Research Article

Yahan Cao and Shengjin Ji*

A note on the edge general position number of cactus graphs

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Abstract: For a given graph G, a subset S of $E(G)$ is an edge general position set of G if no triple of S is contained in a common shortest path. The cardinality of a largest edge general position set of G is called the edge general position number of G, denoted by $\mathrm{gp}_e(G)$. In the paper, sharp upper and lower bounds of the edge general position number are obtained among cactus graphs. Moreover, we characterize all graphs that attained these bounds.

Keywords: general position set; edge general position set; cut vertex; cactus graph

MSC 2020: 05C12; 05C35; 05C70

1 Research background

Let $G = (V(G), E(G))$ be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. As usual, $|V(G)|$ and $|E(G)|$ are called *the order* and *the size* of G, respectively. If $uv \in E(G)$, then we say that u is a *neighbor* of v in G and vice versa. For a vertex $v \in V(G)$, set $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ is regarded as the *open neighborhood* of v. The *degree* of a vertex $v \in V(G)$ is $d_G(v) = |N_G(v)|$. The general position problem in graph theory is to find a largest set of vertices $S \subseteq V(G)$, called a *gp*-set of G, such that no shortest path of G contains three vertices of S, which was first proposed by Manuel and Klavžar [1]. The general position number (gp-number for short) of G, denoted by gp(G), is the cardinality of a gp-set of G. In fact, they researched the basic properties and the bounds of gp-number in some special graphs, meanwhile, they proved that the general position problem is NP-complete. Note that the classical general position problem is traced back close to the celebrated century-old problem named as the no-three-in-line problem, first introduced by Dudeney [2] in 1917. Recently, Payne and Wood [3] extended the no-three-in-line problem to the general position subset selection problem in discrete geometry. For progress in this regard, see [4,5] and references therein.

Now we focus on the general position problem in graph theory. Patkós [6] studied the gp-number of Kneser graphs, and determined the exact value of gp-number of some special Kneser graphs by using a generalization of Bollobás's inequality on intersecting set pair systems. Klavžar et al. [7] and Tian and Xu [8] studied the gp-number of Cartesian products. Tian et al. [9] determined the gp-numbers of maximal outerplanar graphs. For further results, see the recent survey on gp-number [10].

The edge general position set of a graph is the edge version of the general position set of a graph, see the seminal paper [11]. We now introduce formally its definition as follows. Let $G = (V(G), E(G))$ be a graph and $S \subseteq E(G)$. We say that S is *an edge general position set* if no three edges of S lie on a common shortest path. The

***Corresponding author: Shengjin Ji**, School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong 255000, China, E-mail: jishengjin2013@163.com. https://orcid.org/0000-0002-5345-9953
Yahan Cao, School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong 255000, China, E-mail: caoyahanisyoung@163.com. https://orcid.org/0009-0001-6368-1021

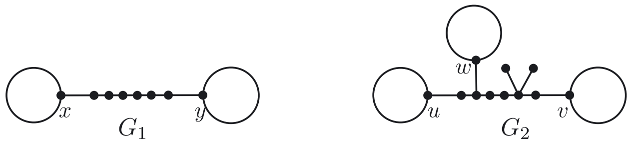

Figure 1: The two examples of cyclic paths.

subset S is also called *a maximum edge general position set* if it has the largest cardinality in all edge general position sets. We also call S a gp_e-set of G for short. *The edge general position number* (gp_e-number for short) of G, denoted by $\mathrm{gp}_e(G)$, is the cardinality of a gp_e-set in G. An edge general position problem is to find a gp_e-set of graphs. And then it should be added that the edge general position set has been recently extended to k-edge general position sets in [12]. Klavžar and Tan [13] obtained the sharp bounds of gp_e-number on Fibonacci and Lucas Cubes. Note that many researchers concerned the extremal problems of cactus graphs, cf. [14–17]. Hence, it is interesting to study the gp_e-number of cactus graphs. In the paper, we continue the research in this direction.

For convenience, we now introduce some notations. *A block* of G is a maximal connected subgraph of G without cut vertex. A connected graph G is called a *cactus graph* if its any block is either a cycle or an edge. Let $\mathscr{C}_n^k$ be the set of all cactus graphs of order n with $k \geq 1$ cycles. Let $\mathscr{C}_n^{k,t}$ be the set of cactus graphs on order n with k cycles and t leaves, where $t \geq 0$ and $k \geq 1$.

Let $G \in \mathscr{C}_n^{k,t}$ be a graph. *A cyclic path* is a path connecting two cycles in G such that except two end vertices, any internal vertex (if it exists) does not belong to any cycle of G. Note that two cyclic paths can overlap, and internal vertices of a cyclic path do not necessarily have degree 2. As shown in Figure 1, (x, y)-path, (u, v)-path, (u, w)-path and (w, v)-path are cyclic paths. We use $\mathscr{P}(G)$ to denote the set of cyclic paths of G. Let P and C be a cyclic path in G and a cycle of G, respectively. For a cut vertex u of G contained in $V(C) \cup V(P)$, by removing all edges in $E(C) \cup E(P)$ incident with u, the component containing u, denoted by T_u, is named as *a root tree* of G with root u if T_u does not contain cycles. *A root tree* of G is a tree with roots on a cycle or a cyclic path. We use $\mathscr{T}(G)$ to denote the set of all root trees of G. Moreover, a vertex w of a root tree is called *a leaf* if w has degree one and is not the root. Clearly, the number of leaves in $\mathscr{T}(G)$ equals t. A vertex $v \in V(G)$ is called a *cut vertex* if removing v increases the number of connected components. *An inner cut vertex* is a vertex of a cycle shared by another cycle or a cyclic path. Clearly, an inner cut vertex is indeed a cut vertex. We use $c(C)$ to denote the number of cut vertices of C and $p_e(C)$ to denote the number of pendant edges of root trees on C such that they are not shared by other cycles or cyclic paths.

