\|_{\infty} (b-x)^m, & c < x < a, \\ C^{<E>}(m) \|f^{(m)}\|_{\infty} (b-a)^m, & a < x < b, \\ C^{<E>}(m) \|f^{(m)}\|_{\infty} (x-a)^m, & b < x < d, \end{cases} \quad (18)$$

where $\|\cdot\|_{\infty}$ denotes the sup-norm on $[c, d]$ and

$$C^{<E>}(m) = \frac{1}{m!} \sum_{k=0}^m \sum_{l=0}^k \binom{m}{k} \binom{k}{l} \left| E_l\left(\frac{1}{2}\right) \right|, m = 0, 1, \dots \quad (19)$$

Proof. Let $c < x < a$, then we have from the first case of (12) that

$$\begin{aligned} R_m^{<E>}[f; a, b](x) &= \frac{1}{m!} \int_x^a f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt + \frac{1}{m!} \int_a^b f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt. \end{aligned} \quad (20)$$

Let $x < t < a$, then

$$\begin{aligned} K_{a,b}^{<E>}(x, t) &= - \sum_{k=0}^m \frac{(m-1)!}{2(m-1-k)!k!} [(a-t)^{m-1-k} + (b-t)^{m-1-k}] h^k E_k\left(\frac{x-a}{h}\right) \end{aligned}$$

so that

$$\begin{aligned} \int_x^a f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt &= - \sum_{k=0}^m \frac{(m-1)!}{2(m-1-k)!k!} h^k E_k\left(\frac{x-a}{h}\right) \\ &\quad \times \int_x^a [(a-t)^{m-1-k} + (b-t)^{m-1-k}] f^{(m)}(t) dt. \end{aligned} \quad (21)$$

Note that the integrands are of type $h(t)f^{(m)}(t)$ with a $h(t)$ that does not change sign in $[x, a]$. By applying the first mean value theorem for integrals, we find for some $\xi_k \in [c, d]$, $k = 0, 1, \dots, m$, that

$$\begin{aligned} \int_x^a f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt &= - \sum_{k=0}^m \frac{(m-1)!}{2(m-1-k)!k!} h^k E_k\left(\frac{x-a}{h}\right) \\ &\quad \times f^{(m)}(\xi_k) \int_x^a [(a-t)^{m-1-k} + (b-t)^{m-1-k}] dt \\ &= - \sum_{k=0}^m \frac{(m-1)!}{2(m-k)!k!} h^k E_k\left(\frac{x-a}{h}\right) \\ &\quad \times f^{(m)}(\xi_k) [-(b-a)^{m-k} + (b-x)^{m-k} + (a-x)^{m-k}] \\ &= -h^m \sum_{k=0}^m \frac{(m-1)!}{2(m-k)!k!} E_k\left(\frac{x-a}{h}\right) \\ &\quad \times f^{(m)}(\xi_k) \left[-1 + \left(\frac{b-x}{h}\right)^{m-k} + \left(\frac{a-x}{h}\right)^{m-k} \right]. \end{aligned} \quad (22)$$

If $a < t < b$, then

$$K_{a,b}^{<E>}[f; a, b](x) = - \sum_{k=0}^m \frac{(m-1)!}{2(m-1-k)!k!} h^k E_k\left(\frac{x-a}{h}\right) (b-t)^{m-1-k} \quad (23)$$

and

$$\begin{aligned} \int_a^b K_{a,b}^{<E>}(x, t) f^{(m)}(t) dt \\ = - \sum_{k=0}^m \frac{(m-1)!}{2(m-1-k)!k!} h^k E_k\left(\frac{x-a}{h}\right) \int_a^b (b-t)^{m-1-k} f^{(m)}(t) dt. \end{aligned}$$

Based on the first mean value theorem for integrals, we get for some $\beta_k \in [c, d]$, $k = 0, 1, \dots, m$, that

$$\begin{aligned} \int_a^b f^{(m)}(t) K_{a,b}^{<E>}(x, t) dt \\ = - \sum_{k=0}^m \frac{(m-1)!}{2(m-1-k)!k!} h^k E_k\left(\frac{x-a}{h}\right) \times f^{(m)}(\beta_k) \int_a^b (b-t)^{m-1-k} dt \\ = - \sum_{k=0}^m \frac{(m-1)!}{2(m-k)!k!} h^k E_k\left(\frac{x-a}{h}\right) \times f^{(m)}(\beta_k) (b-a)^{m-k} \\ = -h^m \sum_{k=0}^m \frac{(m-1)!}{2(m-k)!k!} E_k\left(\frac{x-a}{h}\right) \times f^{(m)}(\beta_k). \end{aligned} \quad (24)$$

Substituting into (20) the left-hand sides of (22), (24) with their respective right-hand sides, we finally give after some calculations

$$\begin{aligned} R_m^{<E>}[f; a, b](x) &= -\frac{h^m}{m!} \sum_{k=0}^m f^{(m)}(\xi_k) \frac{m!}{2(m-k)!k!} E_k\left(\frac{x-a}{h}\right) \\ &\quad \times \left[-1 + \left(\frac{b-x}{h}\right)^{m-k} + \left(\frac{a-x}{h}\right)^{m-k} \right] \\ &\quad - \frac{h^m}{m!} f^{(m)}(\beta_k) \frac{m!}{2(m-k)!k!} E_k\left(\frac{x-a}{h}\right) \end{aligned}$$

and

$$\left| R_m^{<E>}[f; a, b](x) \right| \leq h^m \frac{\|f^{(m)}\|}{m!} \sum_{k=0}^m \frac{m!}{2(m-k)!k!} \left| E_k\left(\frac{x-a}{h}\right) \right| \times 2 \left(\frac{b-x}{h}\right)^{m-k}.$$

In [21], we have the following known identity:

$$E_k(x+y) = \sum_{l=0}^k \binom{k}{l} E_k(x) y^{k-l}, \quad k = 0, 1, \dots, m. \quad (25)$$

From the relations (25) we can easily deduce the following formula:

$$E_k\left(\frac{x-a}{h}\right) = \sum_{l=0}^k \binom{k}{l} E_l\left(\frac{1}{2}\right) \left(\frac{x-a}{h} - \frac{1}{2}\right)^{k-l}, \quad (26)$$

so that we get

$$\begin{aligned} &\left| R_m^{<E>}[f; a, b](x) \right| \\ &\leq h^m \frac{\|f^{(m)}\|}{m!} \sum_{k=0}^m \frac{m!}{(m-k)!k!} \left(\frac{b-x}{h}\right)^{m-k} \sum_{l=0}^k \binom{k}{l} \left| E_l\left(\frac{1}{2}\right) \right| \left| \left(\frac{b-x}{h} - \frac{1}{2}\right)^{k-l} \right| \\ &\leq h^m \frac{\|f^{(m)}\|}{m!} \sum_{k=0}^m \sum_{l=0}^k \binom{m}{k} \binom{k}{l} \left| E_l\left(\frac{1}{2}\right) \right| \left(\frac{b-x}{h}\right)^m. \end{aligned} \quad (27)$$

Similarly, we can prove the remaining expressions of (18). $\square$

Since the algebraic degree of exactness of the operator $P_m^{<E>}[\cdot; a, b]$ is equal to m , we can prove the following desired bounds in an analogous manner.