In the paper, sharp upper and lower bounds of gp_e-number are obtained among cactus graphs with k cycles and t pendant leaves. Moreover, we characterize the structures of these graphs that attain the bounds.

2 Edge general position sets of cactus graphs

In the section, we first give some notations which will be useful in showing our main results. We will obtain the upper bound of gp_e-number in $\mathscr{C}_n^{k,t}$ and characterize the graphs attaining the bound. And then, we will show the lower bounds of gp_e-number in cactus graphs, meanwhile, the extremal graphs are obtained completely.

2.1 Notations

2.1.1 An inner cycle and an outer cycle

We now define several types of cycles of G by means of the cut vertices contained in these cycles. A cycle C_l is *an inner cycle* if there are at least two subgraphs of $G - E(C_l)$ containing cycles, *an outer cycle* otherwise. In particular, an outer cycle with exactly one cut vertex is *an end-block*.

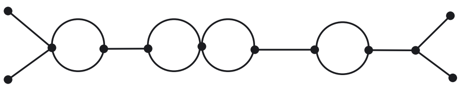

Figure 2: A root chain cactus graph.

2.1.2 A chain cactus and a root chain cactus

A chain cactus G is a cactus graph in which all blocks have at most two cut vertices, and each cut vertex is exactly shared by two blocks. Clearly, a chain graph G has exactly two outer cycles and at most two leaves. We call a graph G' *a root chain cactus* if it is obtained from G by changing at least one outer cycle of G such that it contains exactly one root tree with two leaves at a vertex other than the inner cut vertex, see Figure 2. The subgraph of G formed from two outer cycles and the inner cycles and cyclic paths connecting them is called *a subchain cactus* of G. A subchain cactus of the chain cactus graph presented in Figure 3 is obtained from it by removing two leaves.

2.1.3 A cut-path of C_l and $D_c(C_l)$

Let u_i and u_j be two vertices of C_l, clearly, C_l can be regarded as consisting of two (u_i, u_j)-paths. If all cut vertices lying on C_l belong to one (u_i, u_j)-path, then the path is referred to as a (u_i, u_j)-*cut-path* (or a cut-path for short) and then denote $d_c(u_i, u_j)$ the number of edges contained in it, e.g., $d_c(z_1, z_3) = 4$ see Figure 4. In particular, suppose now that a cycle C_l of G has at least three cut vertices. If there are three cut vertices satisfying the following: two cut vertices x_i and x_j form a (x_i, x_j)-cut-path only containing the third cut vertex x_k while it is the root vertex of a root tree T_{x_k} with a leaf, then we name it a (x_i, x_j)-root-cut-path (or a root-cut-path for short) and denote the number of edges on the path by $d_r(x_i, x_j)$. And then the vertex u is referred to as *a bad vertex* if either $d_r(x_i, x_j) \geq \frac{l}{2} + 1$ for even l or $d_r(x_i, x_j) \geq \lfloor \frac{l}{2} \rfloor + 1$ otherwise. As shown in Figure 4, $d_r(x_1, x_3) = 4$ and x_2 is a bad vertex. Set $D_c(C_l) = \min_{u_i, u_j \in V(C_l)} d_c(u_i, u_j)$ (D_c for short), e.g., $D_c(C^4) = 3$, see Figure 4.

2.1.4 A good cycle, a normal cycle and a bad cycle

We now give a classification of cycles in G under the assumption that if C_l is an outer cycle then $c(C_l) \geq 4$. We call the cycle C_l a *normal I cycle* if D_c is no more than $\frac{l}{2} - 1$ for even l or $\lfloor \frac{l}{2} \rfloor$ for odd l. Conversely, assume now that $D_c \geq \frac{l}{2}$ for even l or $D_c \geq \lfloor \frac{l}{2} \rfloor + 1$ for odd l. The cycle C_l is named as either a *normal II cycle* if there exists a root-cut-path having a bad vertex or a *bad cycle* otherwise. For example, C^3 is a normal II cycle, see Figure 5.

In particular, suppose next that C_l is an outer cycle with $c(C_l) \leq 3$. For $c(C_l) = 1$, clearly, C_l is an end-block. Then C_l is either a *normal I cycle* for even order, or a *good cycle* otherwise. Assume now that $2 \leq c(C_l) \leq 3$.

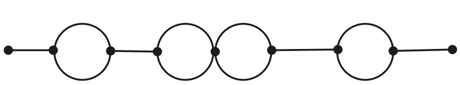

Figure 3: A chain cactus graph.

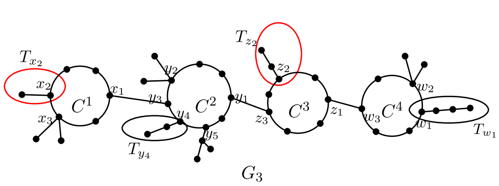

Figure 4: Illustrative examples of cut-path, root cut-path and D_c.