Theorem 3. *If $f \in C^{m+1}[c, d]$ and $x \in [c, d]$, then for the remainder we have*

$$\left| R_m^{<E>}[f; a, b](x) \right| \leq \begin{cases} C^{<E>}(m+1) \|f^{(m+1)}\|_{\infty} (b-x)^{m+1}, & c < x < a, \\ C^{<E>}(m+1) \|f^{(m+1)}\|_{\infty} (b-a)^{m+1}, & a < x < b, \\ C^{<E>}(m+1) \|f^{(m+1)}\|_{\infty} (x-a)^{m+1}, & b < x < d, \end{cases} \quad (28)$$

where

$$C^{<E>}(m+1) = \frac{1}{(m+1)!} \sum_{k=0}^m \sum_{l=0}^k \binom{m+1}{k} \binom{k}{l} \left| E_l\left(\frac{1}{2}\right) \right|. \quad (29)$$

3 A kind of Shepard-Euler operators with derivatives

Suppose that $x_1 < x_2 < \dots < x_{N-1} < x_N$ are fixed points in an interval $I = [x_1, x_N] \subset \mathbb{R}$ and $x_{N+1} = x_{N-1}$. For each fixed $\mu > 0$ and $m = 1, 2, \dots$, by combining the Shepard operator with the extension in the Euler polynomials, we first construct a kind of Shepard-Euler operators $\tilde{S}_{E_m}$ with derivatives of function f at endpoints as follows

$$\tilde{S}_{E_m}[f](x) = \sum_{i=1}^N A_{\mu,i}(x) P_m^{<E>}[f; x_i, x_{i+1}](x), \quad x \in I, \quad (30)$$

where $P_m^{<E>}[f; x_i, x_{i+1}](x)$ is the natural extension of the polynomial approximation term defined in (7).

Theorem 4. *The operator $\tilde{S}_{E_m}$ reproduces all univariate polynomials of degree no more than m .*

Proof. The argument $\tilde{S}_{E_m}[p] = p$ follows from the well-known property

$$\sum_{i=1}^N A_{\mu,i}(x) = 1 \quad (31)$$

and

$$P_m^{<E>}[p; x_i, x_{i+1}](x) = p \quad \text{for } i = 1, 2, \dots, N,$$

where $p \in \mathbb{P}_m$. □

To study the convergence rates of the two kinds of operators $\tilde{S}_{E_m}$ and S_{E_m} , we make use of the following notations

$$\begin{aligned} I_\rho(x) &= [x - \rho, x + \rho], \quad \rho > 0, \\ r &= \inf\{\rho > 0: \forall x \in I, I_\rho(x) \cap X \neq \emptyset\}, \\ M &= \max_{x \in I} \#(I_r(x) \cap X), \end{aligned}$$

where $X = \{x_1, x_2, \dots, x_N\}$ and $\#(\cdot)$ denotes the cardinality function. So M denotes the maximum number of points from X contained in an interval $I_r(x)$. For the operators $\tilde{S}_{E_m}$ we then give the error estimates as follows.

Theorem 5. *Let $f(x) \in C^m(I)$. Then*

$$\|\tilde{S}_{E_m}[f](x) - f(x)\|_\infty \leq C^{<E>} M \|f^{(m)}\|_\infty \varepsilon_\mu^{m-1}(r), \quad (32)$$

where

$$\varepsilon_\mu^{m-1}(r) = \begin{cases} |\ln r|^{-1}, & \mu = 1, \\ r^{\mu-1}, & 1 < \mu < m+1, \\ r^{\mu-1} |\ln r|, & \mu = m+1, \\ r^m, & \mu > m+1, \end{cases} \quad (33)$$

$C^{(E)}$ is a positive constant independent of x , and X , and r is given above.

Proof. Assume that each pair $x_i, x_{i+1} \in I$ is fixed and suppose $x_i < x_{i+1}$. For each $x \in I$, we make use of the following settings

$$d[x_i, x_{i+1}](x) = \begin{cases} x_{i+1} - x, & x \leq x_i, \\ x_{i+1} - x_i, & x_i \leq x \leq x_{i+1}, \\ x - x_i, & x_{i+1} \leq x, \end{cases} \quad (34)$$

$$d^m[x_i, x_{i+1}](x) = (d[x_i, x_{i+1}](x))^m.$$

Based on (2) and (30), we obtain

$$\begin{aligned} |\tilde{S}_{E_m} - f(x)| &\leq \sum_{i=1}^N A_{\mu,i} P_m^{<E>}[f; x_i, x_{i+1}] - f(x) \\ &\leq \sum_{i=1}^N A_{\mu,i} |P_m^{<E>}[f; x_i, x_{i+1}] - f(x)| \\ &\leq C^{<E>}(m) \|f^{(m)}\|_{\infty} s_{\mu}^m(x), \end{aligned}$$

where

$$s_{\mu}^m(x) = \frac{\sum_{i=1}^N |x - x_i|^{-\mu} d^m[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}}. \quad (35)$$

From [9], we can give the following prove:

$$s_{\mu}^m(x) \leq C^{<E>} M \varepsilon_{\mu}^{m-1}(r).$$

Suppose that

$$n = \left\lceil \frac{x_N - x_0}{2r} \right\rceil + 1,$$

$$Q_{\rho}(u) = (u - \rho, u + \rho], \quad u \in I, \rho > 0,$$

$$T_j = Q_r(x - 2rj) \cup Q_h(x + 2rj), \quad j = 0, 1, \dots, n,$$

where the set $\cup_{j=-n}^n Q_h(x + 2rj)$ denotes the covering of I with half open intervals. Thus, for every $i \in \{1, 2, \dots, N\}$ there exists a unique $j \in \{0, 1, \dots, n\}$ such that $x_i \in T_j$. Then, we obtain the following inequalities

$$\begin{aligned} (2j-1)r &\leq |x - x_i| \leq (2j+1)r, \\ (2(j-1)-1)r &\leq |x - \tau_i| \leq (2(j+1)+1)r, \end{aligned} \quad (36)$$

where $j = 2, 3, \dots, n$ and $\tau_i \in [x_{i-1}, x_{i+1}]$. Therefore, we find from (34)

$$d[x_i, x_{i+1}](x) \leq (2(j+1)+1)r, \quad (37)$$

On the other hand, we also find from the definition of M

$$\begin{aligned} 1 &\leq \#(X \cap T_0) \leq M, \\ 1 &\leq \#(X \cap T_j) \leq 2M, \quad j = 1, 2, \dots, n. \end{aligned} \quad (38)$$

Let us denote by x_d the node closest to x_i since

$$\frac{|x - x_d|^{-\mu}}{\sum_{k=1}^N |x - x_k|^{-\mu}} \leq 1.$$

By applying (36) and (37), we have

$$\begin{aligned}
 s_{\mu}^m(x) &\leq \sum_{x_i \in T_0} \frac{|x - x_i|^{-\mu} d^m[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}} + \sum_{j=1}^n \sum_{x_i \in T_j} \frac{|x - x_i|^{-\mu} d^m[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}} \\
 &\leq \sum_{x_i \in T_0} d^m[x_i, x_{i+1}](x) + |x - x_d|^{\mu} \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-\mu} d^m[x_i, x_{i+1}](x) \\
 &\leq M(3r)^m + 2Mr^{\mu} \sum_{j=1}^n ((2j-1)r)^{-\mu} ((2j+3)r)^m \\
 &\leq M5^m r^m \left(1 + 2 \sum_{j=1}^n j^{m-\mu} \right),
 \end{aligned}$$

where the last inequality follows from

$$\begin{cases} 2j-1 \geq j, & j = 1, 2, \dots, \\ 2j+3 \leq 5j, & j = 1, 2, \dots \end{cases}$$

Case 1: ($\mu > 1$)

If $1 < \mu < m+1$, then

$$r^m \left(1 + 2 \sum_{j=1}^n j^{m-\mu} \right) = \mathcal{O}(r^{\mu-1}).$$

If $\mu = m+1$, then

$$\sum_{j=1}^n j^{m-\mu} = \mathcal{O}(|\ln r|).$$

If $\mu > m+1$, then $\sum_{j=1}^n j^{m-\mu}$ is bounded.

Case 2: ($\mu = 1$)

$$\begin{aligned}
 s_u^m(x) &= s_1^m(x) \\
 &\leq \frac{\sum_{i=1}^N |x - x_i|^{-1} d^m[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-1}} \\
 &\leq \frac{\sum_{x_i \in T_0} d^m[x_i, x_{i+1}](x) + \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-1} d^m[x_i, x_{i+1}](x)}{\sum_{x_k \in T_0} |x - x_k|^{-1} + \sum_{j=1}^n \sum_{x_k \in T_j} |x - x_k|^{-1}} \\
 &\leq \frac{\sum_{x_i \in T_0} d^m[x_i, x_{i+1}](x) + \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-1} d^m[x_i, x_{i+1}](x)}{\sum_{x_k \in T_0} |x - x_k|^{-1} + \frac{C}{r} |\ln r|} \\
 &\leq \sum_{x_i \in T_0} d^m[x_i, x_{i+1}](x) + \frac{C_1 r}{|\ln r|} \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-1} d^m[x_i, x_{i+1}](x)
 \end{aligned}$$

$$\begin{aligned}
&\leq M(3r)^m + \frac{C_1}{|\ln r|} \sum_{j=1}^n \sum_{x_i \in T_j} (2j-1)^{-1} (2j+3)^m r^m \\
&\leq M5^m r^m \left(1 + \frac{C_2}{|\ln r|} \sum_{j=1}^n j^{m-1} \right) \\
&\leq M5^m r^m \left(1 + \frac{C_2}{|\ln r|} \mathcal{O}(r^{-m}) \right) \\
&= \mathcal{O}(|\ln r|^{-1}).
\end{aligned}$$

□

In an analogous manner we can prove the following theorem.

Theorem 6. Let $f(x) \in C^{m+1}(I)$. Then

$$\|\tilde{S}_{E_m}[f](x) - f(x)\|_\infty \leq C^{<E>} M \|f^{(m+1)}\|_\infty \varepsilon_\mu^m(r), \quad (39)$$

where

$$\varepsilon_\mu^m(r) = \begin{cases} |\ln r|^{-1}, & \mu = 1, \\ r^{\mu-1}, & 1 < \mu < m+2, \\ r^{\mu-1} |\ln r|, & \mu = m+2, \\ r^{m+1}, & \mu > m+2, \end{cases} \quad (40)$$

and $C^{(E)}$ is a positive constant independent of x and X .

Because of disadvantage with the derivatives in the operator $\tilde{S}_{E_m}$, we give the following modification operator S_{E_m} .

4 A kind of improved Shepard-Euler operators without derivatives

Although the operator $\tilde{S}_{E_m}$ possesses the degree of exactness not greater than m , they require the derivatives of the function f at the nodes, which are very difficult to measure in practice. By using divided difference operator $D_A^k f$ in the following Definition 1 to substitute the derivatives $f^{(k)}$ in the operator $\tilde{S}_{E_m}$, we define a kind of improved Shepard-Euler operators S_{E_m} without derivatives of function f at endpoints.

Definition 1 (see [18]). Let $\mathcal{F} = \{f: \mathbb{R} \rightarrow \mathbb{R}\}$ and let A be a discrete subset of $\mathbb{R}$, $k \in \mathbb{N}$. Suppose that D^k is the order k derivative. An operator $D_A^k: \mathcal{F} \rightarrow \mathcal{F}$ is said to be a $\mathbb{P}_m$ -exact A -discretization of D^k if and only if

(i) There exists a real vector $\lambda = (\lambda_a)_{a \in A}$ s.t. for any $f \in \mathcal{F}$,

$$D_A^k f(\cdot) = \sum_{a \in A} \lambda_a f(\cdot + a), \quad k = 1, \dots, m; \quad (41)$$

(ii) For any $p \in \mathbb{P}_m$,

$$D_A^k p = D^k p. \quad (42)$$

In such situation, we also say that $D_A^k f$ is a $\mathbb{P}_m$ -exact A -discretization of $D^k f$. Let the points be distinct in the set A , then D_A^k is determined uniquely.

Suppose that $|\cdot|$ denotes the number of elements in set and assume that the points in set A are distinct, and $|A| = m + 1$. Then by Definition 1 and [18], a $\mathbb{P}_m$ -exact A -discretization of the order k derivative $f^{(k)}$ is

$$D_A^k f(x) = \sum_{a \in A} \lambda_a f(x + a), \quad k = 1, 2, \dots, m, \quad (43)$$

where

$$\lambda_a = \begin{cases} \frac{k!}{\prod_{c \in A \setminus \{a\}} (a - c)}, & k = m, \\ \frac{(-1)^{m-k} k! \sum_{\substack{A' \subset A \setminus \{a\} \\ |A'| = m-k}} \prod_{c \in A'} (c)}{\prod_{c \in A \setminus \{a\}} (a - c)}, & k < m. \end{cases} \quad (44)$$

According to the location of each pair x_i, x_{i+1} ($i = 1, 2, \dots, N$), we choose suitable sets A_i , and substitute $f^{(k)}(x_i)$, $f^{(k)}(x_{i+1})$ in (30) by $D_{A_i}^k f(x_i)$, $D_{A_i}^k f(x_{i+1})$, respectively, then a kind of improved Shepard-Euler operators S_{E_m} can be written as

$$\begin{aligned} S_{E_m}[f](x) &= \sum_{i=1}^N \frac{f(x_i) + f(x_{i+1})}{2} E_0\left(\frac{x - x_i}{x_{i+1} - x_i}\right) A_{\mu,i}(x) \\ &\quad + \sum_{i=1}^N \left[\sum_{k=1}^m \frac{\sum_{a \in A_i} \lambda_a f(x_i + a) + \sum_{a \in A_i} \lambda_a f(x_{i+1} + a)}{2k!} (x_{i+1} - x_i)^k \right. \\ &\quad \left. \times E_k\left(\frac{x - x_i}{x_{i+1} - x_i}\right) \right] A_{\mu,i}(x). \end{aligned} \quad (45)$$

Theorem 7. The operator S_{E_m} reproduces all univariate polynomials of degree no more than m .