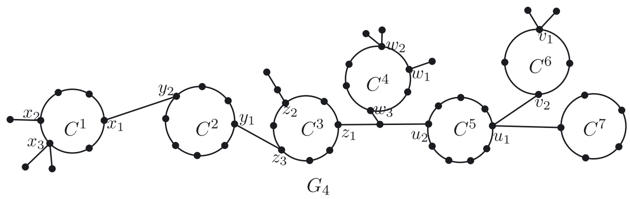

Figure 5: Illustrative examples of different kinds of cycles.

Let the order of C_l be even. The cycle C_l is *a normal I cycle* with the condition $D_c \leq \frac{l}{2} - 1$. Under the condition $D_c \geq \frac{l}{2} + 1$, C_l is *a normal II cycle* if it contains a root-cut-path with a bad vertex or *a bad cycle* otherwise. For the case $D_c = \frac{l}{2}$ and $c(C_l) = 3$, we call that C_l is *a normal II cycle* if there exists a root-cut-path with a bad vertex, *a bad cycle* otherwise, such as, a normal II cycle C^1 and a bad cycle C^4 of G_4 shown in Figure 5. For the case $D_c = \frac{l}{2}$ and $c(C_l) = 2$, we call C_l either *a bad cycle* if $p_e(C_l) \geq 2$ or *a normal II cycle* otherwise. We can check that C^6 of G_4 is a bad cycle, see Figure 5.

Suppose the order of C_l is odd. The cycle C_l is *a normal I cycle* if $D_c \leq \lfloor \frac{l}{2} \rfloor$. Moreover, C_l with $D_c \geq \lfloor \frac{l}{2} \rfloor + 1$ and $c(C_l) = 3$ is *a normal II cycle* if a root-cut-path of C_l has a bad vertex, *a bad cycle* otherwise.

2.2 The upper bounds of the edge general position number

From [18], we know the following proposition.

Proposition 2.1. $\mathrm{gp}_e(C_n) = n$ *if* $n \in \{3, 4, 5\}$, *and* $\mathrm{gp}_e(C_n) = 4$ *otherwise.*

Observation 2.1. For arbitrary graph $G \in \mathscr{C}_n^{k,t}$ with $k \geq 2$, let $\mathscr{T}(G)$ be the set of root trees with t leaves in G, then there exists a gp_e-set S such that $|S \cap E(\mathscr{T}(G))| \leq t$.

Lemma 2.1. *Let* $G \in \mathscr{C}_n^{k,t}$ *be a graph with* $k \geq 2$ *and* $\mathscr{P}(G)$ *be the set of cyclic paths of* G*. Then there exists a* gp_e*-set* S *such that* $|S \cap E(\mathscr{P}(G))| = 0$.

Proof. Suppose $G \in \mathscr{C}_n^{k,t}$ is a cactus graph. Assume that S is a gp_e-set of G having edges from outer cycles and $\mathscr{T}(G)$ as more as possible. Let $\mathscr{P}(G)$ be the set of cyclic paths of G and an edge $e \in E(\mathscr{P}(G))$. By contradiction, assume that S is a gp_e-set of G with $e \in S$. Let H_1 and H_2 be two components of $G - \{e\}$. Observe that H_i contains at least one outer cycle from the definition of a cyclic path, and every shortest path from $V(H_1)$ to $V(H_2)$ goes through e. In addition, if C is an end-block of G, then we observe that either $|S \cap E(C)| \leq 3$ for odd order of C or $|S \cap E(C)| \leq 2$ for even order of C. Based on the types of outer cycles, we will take three cases to proceed the proof.

Case 1 H_1 and H_2 contain end-blocks.

Let C_i be an end-block of H_i and u_i be the unique cut vertex for $i = 1, 2$. We first consider that the lengths of C_1 and C_2 have the same parity. If $|V(C_1)|$ and $|V(C_2)|$ are odd, then they are good cycles. From the maximum of S and $e \in S$, we deduce that one of $|S \cap E(C_1)|$ and $|S \cap E(C_2)|$ equals 1 and the other equals 3. Without loss of generality, assume that $|S \cap E(C_1)| = 1$ and $|S \cap E(C_2)| = 3$. Let $S_1 = (S - \{e\}) \cup \{e_1, e_2\}$, where e_1 and e_2 are two edges incident with u_1 in C_1, as shown in Figure 6. It follows that S_1 is a new edge general position set of G larger

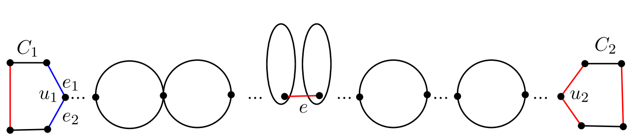

Figure 6: Used to illustrate Case 1, both C_1 and C_2 are odd cycles.

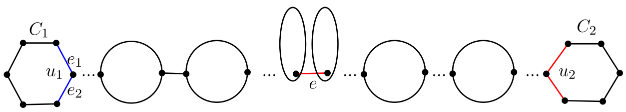

Figure 7: Used to illustrate Case 1, both C_1 and C_2 are even cycles.

than S, a contradiction. If $|V(C_1)|$ and $|V(C_2)|$ are even, then they are normal I cycles. By the choice of S, we obtain that one of $|S \cap E(C_1)|$ and $|S \cap E(C_2)|$ equals 0 and the other equals 2. Without loss of generality, assume that $|S \cap E(C_1)| = 0$ and $|S \cap E(C_2)| = 2$. Let $S_2 = (S - \{e\}) \cup \{e_1, e_2\}$, where e_1 and e_2 are two edges incident with u_1 in C_1, as shown in Figure 7. Obviously, $|S_2| > |S|$. In addition, observe that S_2 is also an edge general position set of G, contradicting our assumption. We now assume that the lengths of C_1 and C_2 have different parity. Using the similar way of the first case, we also have done. So we omit the process here.