Proof. Since $D_A^k f$ are the $\mathbb{P}_m$ -exact A -discretization of the derivative of the k th order $f^{(k)}$, $1 \leq k \leq m$, then for any $f \in \mathbb{P}_m$ we have

$$D_A^k f(x) = f^{(k)}(x), \quad x \in \mathbb{R},$$

so that

$$\begin{aligned} S_{E_m}[f](x) &= \sum_{i=1}^N \frac{f(x_i) + f(x_{i+1})}{2} E_0\left(\frac{x - x_i}{x_{i+1} - x_i}\right) A_{\mu,i}(x) \\ &\quad + \sum_{i=1}^N \left[\sum_{k=1}^m \frac{\sum_{a \in A_i} \lambda_a f(x_i + a) + \sum_{a \in A_i} \lambda_a f(x_{i+1} + a)}{2k!} (x_{i+1} - x_i)^k \right. \\ &\quad \left. \times E_k\left(\frac{x - x_i}{x_{i+1} - x_i}\right) \right] A_{\mu,i}(x) \\ &= \sum_{i=1}^N \frac{f(x_i) + f(x_{i+1})}{2} E_0\left(\frac{x - x_i}{x_{i+1} - x_i}\right) A_{\mu,i}(x) \\ &\quad + \sum_{i=1}^N \left[\sum_{k=1}^m \frac{f^{(k)}(x_i) + f^{(k)}(x_{i+1})}{2k!} (x_{i+1} - x_i)^k E_k\left(\frac{x - x_i}{x_{i+1} - x_i}\right) \right] A_{\mu,i}(x). \end{aligned} \quad (46)$$

According to (46) and the proof of Theorem 4, we can obtain $S_{E_m}[f](x) = f(x)$, when $f(x) = 1, x, \dots, x^m$. Therefore, we have proved that the operators S_{E_m} satisfy the m th degree polynomial reproduction property. $\square$

For the Shepard-Euler univariate operator S_{E_m} we then have the following desired error estimates.

Theorem 8. Let $f(x) \in C^{m+1}(I)$. Then

$$\|S_{E_m}[f](x) - f(x)\|_\infty \leq \bar{C}^{<E>} M \|f^{(m+1)}\|_\infty \varepsilon_\mu^m(r), \quad (47)$$

where

$$\varepsilon_\mu^m(r) = \begin{cases} |\ln r|^{-1}, & \mu = 1, \\ r^{\mu-1}, & 1 < \mu < m+2, \\ r^{\mu-1} |\ln r|, & \mu = m+2, \\ r^{m+1}, & \mu > m+2, \end{cases} \quad (48)$$

and $\bar{C}^{<E>}$ is a positive constant independent of x and X .

Proof. Consider

$$\begin{aligned} |S_{E_m}[f](x) - f(x)| &= |[S_{E_m}[f](x) - \tilde{S}_{E_m}[f](x)] + [\tilde{S}_{E_m}[f](x) - f(x)]| \\ &\leq |\tilde{S}_{E_m}[f](x) - f(x)| + |S_{E_m}[f](x) - \tilde{S}_{E_m}[f](x)|. \end{aligned} \quad (49)$$

The first term of the right-hand sides in (49) has been obtained from the Theorem 6:

$$|\tilde{S}_{E_m}[f](x) - f(x)| \leq \underline{C}^{<E>} \|f^{(m+1)}\|_\infty s_\mu^{m+1}(x), \quad (50)$$

where

$$s_\mu^{m+1}(x) = \frac{\sum_{i=1}^N |x - x_i|^{-\mu} d^{m+1}[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}}.$$

Based on (36) and (37), we get

$$\begin{aligned} s_\mu^{m+1}(x) &\leq \sum_{x_i \in T_0} \frac{|x - x_i|^{-\mu} d^{m+1}[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}} + \sum_{j=1}^n \sum_{x_i \in T_j} \frac{|x - x_i|^{-\mu} d^{m+1}[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}} \\ &\leq \sum_{x_i \in T_0} d^{m+1}[x_i, x_{i+1}](x) + |x - x_d|^\mu \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-\mu} d^{m+1}[x_i, x_{i+1}](x) \\ &\leq M(3r)^{m+1} + 2Mr^\mu \sum_{j=1}^n ((2j-1)r)^{-\mu} ((2j+3)r)^{m+1} \\ &\leq M5^{m+1} r^{m+1} \left(1 + 2 \sum_{j=1}^n j^{m+1-\mu} \right). \end{aligned} \quad (51)$$

$$\leq M5^{m+1} r^{m+1} \left(1 + 2 \sum_{j=1}^n j^{m+1-\mu} \right). \quad (52)$$

Next, we need to prove the first term of the right-hand sides in (49). We denote by $r_{\max}$, $r_{\min}$ the maximum and the minimum distance between adjacent nodes respectively. Let $\kappa = \frac{r_{\max}}{r_{\min}} \geq 1$. Let C_1 , C_2 be a constant, then according to [18], we get

$$\sum_{a \in A} |\lambda_a| \frac{|a|^{m+1}}{(m+1)!} \leq \sum_{a \in A} |\lambda_a| \frac{1}{(m+1)!} (\max_{a \in A} |A|)^{m+1} \leq C_1 r^{m+1-k} \kappa^m \|f^{(m+1)}\|_\infty. \quad (53)$$

Therefore, we have

$$|D_{A_i}^k f(x_i) - f^{(k)}(x_i)| \leq C_1 r^{m+1-k} \kappa^m \|f^{(m+1)}(x)\|_\infty, \quad (54)$$

and

$$|D_{A_i}^k f(x_{i+1}) - f^{(k)}(x_{i+1})| \leq C_2 r^{m+1-k} \kappa^m \|f^{(m+1)}(x)\|_\infty. \quad (55)$$