Case 2 H_1 and H_2 contain no end-blocks.

Assume that C_1' and C_2' are two outer cycles of H_1 and H_2, respectively. Let u_i be an inner cut vertex of C_i' for $i = 1, 2$. Since they are not end-blocks, C_i' contains at least one root of some root tree for $i = 1, 2$. Let L_i be the set of pendant edges belonging to root trees on C_i for $i = 1, 2$. From the maximum of S and $e \in S$, we conclude that at least one of $S \cap L_1 = \emptyset$ and $S \cap L_2 = \emptyset$ is valid.

We now consider that $S \cap L_1 = \emptyset$ and $S \cap L_2 = \emptyset$ hold simultaneously. So we deduce that $|S \cap E(C_i')| \in \{2, 3\}$ according to the parity of the lengths of these two cycles. In particular, if $|S \cap E(C_1')| = |S \cap E(C_2')| = 3$, then we will find three edges of S lying a shortest path, a contradiction. If there is one cycle, say C_1', such that $|S \cap E(C_1')| = 2$, then we deduce that C_1' has exactly two cut vertices with $D_c(C_1') = \frac{|C_1'|}{2}$. So there is a contradiction again. We thus assume that one of $S \cap L_1 = \emptyset$ and $S \cap L_2 = \emptyset$ is correct, say $S \cap L_1 = \emptyset$ and $S \cap L_2 \neq \emptyset$. Hence, $|S \cap E(C_1')| \in \{2, 3\}$ and $|S \cap E(L_2)| \geq 2$. Using the similar argument of the first case, we obtain that S contains three edges lying on a shortest path through e of G. We thus get a contradiction.

Case 3 One of H_1 and H_2 contains end-blocks.

Assume that H_1 contains an end-block, denoted by C_3. Let u_3 be the inner cut vertex in C_3. Then C_3 is a normal I cycle for even order or a good cycle otherwise. Let C_4 be an outer cycle in H_2 and L_3 be the set of pendant edges belonging to root trees of C_4. Then one of $S \cap L_3 = \emptyset$ and $S \cap L_3 \neq \emptyset$ holds. For each case, the choice of S and $e \in S$ imply that $|S \cap C_3| \in \{0, 1\}$. Let $S_3 = (S - \{e\}) \cup \{e_{31}, e_{32}\}$ with $|S_3| > |S|$, where e_{31} and e_{32} incident with u_3 of C_3. It follows that S_3 is also an edge general position set of G, a contradiction.

Therefore, we finish the proof. □

Lemma 2.2. *Suppose $G \in \mathscr{C}_n^{k,t}$ with $k \geq 2$ cycles. Let C_0 be a cycle of G. Then there exists a gp_e-set S such that $|E(C_0) \cap S| \leq 3$ and $|E(C_0) \cap S| \in \{0, 2, 3\}$. In addition, the following assertions hold.*

(i) $|E(C_0) \cap S| = 3$ if and only if C_0 is a good cycle.
(ii) $|E(C_0) \cap S| = 2$ if and only if C_0 is a normal I cycle or a normal II cycle.
(iii) $|E(C_0) \cap S| = 0$ if and only if C_0 is a bad cycle.

Proof. Let G be a cactus graph with $k \geq 2$ cycles and t leaves. Let S be a gp_e-set containing pendant edges and these edges from outer cycles as more as possible. By means of Lemma 2.1, the elements of S are derived from cycles and root trees of G. Assume that C_0 is a cycle of G. We first prove the following claim. □

Claim 1. $|S \cap E(C_0)| \leq 3$.

Proof. By contradiction, suppose $|S \cap E(C_0)| \geq 4$. Together with Proposition 2.1, we conclude that $|S| = |S \cap E(C_0)| \in \{4, 5\}$.

If C_0 is an inner cycle, then we deduce that G is a chain cactus by the maximum of S. Assume that C_1 and C_2 are two outer cycles of G with two inner cut vertices u_1 and u_2, respectively. In addition, by the choice of S,

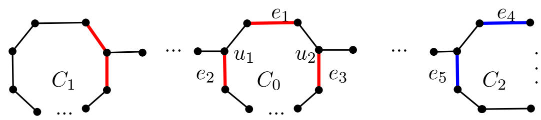

Figure 8: Used to illustrate Case 1.

$|S \cap E(C_i)| = 0$ for $i = 1, 2$. Clearly, C_0 is an even cycle. Otherwise, $D_c(C_0) \le \lfloor \frac{|C_0|}{2} \rfloor$. Then by choosing the two end edges of cut-path of C_0 with length $|C_0| - D_c(C_0)$, say e_0, e_0', we get a new edge set S' from S by removing all edges from C_0 and adding two edges e_0, e_0' and four edges from C_1 and C_2 respectively incident with u_1 and u_2. Evidently, S' is an edge general position set of G with $|S'| > |S|$, we get a contradiction with the maximum of S. Hence, $|S| = 4$, which infers that C_1 and C_2 are even cycles. We thus deduce that $D_c(C_0) = \frac{|C_0|}{2}$. For each C_i, if C_i has two cut vertices, then $D_c(C_i) = \frac{|C_i|}{2}$, which contradicts the choice of S.

If C_0 is an outer cycle with the inner cut vertex u_0, then by the choice of S we have that G is a chain cactus graph. In other words, all inner cycles are bad. Let C_0' be another outer cycle with an inner cut vertex u_0'. But we find that $|S \cap E(C_0')| = 0$, which contradicts the choice of S.

Hence, $|S \cap E(C_0)| \le 3$ is true. □

We now divide the following cases to finish the remaining proof.

Case 1 C_0 is an inner cycle of G.

In the case, we know that $k \ge 3$. Note that, for each inner cycle C_0, there are two outer cycles C_1 and C_2 of G for which C_0 is lying on its unique subchain cactus between C_1 and C_2. Let u_1 and u_2 be the two inner cut vertices of C_0 belonging to the subchain cactus. By the maximum of S, we can claim that $|S \cap E(C_0)| \le 2$. Assume to the contrary that $|S \cap E(C_0)| \ge 3$. Recall that $|S \cap E(C_0)| \le 3$. So $|S \cap E(C_0)| = 3$. Set $\{e_1, e_2, e_3\} \subseteq S \cap E(C_0)$. So there exists an edge, say e_1, such that it lies on a (u_1, u_2)-path with length no more than $\frac{|C_0|}{2} - 1$ for even order (or $\frac{|C_0|-1}{2}$ for odd order). In addition, the choice of S implies that one of $|S \cap E(C_1)|$ and $|S \cap E(C_2)|$ equals zero, say $|S \cap E(C_2)| = 0$. Set $S' = (S - e_1) \cup \{e_4, e_5\}$, where $e_4, e_5 \in E(C_2)$ with the same distance to e_1, as shown in Figure 8. Evidently, S' is an edge general position set with larger size than that of S, a contradiction.

Observe that, if $D_c(C_0) \le \frac{|C_0|}{2} - 1$ for even order (or $\frac{|C_0|-1}{2}$ for odd order), we obtain that $|S \cap E(C_0)| = 2$. In fact, there is a (w_1, w_2)-cut-path with length $D_c(C_0)$ in C_0. Conversely, for another (w_1, w_2)-path of C_0, we can choose its two end edges as the elements of S. Thus, C_0 is a normal I cycle. Furthermore, assume that $D_c(C_0) \ge \frac{|C_0|}{2}$ for even order (or $\frac{|C_0|-1}{2} + 1$ for odd order). If there is a bad vertex u for which a root-cut-path containing u has length greater than $\frac{|C_0|}{2} + 1$ for even order (or $\frac{|C_0|-1}{2} + 1$ for odd order). Then $|S \cap E(C_0)| = 2$ and C_0 is a normal II cycle. Otherwise, $|S \cap E(C_0)| = 0$ and C_0 is a bad cycle. Hence, in the case, $|E(C_0) \cap S| \ne 1$.

Case 2 C_0 is an outer cycle of G.

Note that $c(C_0) \ge 1$. Assume now that $c(C_0) = 1$, which implies that C_0 is an end-block. The choice of S results in either $|S \cap E(C_0)| = 3$ for odd order or $|S \cap E(C_0)| = 2$ for even order.

We now assume that $c(C_0) \ge 4$. According to the values of $D_c(C_0)$, we conclude that $|S \cap E(C_0)| = 2$ with $D_c(C_0) \le \frac{|C_0|}{2} - 1$ for even order (or $\frac{|C_0|-1}{2}$ for odd order). Thus, C_0 is a normal I cycle. Suppose that $D_c(C_0) \ge \frac{|C_0|}{2}$ for even order (or $\frac{|C_0|-1}{2} + 1$ for odd order). If there is a bad vertex u such that a root-cut-path containing u has the length no less than $\frac{|C_0|}{2} + 1$ for even order (or $\frac{|C_0|-1}{2} + 1$ for odd order). Then $|S \cap E(C_0)| = 2$ and C_0 is a normal II cycle. Otherwise, $|S \cap E(C_0)| = 0$ and C_0 is a bad cycle.

Assume next that $c(C_0) = 3$. By the values of $D_c(C_0)$, we deduce that $|S \cap E(C_0)| = 2$ with $D_c(C_0) \le \frac{|C_0|}{2} - 1$ for even order (or $\frac{|C_0|-1}{2}$ for odd order) and C_0 is a normal I cycle. Suppose now that $D_c(C_0) \ge \frac{|C_0|}{2}$ for even order (or $\frac{|C_0|-1}{2} + 1$ for odd order). If there is a root-cut-path with a bad vertex, then $|S \cap E(C_0)| = 2$ and C_0 is a normal II cycle. Otherwise, $|S \cap E(C_0)| = 0$ by the choice of S, so C_0 is a bad cycle.