Let $C_{1,0}, C_{1,1}, C_{2,0}, C_{2,1}, C_{2,2}, \dots, C_{m,0}, C_{m,1}, \dots, C_{m,m}, \bar{C}, C, C', C_0 = \|f^{(m+1)}\|_\infty$ be constants. Then

$$\begin{aligned} & \left| S_{E_m}[f](x) - \tilde{S}_{E_m}[f](x) \right| \\ &= \left| \sum_{i=1}^N \sum_{k=1}^m \frac{[D_{A_i}^k f(x_i) - f^{(k)}(x_i)] + [D_{A_i}^k f(x_{i+1}) - f^{(k)}(x_{i+1})]}{2k!} (x_{i+1} - x_i)^k \right. \\ & \quad \left. \times E_k \left(\frac{x - x_i}{x_{i+1} - x_i} \right) A_{\mu,i}(x) \right| \\ &\leq \left| \sum_{i=1}^N \sum_{k=1}^m (x_{i+1} - x_i)^k \frac{[D_{A_i}^k f(x_i) - f^{(k)}(x_i)] + [D_{A_i}^k f(x_{i+1}) - f^{(k)}(x_{i+1})]}{2k!} \right. \\ & \quad \left. \times \sum_{l=0}^k \binom{k}{l} E_l \left(\frac{1}{2} \right) \left(\frac{1}{x_{i+1} - x_i} \right)^{k-l} d^{k-l}[x_i, x_{i+1}] A_{\mu,i}(x) \right| \\ &\leq (C_1 + C_2) C_0 \kappa^m \sum_{i=1}^N \sum_{k=1}^m \sum_{l=0}^k r^{m+1-k} \frac{(x_{i+1} - x_i)^k}{2k!} \binom{k}{l} \left| E_l \left(\frac{1}{2} \right) \right| \\ & \quad \times \left(\frac{1}{x_{i+1} - x_i} \right)^{k-l} d^{k-l}[x_i, x_{i+1}] A_{\mu,i}(x) \\ &\leq C_{1,0} r^m \kappa^m \sum_{i=1}^N d[x_i, x_{i+1}] A_{\mu,i}(x) + C_{1,1} r^m \kappa^m \sum_{i=1}^N r A_{\mu,i}(x) + C_{2,0} r^{m-1} \kappa^m \\ & \quad \times \sum_{i=1}^N d^2[x_i, x_{i+1}] A_{\mu,i}(x) + C_{2,1} r^m \kappa^{m-1} \sum_{i=1}^N r d[x_i, x_{i+1}] A_{\mu,i}(x) \\ & \quad + C_{2,2} r^{m-1} \kappa^m \sum_{i=1}^N r^2 A_{\mu,i}(x) + \dots + C_{m,0} r \kappa^m \sum_{i=1}^N d^m[x_i, x_{i+1}] A_{\mu,i}(x) \\ & \quad + C_{m,1} r \kappa^m \sum_{i=1}^N r d^{m-1}[x_i, x_{i+1}] A_{\mu,i}(x) + \dots + C_{m,m} r \kappa^m \sum_{i=1}^N r^m A_{\mu,i}(x) \\ &\leq (C_{1,1} + C_{2,2} + \dots + C_{m,m}) \kappa^m r^{m+1} \sum_{i=1}^N A_{\mu,i}(x) \\ & \quad + (C_{1,0} + C_{2,1} + \dots + C_{m,m-1}) \kappa^m r^m \sum_{i=1}^N d[x_i, x_{i+1}](x) A_{\mu,i}(x) \\ & \quad + (C_{2,0} + C_{3,1} + \dots + C_{m,m-2}) \kappa^m r^{m-1} \sum_{i=1}^N d^2[x_i, x_{i+1}](x) A_{\mu,i}(x) \\ & \quad + \dots + C_{m,0} \kappa^m r \sum_{i=1}^N d^m[x_i, x_{i+1}](x) A_{\mu,i}(x). \end{aligned}$$

By applying (36) and (37), we have

$$\begin{aligned}
& \left| S_{E_m}[f](x) - \tilde{S}_{E_m}[f](x) \right| \\
& \leq (C_{1,1} + C_{2,2} + \cdots + C_{m,m}) \kappa^m r^{m+1} + (C_{1,0} + C_{2,1} + \cdots + C_{m,m-1}) \kappa^m r^m \\
& \quad \times \left(\sum_{x_i \in T_0} \frac{|x - x_i|^{-\mu} d[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}} + \sum_{j=1}^n \sum_{x_i \in T_j} \frac{|x - x_i|^{-\mu} d[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}} \right) \\
& \quad + (C_{2,0} + C_{3,1} + \cdots + C_{m,m-2}) \kappa^m r^{m-1} \\
& \quad \times \left(\sum_{x_i \in T_0} \frac{|x - x_i|^{-\mu} d^2[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}} + \sum_{j=1}^n \sum_{x_i \in T_j} \frac{|x - x_i|^{-\mu} d^2[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}} \right) \\
& \quad + \cdots + C_{m,0} \kappa^m r \\
& \quad \times \left(\sum_{x_i \in T_0} \frac{|x - x_i|^{-\mu} d^m[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}} + \sum_{j=1}^n \sum_{x_i \in T_j} \frac{|x - x_i|^{-\mu} d^m[x_i, x_{i+1}](x)}{\sum_{k=1}^N |x - x_k|^{-\mu}} \right) \\
& \leq (C_{1,1} + C_{2,2} + \cdots + C_{m,m}) \kappa^m r^{m+1} + (C_{1,0} + C_{2,1} + \cdots + C_{m,m-1}) \kappa^m r^m \\
& \quad \times \left(\sum_{x_i \in T_0} d[x_i, x_{i+1}](x) + |x - x_d|^\mu \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^\mu d[x_i, x_{i+1}](x) \right) \\
& \quad + (C_{2,0} + C_{3,1} + \cdots + C_{m,m-2}) \kappa^m r^{m-1} \\
& \quad \times \left(\sum_{x_i \in T_0} d^2[x_i, x_{i+1}](x) + |x - x_d|^\mu \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-\mu} d^2[x_i, x_{i+1}](x) \right) \\
& \quad + \cdots + C_{m,0} \kappa^m r \\
& \quad \times \left(\sum_{x_i \in T_0} d^m[x_i, x_{i+1}](x) + |x - x_d|^\mu \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-\mu} d^m[x_i, x_{i+1}](x) \right) \\
& \leq (C_{1,1} + C_{2,2} + \cdots + C_{m,m}) \kappa^m r^{m+1} + (C_{1,0} + C_{2,1} + \cdots + C_{m,m-1}) \kappa^m r^m \\
& \quad \times \left(M(3r) + 2Mr^\mu \sum_{j=1}^n ((2j-1)r)^{-\mu} ((2j+3)r) \right) \\
& \quad + (C_{2,0} + C_{3,1} + \cdots + C_{m,m-2}) \kappa^m r^{m-1} \\
& \quad \times \left(M(3r)^2 + 2Mr^\mu \sum_{j=1}^n ((2j-1)r)^{-\mu} ((2j+3)r)^2 \right) \\
& \quad + \cdots + C_{m,0} \kappa^m r \times \left(M(3r)^m + 2Mr^\mu \sum_{j=1}^n ((2j-1)r)^{-\mu} ((2j+3)r)^m \right)
\end{aligned}$$

$$\begin{aligned}
&\leq (C_{1,1} + C_{2,2} + \cdots + C_{m,m})\kappa^m r^{m+1} + (C_{1,0} + C_{2,1} + \cdots + C_{m,m-1})\kappa^m r^m \\
&\quad \times \left(M(5r) \left(1 + 2 \sum_{j=1}^n j^{1-\mu} \right) \right) \\
&\quad + (C_{2,0} + C_{3,1} + \cdots + C_{m,m-2})\kappa^m r^{m-1} \left(M(5r)^2 \left(1 + 2 \sum_{j=1}^n j^{2-\mu} \right) \right) \\
&\quad + \cdots + C_{m,0}\kappa^m r \left(M(5r)^m \left(1 + 2 \sum_{j=1}^n j^{m-\mu} \right) \right) \\
&\leq \bar{C} \|f^{(m+1)}\|_{\infty} M \left(5^m r^{m+1} \left(1 + 2 \sum_{j=1}^n j^{m-\mu} \right) \right).
\end{aligned}$$