We now consider the case $c(C_0) = 2$. It is clear that $D_c(C_0) \le \frac{|C_0|}{2}$ for even order (or $\frac{|C_0|-1}{2}$ for odd order). If C_0 is an odd cycle, then we deduce that $|S \cap E(C_0)| = 2$ and C_0 is a normal I cycle, as shown in Figure 9. Assume that C_0 is an even cycle. We can verify that, if $D_c(C_0) \le \frac{|C_0|}{2} - 1$, then $|S \cap E(C_0)| = 2$ and C_0 is a normal I cycle,

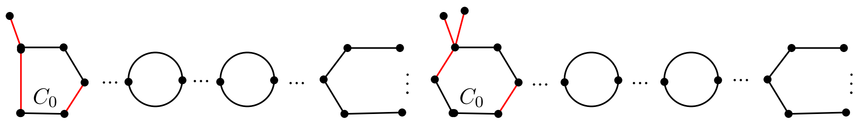

Figure 9: Used to illustrate Case 2 with $c(C_0) = 2$ and C_0 is a normal I cycle.

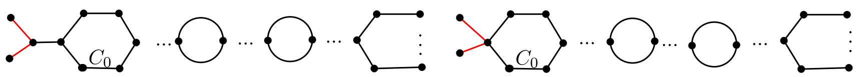

Figure 10: Used to illustrate Case 2 with $c(C_0) = 2$ and C_0 is a bad cycle.

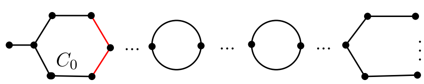

Figure 11: Used to illustrate Case 2 with $c(C_0) = 2$ and C_0 is a normal II cycle.

as shown in Figure 9. Otherwise $D_c(C_0) \geq \frac{|C_0|}{2}$. We deduce that C_0 is either a normal II cycle with $|S \cap E(C_0)| = 2$ and $p_e(C_0) = 1$ or a bad cycle with $|S \cap E(C_0)| = 0$ and $p_e(C_0) \geq 2$, as shown in Figures 10 and 11. Clearly, in the case, $|E(C_0) \cap S| \neq 1$.

Therefore, we have done as required. □

Based on the above conclusions, we deduce the following result.

Theorem 2.1. *For $k \geq 2$, let $G \in \mathscr{C}_n^{k,t}$ be a graph with r odd cycles and $k - r$ even cycles. Then $\mathrm{gp}_e(G) \leq 2(k - r) + 3r + t$ with equality only if all odd cycles are good and all even cycles are normal I.*

Proof. Suppose that $G \in \mathscr{C}_n^{k,t}$ is a graph with r odd cycles and $k - r$ even cycles and let S be a gp_e-set of G containing as many pendant edges and edges from end-blocks as possible. Let C_l be a cycle of G with length l. Combining Lemmas 2.1, 2.2 and Observation 2.1, we obtain that

$$|S| \leq 2(k - r) + 3r + t.$$

Hence, we next show the second part of the conclusion. Assume now that G is a cactus graph such that $\mathrm{gp}_e(G) = |S| = 2(k - r) + 3r + t$. From Lemma 2.2, we have that $|S \cap E(C_l)| \leq 3$, and then, $|S \cap E(C_l)| \leq 2$ for even l. We first claim that each odd cycle C_l is an end-block. If it is not an end-block, then $|S \cap E(C_l)| \leq 2$ by Lemma 2.2. So we get that.

$$\begin{aligned}|S| &= |S \cap E(C_l)| + |S \cap (E(G) - E(C_l))| \\ &\leq 2 + |S \cap (E(G) - E(C_l))| \\ &\leq 2 + 2(k - r) + 3(r - 1) + t \\ &< 2(k - r) + 3r + t,\end{aligned}$$

a contradiction. We next claim that each even cycle C_l is a normal I cycle. Contrary to our claim, suppose that C_l is not normal I. Hence, C_l is either a normal II cycle or a bad cycle. From by Lemma 2.2, we deduce that either $|S| \leq\leq 2 + 2(k - r) + 3(r - 1) + t - 1 < 2(k - r) + 3r + t$ for the first case, or $|S| \leq\leq 0 + 2(k - r) + 3(r - 1) + t < 2(k - r) + 3r + t$ otherwise. We get a contradiction.

Combining the above two cases, $|S| = 2(k - r) + 3r + t$ implies that G contains r good odd cycle and $k - r$ even normal I cycle. In other words, all root trees are lying on cut-paths with length less than a half of the order of each even cycle. Consequently, $|S \cap E(\mathscr{T}(G))| = t$. Therefore, we verify the conclusion as claimed. □

The following two results hold directly from Theorem 2.1.

Corollary 2.2. *Let $G \in \mathscr{C}_n^{k,t}$ be a graph with $k \geq 2$, then $\mathrm{gp}_e(G) \leq 3k + t$ with equality only if G has k good odd cycles.*

Corollary 2.3. *Suppose that $G \in \mathscr{C}_n^{k,t}$ is a graph with k (≥ 2) even cycles. We have $\mathrm{gp}_e(G) \leq 2k + t$ with equality if and only if all even cycles of G are normal I.*

Theorem 2.4. *Let $G \in \mathscr{C}_n^{k,t}$ be a graph with $k \geq 1$, then*

$$\mathrm{gp}_e(G) \leq \max\{5,\ 3k + t\},$$

where equality holds if and only if all cycles of G are good odd cycles.

Proof. Suppose that $G \in \mathscr{C}_n^{k,t}$ is a graph with $k \geq 1$ and $t \geq 0$. Let C_l be a cycle of G with length l. We will consider three cases to proceed with the proof.

Case 1 $k = 1$ and $t \leq 1$.

In fact, G is a unicyclic graph with a unique cycle C_l. Clearly, $3k + t \leq 4 < 5$. It is easy to check that $\mathrm{gp}_e(G) \leq 5$ with equality holds if and only if $G \cong C_5$. At the time, C_5 is a good odd cycle of G.