By applying formulas (50), (52), and the results above, we can obtain

$$\begin{aligned}
&|S_{E_m}[f](x) - f(x)| \\
&\leq \underline{C}^{<E>} M \|f^{(m+1)}\|_{\infty} \left(5^{m+1} r^{m+1} \left(1 + 2 \sum_{j=1}^n j^{m+1-\mu} \right) \right) + \bar{C} \|f^{(m+1)}\|_{\infty} M \\
&\quad \times \left(5^m r^{m+1} \left(1 + 2 \sum_{j=1}^n j^{m-\mu} \right) \right) \\
&\leq CM \|f^{(m+1)}\|_{\infty} r^{m+1} \left(1 + 2 \sum_{j=1}^n j^{m+1-\mu} \right).
\end{aligned}$$

Case 1: ($\mu > 1$)

If $1 < \mu < m+2$, then

$$r^{m+1} \left(1 + 2 \sum_{j=1}^n j^{m+1-\mu} \right) = \mathcal{O}(r^{\mu-1}).$$

If $\mu = m+2$, then

$$\sum_{j=1}^n j^{m+1-\mu} = \mathcal{O}(|\ln r|).$$

If $\mu > m+2$, then $\sum_{j=1}^n j^{m+1-\mu}$ is bounded.

Case 2: ($\mu = 1$)

$$\begin{aligned}
&|S_{E_m}[f](x) - f(x)| \\
&\leq |\tilde{S}_{E_m}[f(x) - f(x)]| + |S_{E_m}[f](x) - \tilde{S}_{E_m}[f](x)| \\
&\leq (C_{1,1} + C_{2,2} + \cdots + C_{m,m})\kappa^m r^{m+1} + (C_{1,0} + C_{2,1} + \cdots + C_{m,m-1})\kappa^m r^m \\
&\quad \times \left(\frac{\sum_{x_i \in T_0} |x - x_i|^{-1} d[x_i, x_{i+1}](x) + \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-1} d[x_i, x_{i+1}](x)}{\sum_{x_k \in T_0} |x - x_k|^{-1} + \frac{C'}{r} |\ln r|} \right) \\
&\quad + (C_{2,0} + C_{3,1} + \cdots + C_{m,m-2})\kappa^m r^{m-1}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{\sum_{x_i \in T_0} |x - x_i|^{-1} d^2[x_i, x_{i+1}](x) + \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-1} d^2[x_i, x_{i+1}](x)}{\sum_{x_k \in T_0} |x - x_k|^{-1} + \frac{C'}{r} |\ln r|} \right) \\
& + \cdots + C_{m,0} \kappa^m r \\
& \times \left(\frac{\sum_{x_i \in T_0} |x - x_i|^{-1} d^m[x_i, x_{i+1}](x) + \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-1} d^m[x_i, x_{i+1}](x)}{\sum_{x_k \in T_0} |x - x_k|^{-1} + \frac{C'}{r} |\ln r|} \right) \\
& + \underline{C}^{<E>} \|f^{(m+1)}\|_\infty \\
& \times \left(\frac{\sum_{x_i \in T_0} |x - x_i|^{-1} d^{m+1}[x_i, x_{i+1}](x) + \sum_{j=1}^n \sum_{x_i \in T_j} |x - x_i|^{-1} d^{m+1}[x_i, x_{i+1}](x)}{\sum_{x_k \in T_0} |x - x_k|^{-1} + \frac{C'}{r} |\ln r|} \right) \\
& \leq (C_{1,1} + C_{2,2} + \cdots + C_{m,m}) \kappa^m r^{m+1} \\
& + (C_{1,0} + C_{2,1} + \cdots + C_{m,m-1}) \kappa^m r^m \\
& \times \left(M(3r) + \frac{C' r}{|\ln r|} \sum_{j=1}^n \sum_{x_i \in T_j} (2j-1)^{-1} ((2j+3)r) \right) \\
& + (C_{2,0} + C_{3,1} + \cdots + C_{m,m-2}) \kappa^m r^{m-1} \\
& \times \left(M(3r)^2 + \frac{C' r}{|\ln r|} \sum_{j=1}^n \sum_{x_i \in T_j} (2j-1)^{-1} ((2j+3)r)^2 \right) \\
& + C_{m,0} \kappa^m r \left(M(3r)^m + \frac{C' r}{|\ln r|} \sum_{j=1}^n \sum_{x_i \in T_j} (2j-1)^{-1} ((2j+3)r)^m \right) \\
& + \underline{C}^{<E>} \|f^{(m+1)}\|_\infty \left(M(3r)^{m+1} + \frac{C' r}{|\ln r|} \sum_{j=1}^n \sum_{x_i \in T_j} (2j-1)^{-1} ((2j+3)r)^{m+1} \right) \\
& \leq (C_{1,1} + C_{2,2} + \cdots + C_{m,m}) \kappa^m r^{m+1} + (C_{1,0} + C_{2,1} + \cdots + C_{m,m-1}) \kappa^m r^m \\
& \times \left(M(5r) \left(1 + \frac{C'}{|\ln r|} \sum_{j=1}^n j^0 \right) \right) \\
& + (C_{2,0} + C_{3,1} + \cdots + C_{m,m-2}) \kappa^m r^{m-1} \left(M(5r)^2 \left(1 + \frac{C'}{|\ln r|} \sum_{j=1}^n j \right) \right) \\
& + C_{m,0} \kappa^m r \left(M(5r)^m \left(1 + \frac{C'}{|\ln r|} \sum_{j=1}^n j^{m-1} \right) \right) \\
& + \underline{C}^{<E>} \|f^{(m+1)}\|_\infty \left(M(5r)^{m+1} \left(1 + \frac{C'}{|\ln r|} \sum_{j=1}^n j^m \right) \right) \\
& = \mathcal{O}(|\ln r|^{-1}).
\end{aligned}$$

□

5 Numerical examples

5.1 Verification of approximation capacity

We consider the following functions on the interval $[0,1]$:

$$\begin{aligned}
 \text{Saddle} \quad f_1 &= \frac{1.25}{6 + 6(3x - 1)^2}, \\
 \text{Sphere} \quad f_2 &= \frac{\sqrt{64 - 81(x - 0.5)^2}}{9} - 0.5, \\
 \text{Cliff} \quad f_3 &= \frac{\tanh(-9x + 1)}{2} + 0.5, \\
 \text{Gentle} \quad f_4 &= \frac{\exp\left(-\frac{81}{16}(x - 0.5)^2\right)}{3}, \\
 \text{Steep} \quad f_5 &= \frac{\exp\left(-\frac{81}{4}(x - 0.5)^2\right)}{3}, \\
 \text{Exponential} \quad f_6 &= 0.75 \exp\left(-\frac{(9x - 2)^2}{4}\right) + 0.75 \exp\left(-\frac{(9x + 1)^2}{49}\right) \\
 &\quad + 0.5 \exp\left(-\frac{(9x - 7)^2}{4}\right) + 0.2 \exp(-(9x - 4)^2).
 \end{aligned}$$

These functions were first introduced in [9] and result from adapting to the univariate case test functions generally used in the multivariate interpolation of large sets of scattered data in [22]. For each function f_i , $i = 1, 2, \dots, 6$ we will compare the numerical results of our new operators $\tilde{S}_{E_m}[f]$ and $S_{E_m}[f]$ with the Shepard-Bernoulli operator $S_{B_m}[f]$ in [9].

We adopt uniform grids of 17 points for S_{B_1} , $\tilde{S}_{E_1}$, and S_{E_1} ; at the same time we use grids of 11 points for S_{B_2} , $\tilde{S}_{E_2}$, and S_{E_2} , and finally grids of 8 points for S_{B_3} , $\tilde{S}_{E_3}$, and S_{E_3} . To have as accurate an estimation of the error as possible, we compute the approximation functions at the points $\frac{i}{101}$, $i = 1, 2, \dots, 100$. Tables 1–6 show mean and max absolute errors for different values of the parameters μ and m . Numerical results show that the approximation accuracy of our new operators $\tilde{S}_{E_m}$ and S_{E_m} is comparable with the accuracy of Shepard-Bernoulli operator S_{B_m} .

In Figure 1, we plot the absolute error graphs of $S_{E_m} f_i$ ($i = 1, 2, \dots, 6$) as the parameters $\mu = 3$ and $m = 2$.

Table 1: Saddle.

(μ, m)	$S_{B_m} f_1$		$\tilde{S}_{E_m} f_1$		$S_{E_m} f_1$	
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2,1)	0.001050	0.004954	0.001067	0.005139	0.001050	0.004955
(2,2)	0.001062	0.004715	0.001100	0.004430	0.001004	0.003710
(2,3)	0.001490	0.005153	0.001516	0.005141	0.001771	0.007104
(3,1)	0.000476	0.003314	0.000496	0.003220	0.000477	0.003315
(3,2)	0.000333	0.002302	0.000313	0.001076	0.000280	0.001071
(3,3)	0.000206	0.001096	0.000391	0.001568	0.000550	0.003165
(4,1)	0.000457	0.003233	0.000476	0.003158	0.000457	0.003234
(4,2)	0.000259	0.001908	0.000259	0.001042	0.000295	0.001007
(4,3)	0.000136	0.001460	0.000358	0.001649	0.000505	0.003001

Table 2: Sphere.

(μ, m)	$S_{B_m} f_2$		$\tilde{S}_{E_m} f_2$		$S_{E_m} f_2$	
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2,1)	0.002145	0.005623	0.002151	0.005592	0.002145	0.005624
(2,2)	0.000312	0.000842	0.000326	0.000839	0.000222	0.000534
(2,3)	0.000586	0.002344	0.000583	0.002392	0.000176	0.000888
(3,1)	0.000583	0.001620	0.000586	0.001576	0.000584	0.001620
(3,2)	0.000058	0.000247	0.000825	0.000171	0.000056	0.000198
(3,3)	0.000079	0.000323	0.000082	0.000315	0.000055	0.000167
(4,1)	0.000510	0.001447	0.000513	0.001512	0.000510	0.001448
(4,2)	0.000039	0.000255	0.000044	0.000171	0.000053	0.000209
(4,3)	0.000025	0.000113	0.000039	0.000130	0.000063	0.000195

Table 3: Cliff.

(μ, m)	$S_{B_m} f_3$		$\tilde{S}_{E_m} f_3$		$S_{E_m} f_3$	
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2,1)	0.006604	0.038815	0.006867	0.041552	0.006604	0.038816
(2,2)	0.004710	0.031367	0.005681	0.032557	0.005213	0.042274
(2,3)	0.013455	0.062821	0.019834	0.067441	0.012252	0.084868
(3,1)	0.002522	0.021627	0.002466	0.023453	0.002523	0.021627
(3,2)	0.002466	0.027527	0.002539	0.020448	0.003466	0.028998
(3,3)	0.002138	0.016732	0.008360	0.049557	0.009664	0.072554
(4,1)	0.002405	0.021752	0.002307	0.021369	0.002406	0.021753
(4,2)	0.002170	0.034048	0.002481	0.021815	0.003451	0.028254
(4,3)	0.001542	0.024101	0.007482	0.049916	0.009573	0.071607

Table 4: Gentle.

(μ, m)	$S_{B_m} f_4$		$\tilde{S}_{E_m} f_4$		$S_{E_m} f_4$	
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2,1)	0.002590	0.007116	0.002585	0.007136	0.002591	0.007116
(2,2)	0.001897	0.005956	0.001897	0.005475	0.001640	0.004237
(2,3)	0.001138	0.006015	0.000982	0.005419	0.000096	0.009007
(3,1)	0.000681	0.003277	0.000695	0.003216	0.000681	0.003278
(3,2)	0.000378	0.001727	0.000355	0.001089	0.000290	0.000703
(3,3)	0.000175	0.000940	0.000174	0.000854	0.000376	0.001751
(4,1)	0.000618	0.002978	0.000630	0.002926	0.000618	0.002979
(4,2)	0.000270	0.001163	0.000257	0.000606	0.000279	0.000622
(4,3)	0.000089	0.000575	0.000162	0.000425	0.000329	0.000983

5.2 Comparison of computational cost

Suppose that Exponential. $f_6(x)$ is still the approximated function, then we choose the different shape parameter μ and different positive integer m to compare the computational cost of our operators $\tilde{S}_{E_m}[f_6]$ and $S_{E_m}[f_6]$ with that of $S_{B_m}[f_6]$. The laptop which we use to compute the max errors have the following properties: processor type is Intel Core i5-5200U, CPU frequency is 2.2 GHz, memory capacity is 4 GB. In Tables 7 and 8, we observe the computational time(s) of each operator.

Table 5: Steep.

(μ, m)	$S_{B_m} f_5$		$\tilde{S}_{E_m} f_5$		$S_{E_m} f_5$	
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2,1)	0.002358	0.012532	0.002542	0.013477	0.002359	0.012532
(2,2)	0.002950	0.015868	0.003583	0.013701	0.003489	0.011714
(2,3)	0.004950	0.019728	0.004996	0.022482	0.004876	0.018069
(3,1)	0.001930	0.011016	0.002034	0.010647	0.001930	0.011016
(3,2)	0.001501	0.009079	0.001521	0.004408	0.001338	0.004893
(3,3)	0.000909	0.005278	0.002128	0.007668	0.003780	0.012462
(4,1)	0.001945	0.011413	0.002014	0.010619	0.001945	0.014433
(4,2)	0.001323	0.008184	0.001322	0.004414	0.001390	0.004782
(4,3)	0.000815	0.006381	0.002138	0.006357	0.003766	0.012590

Table 6: Exponential.