Case 2 $k = 1$ and $t \geq 2$.

Note that C_l is a unique cycle of G. In addition, $5 \leq 3k + t$. By direct checking, we obtain that $\mathrm{gp}_e(G) \leq 3k + t$ with equality if and only if the length of C_l is odd and C_l has a unique root tree with t leaves. So C_l is a good odd cycle.

Case 3 $k \geq 2$.

Observe that $5 < 3k + t$. Meanwhile, from Corollary 2.2, we obtain $\mathrm{gp}_e(G) \leq 3k + t$ with equality only if G contains k good odd cycles. □

2.3 The lower bounds of the edge general position number

Note that if a chain cactus G has two even end-blocks and its every inner cycle is bad and has two cut vertices, then $\mathrm{gp}_e(G) = 4$, as an example $G \cong G_k$ see Figure 12. The graph G_k also appeared in [18, Figure 2]. Observe that a cactus graph G with at least two outer cycles has $\mathrm{gp}_e(G) \geq 4$. In addition, if G has t leaves, then there is an edge general position set of G consisting of t pendant edges. It follows that $\mathrm{gp}_e(G) \geq t$. Observe that $\mathrm{gp}_e(G_{k,t}) = t$, where $G_{k,t}$ contains $k - 2$ triangles, 2 C_4 and t leaves such that each cycle has at least two leaves, as shown in Figure 12. Are they the lower bounds of the cactus graphs? In the following subsection, we will confirm the observations and obtain two sharp lower bounds of the cactus graphs.

Theorem 2.5. *Let $G \in \mathscr{C}_n^{k,t}$ be a graph with $k \geq 2$ and $t \geq 4$. We have that $\mathrm{gp}_e(G) \geq t$ with equality if and only if all cycles of G are bad.*

Proof. Let $G \in \mathscr{C}_n^{k,t}$ be a graph with $\mathrm{gp}_e(G)$ as small as possible. Recall that $\mathscr{T}(G)$ represents a set of root trees of G and L represents the set of pendant edges in $\mathscr{T}(G)$. Let S be a gp_e-set of G such that it contains as many pendant edges as possible. We observe that L is actually an edge general position set, which infers $\mathrm{gp}_e(G) \geq t$. On

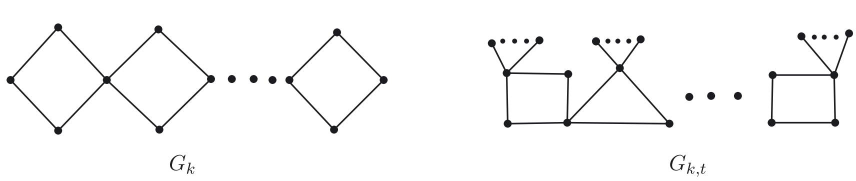

Figure 12: Two examples used in Theorems 2.5 and 2.6.

the other hand, we can check that $\mathrm{gp}_e(G_{k,t}) = t$ in which L is indeed a gp_e-set, where $G_{k,t}$ is illustrated in Figure 6. Hence, $|S| = t$ by the choice of G. In fact, we can claim that $S = L$. We assume to the contrary that $|S \cap L| \leq t-1$.

Case 1 $|S \cap L| = t-1$.

From the assumption, we know that all outer cycles of G are not end-block. Assume that e_1 is the unique pendant edge with $e_1 \in (L-S)$ and e_2 is the unique edge with $e_2 \in (S-L)$. If e_1 and e_2 are lying on the same pendant path of G, then $S' = (S-e_2) \cup \{e_1\}$ is also a gp_e-set of G. We get a contradiction with the choice of S. Hence, e_2 is one edge of some cycle in G. Clearly, e_2 belongs to a normal I cycle or a normal II cycle. (Otherwise, it is contained in some outer cycle, it follows that G has a bigger edge general position set $S \cup \{e_1\}$, a contradiction.) By Lemma 2.2, we also get a contradiction with the choice of S.

Case 2 $|S \cap L| \leq t-2$.

Let e_1 and e_2 be two pendant edges not contained in S. From Case 1, we can assume that the edges in S that are not pendant edges lie on cycles. Let e_3 and e_4 be two elements of S not contained in L. The strategy of choosing these two edges is to make them come from the same cycle as much as possible. We can assume that e_3 and e_4 are contained in some cycle by Case 1. Hence, the cycle is either an outer cycle or an inner cycle of G, say C_0. So, from Lemma 2.2 we get a contradiction to the choice of S. Therefore, we confirm that $S = L$. Together with Lemma 2.2, we deduce that all cycles of G are bad. □

Theorem 2.6. *If $G \in \mathscr{C}_n^k$ has $k \geq 2$ cycles, then $\mathrm{gp}_e(G) \geq 4$ with equality only if G is either a chain cactus or a root chain cactus for which each cycle is even. In particular, each cycle with two cut vertices has D_c equaling half the number of its vertices.*

Proof. Let $G \in \mathscr{C}_n^k$ be a graph with the minimum gp_e-number. Suppose S is a gp_e-set of G. Recall that $\mathrm{gp}_e(G_k) = 4$, so $\mathrm{gp}_e(G) \leq 4$. In addition, G has at least two outer cycles, which implies that $\mathrm{gp}_e(G) \geq 4$. Hence, $\mathrm{gp}_e(G) = |S| = 4$. Let t denote the number of leaves in G. Clearly, $t \leq 4$. (Otherwise, $\mathrm{gp}_e(G) \geq 5$.) Furthermore, G has exactly two even outer cycles for which each outer cycle includes at most two root trees.