(μ, m)	$S_{B_m} f_6$		$\tilde{S}_{E_m} f_6$		$S_{E_m} f_6$	
	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}	ϵ_{mean}	ϵ_{max}
(2,1)	0.007669	0.034757	0.008035	0.036003	0.007670	0.034958
(2,2)	0.005271	0.025436	0.008158	0.026378	0.007703	0.030519
(2,3)	0.025296	0.067861	0.022226	0.059885	0.019342	0.072922
(3,1)	0.005122	0.021099	0.005439	0.022252	0.005123	0.021100
(3,2)	0.004379	0.024620	0.005992	0.015731	0.006238	0.020148
(3,3)	0.003523	0.020488	0.015247	0.058286	0.017772	0.046858
(4,1)	0.005026	0.022762	0.005276	0.021013	0.005026	0.022762
(4,2)	0.004233	0.024080	0.005834	0.015174	0.006435	0.019645
(4,3)	0.003020	0.018326	0.015702	0.058286	0.017589	0.043494

From Tables 7 and 8, we find that for the different μ and m , our Shepard-Euler operator $S_{E_m}[f_6]$ without derivatives cost a less time(s) than the Shepard-Bernoulli operator $S_{B_m}[f_6]$. Hence, we believe that it is really a good idea to move up to high-order scheme.

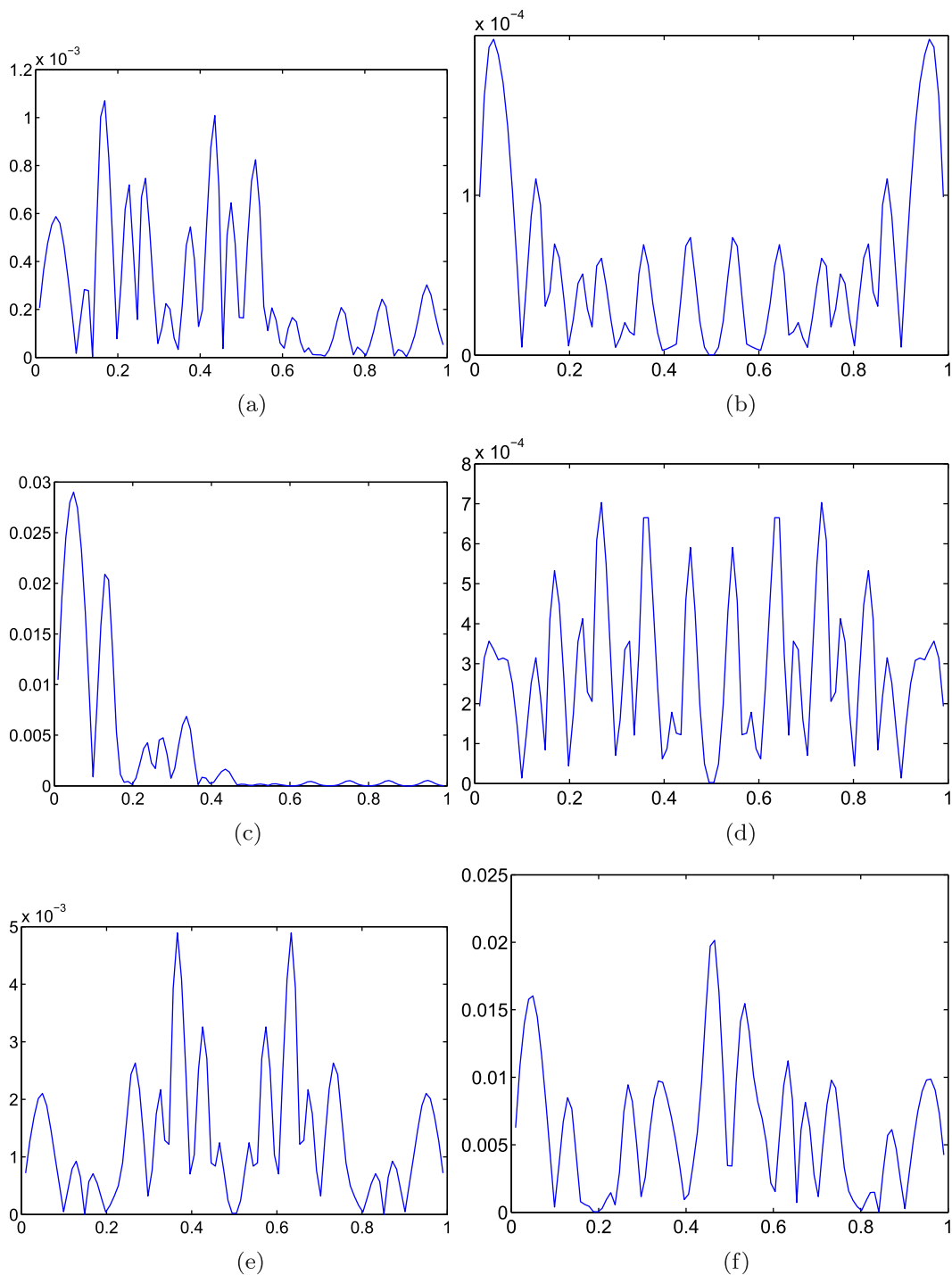


Figure 1: Absolute error graphs using operator S_{E_m} with $m = 2$ and $\mu = 3$ for functions $f_i, i = 1, 2, \dots, 6$. (a) Absolute error graph of $S_{E_2}f_1$. (b) Absolute error graph of $S_{E_2}f_2$. (c) Absolute error graph of $S_{E_2}f_3$. (d) Absolute error graph of $S_{E_2}f_4$. (e) Absolute error graph of $S_{E_2}f_5$. (f) Absolute error graph of $S_{E_2}f_6$.

Table 7: Computational time(s) of the max errors of operators approximating f_6 as $m = 2$.

(μ, m)	(2, 2)	(3, 2)	(4, 2)
$S_{B_m} f_6$	21.023985	21.106101	21.276911
$\tilde{S}_{E_m} f_6$	42.213025	42.838780	44.621109
$S_{E_m} f_6$	0.146423	0.124507	0.105130

Table 8: Computational time(s) of the max errors of operators approximating f_6 as $m = 3$.

(μ, m)	(2, 3)	(3, 3)	(4, 3)
$S_{B_m} f_6$	31.562475	30.887780	32.220974
$\tilde{S}_{E_m} f_6$	47.240185	47.940174	47.033827
$S_{E_m} f_6$	0.202907	0.248268	0.217616

6 Conclusions

In this paper, a kind of univariate Shepard-Euler operators $\tilde{S}_{E_m}$ is constructed by combining a known Shepard operator with the expansion in univariate Euler polynomials. However, it requires the derivatives of approximated function at endpoints, which is not very convenient for practical purposes. Using linear combinations of the shifts of approximated function to approximate the derivatives of approximated function, we propose another kind of improved Shepard-Euler operators S_{E_m} which do not need values of the derivatives at nodes. Meanwhile, we have also proven that the operator S_{E_m} possesses the m th degree polynomial reproduction property and good convergence capacity. Furthermore, Numerical results show that it uses less computational cost. So the method is also applicable for people in applications.

In the following work, on the one hand, we want to use it to the fitting of discrete solutions of the initial value problems of ODEs. On the other hand, the univariate Shepard-Euler operators can be extended to the multivariate case.

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