Case 1 Outer cycles contain no root trees.

By our assumption, each outer cycle of G is an end-block. Hence, G is a chain cactus with two even end-blocks. Otherwise, $\mathrm{gp}_e(G) \geq 5$ by Lemma 2.2, a contradiction. Evidently, S contains four proper edges from two end-blocks. Recall that $|S| = 4$, which infers that all inner cycles contribute 0 edge to S. By Lemma 2.2, all inner cycles are bad and even.

Case 2 An outer cycle contains a root tree, say T.

Observe that T has at most 2 leaves. We first claim that all inner cycles are bad. Otherwise, there is an inner cycle which contributes two edges to S by Lemma 2.2. Together with proper four edges in two outer cycles, we get $|S| \geq 6$, a contradiction. Hence, the four edges of S are derived from the two outer cycles, where one outer cycle, denoted by C_0, has the root tree T. Mark the root of T as u' and the inner cut vertex of C_0 as v_0. We can deduce that u' and v_0 are diagonal of C_0.

Case 3 An outer cycle contains two root trees, denoted by T_1 and T_2.

Let C_0 be the outer cycle. Let v_1 and v_2 be the roots of T_1 and T_2, and let v_3 be the inner cut vertex of C_0. Using the same argument as in Case 1, we deduce that all inner cycles of G are bad. Observe that T_1 and T_2 are pendant paths by the minimality of S. We find that C_0 is a normal II cycle by the three cut vertices v_1, v_2 and v_3. But we can get another edge general position set with size larger than $|S|$, a contradiction.

Combining the above three cases, the conclusion is verified. □

3 Conclusions

As we know, the topological indices and other graph invariants have been explored on cactus graphs. In this paper, we research the edge general position number of cactus graphs, and bound it with the number of cycles and pendant vertices. Moreover, we obtain the lower bound and the upper bound by means of the number of good cycles, bad cycles and pendant vertices. We think that determining the formula of the edge general position number of cactus graphs regarding the number of cycles and pendant vertices is an interesting work in the future.

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References

[1] P. Manuel and S. Klavžar, *A general position problem in graph theory*, Bull. Aust. Math. Soc. **98** (2018), no. 2, 177—187.
[2] H. Dudeney, *Amusements in Mathematics*, Nelson, Edinburgh, 1917.
[3] M. Payne and D. R. Wood, *On the general position subset selection problem*, SIAM J. Discrete Math. **27** (2013), no. 4, 1727—1733.
[4] V. Froese, I. Kanj, A. Nichterlein, and R. Niedermeier, *Finding points in general position*, Int. J. Comput. Geom. Appl. **27** (2017), no. 04, 277—296.
[5] A. Misiak, Z. Stępień, A. Szymaszkiewicz, L. Szymaszkiewicz, and M. Zwierzchowski, *A note on the no-three-in-line problem on a torus*, Discrete Math. **339** (2016), no. 1, 217—221.
[6] B. Patkós, *On the general position problem on Kneser graphs*, Ars. Math. Contemp. **18** (2020), 273—280.
[7] S. Klavžar, B. Patkós, G. Rus, and I. G. Yero, *On general position sets in Cartesian products*, Results Math. **76** (2021), no. 3, 123.
[8] J. Tian and K. Xu, *The general position number of Cartesian products involving a factor with small diameter*, Appl. Math. Comput. **403** (2021), 126206.
[9] J. Tian, K. Xu, and D. Chao, *On the general position numbers of maximal outerplanar graphs*, Bull. Malays. Math. Sci. Soc. **46** (2023), 198.
[10] U. S. V. Chandran, S. Klavžar, and J. Tuite, *The general position problem: a survey* (2025), arXiv:2501.19385, https://doi.org/10.48550/arXiv.2501.19385.
[11] P. Manuel, R. Prabha, and S. Klavžar, *The edge general position problem*, Bull. Malays. Math. Sci. Soc. **45** (2022), no. 6, 2997—3009.
[12] P. Manuel, R. Prabha, and S. Klavžar, *Generalization of edge general position problem*, Art Discrete Appl. Math. **8** (2024), #P1.02.
[13] S. Klavžar and E. Tan, *Edge general position sets in Fibonacci and Lucas cubes*, Bull. Malays. Math. Sci. Soc. **46** (2023), no. 4, 120.
[14] I. Gutman, S. Li, and W. Wei, *Cacti with n-vertices and t cycles having extremal Wiener index*, Discrete Appl. Math. **232** (2017), 189—200.
[15] J. Li, K. Xu, T. Zhang, H. Wang, and S. Wagner, *Maximum number of subtrees in cacti and block graphs*, Aequationes Math. **96** (2022), no. 5, 1027—1040.
[16] Y. Yao, M. He, and S. Ji, *On the general position number of two classes of graphs*, Open. Math. **20** (2022), no. 1, 1021—1029.
[17] S. Wang, *On extremal cacti with respect to the Szeged index*, Appl. Math. Comput. **309** (2017), 8592.
[18] J. Tian, S. Klavžar, and E. Tan, *Extremal edge general position sets in some graphs*, Graphs Combin. **40** (2024), no. 2, 40.