

Research Article

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Pedal and negative pedal surfaces of framed curves in the Euclidean 3-space

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Abstract: We define pedal and negative pedal surfaces of framed curves in the Euclidean 3-space and find that the loci of singularities of them are pedal and negative pedal curves, respectively. Moreover, we give sufficient conditions that pedal and negative pedal surfaces to be framed base surfaces. We also give a sufficient condition that negative pedal curves to be framed base curves.

Keywords: pedal surface; negative pedal surface; negative pedal curve; framed curve; framed surface

MSC 2020: 53A05; 53A04; 57R45

1 Introduction

Curves or surfaces generated by curves are important branches in differential geometry. Some properties of the original curve can be reflected by the new curve or surface. For example, given a smooth regular curve with a vertex in the Euclidean plane, its evolute is singular at the point corresponding to the vertex of the original curve. There are also relations between curves and surfaces generated by the same curve. For a spatial curve, the singularity locus of its focal surface coincides with its evolute [1] (Figure 1).

People first considered that original curves are regular. In fact, there is a limitation. With the development of Legendre curves, people can deal with this kind of curves, which may have singularities, similarly to regular curves [2]. A Legendre curve can be determined by its invariants uniquely up to rigid motions. For a Legendre curve (Legendre immersion), its base curve is called a frontal (front). Lifting the dimension of the space, Honda and Takahashi defined framed curves in the Euclidean n -space [3]. There is more than one way to frame a curve and different frames can be used to research its properties [4]. One can find an adapted frame to make some questions easier. Honda and Takahashi defined the Frenet type frame along a framed base curve in the Euclidean 3-space [1,5]. So far, framed curves with Frenet type frames can be regarded as the generalization of Frenet curves.

Roulettes and pedal curves play an important role in the study of road-wheel pairs. Pedal curves have connections with moving wheels [6]. In astronomy, Kepler discovered the trace of a moving planet is an ellipse. The ellipse is called Kepler's ellipse. Hamilton claimed that the pedal curve of Kepler's ellipse relative to the focus is a circle [7]. That is to say the ellipse is the negative pedal curve of the circle. Later, pedal curves in different spaces have been studied by many researchers. People were first concerned about pedal curves of regular curves. They found the pedal curve is singular if the original curve has inflections or the pedal point lies on

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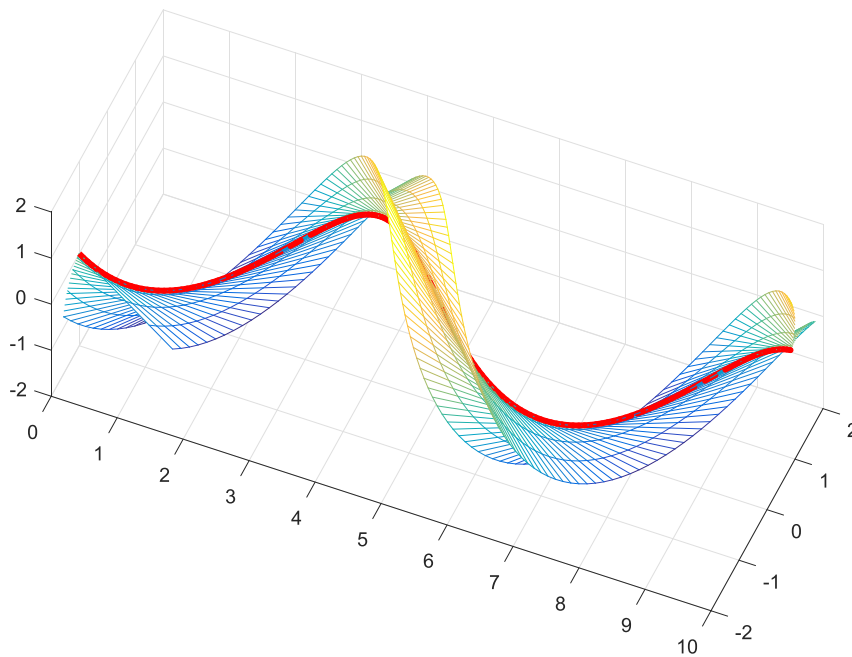


Figure 1: A focal surface (mesh) and an evolute (red).

the original curve. Later, Nishimura defined pedal curves in n -dimensional sphere and studied normal forms for singularities of them [8,9]. Using the theory of Legendre curves, people defined pedal curves of fronts and frontals in the Euclidean plane [10,11]. It contains the case of regular plane curves. However, pedal curves of frontals may be not frontals. Tuncer et al. gave a sufficient condition when the pedal curve of a front is a frontal. We studied the pedal curves of framed curves with Frenet type frames in the Euclidean 3-space [12]. Li and the second author of this paper researched pedal-contrapedal curve pairs of fronts in the unit sphere and discussed their relationships with the involute-evolute curve pairs [13]. As for other spaces, such as Minkowski space and its submanifolds, pedal curves were defined and studied [14–16]. If the orthogonal projection is changed to the slant projection, one can get pedaloids in plane. There are some results about pedaloids in the Euclidean plane and the Lorentz plane [17–19]. Pedal curves in a plane have applications in physics. One can use pedal curves to generate coordinates, called pedal coordinates, to deal with some mechanical problems [20–22].

Naturally, there is an inverse problem: if we know a curve and a fixed point, how can we get a new curve such that its pedal curve relative to the fixed point is the given curve? The new curve is called the negative pedal curve or the primitive. Arnol'd researched the primitive of a hypersurface [23]. It is the envelope of the normal hyperplanes. Izumiya and Takeuchi introduced the primitivoids of plane curves, which are generalizations of primitives [24]. For n dimensional frontals in the $(n + 1)$ dimensional Euclidean space, Janeczko and Nishimura defined their negative pedals [25]. This is the generalization of the case of Arnol'd.

In the Euclidean plane, the pedal curve is the envelope of a family of circles and the primitive is the envelope of a family of lines. Taking the pedal curve and taking the primitive can be regarded as inverse processes. We consider envelopes of a family of spheres and a family of planes in the Euclidean 3-space, call them the pedal surface and the negative pedal surface, respectively. Previously, people usually studied pedal surfaces of surfaces [26]. It is a transformation of the same dimension. We change the correspondence of dimension. Based on our previous work [12], we research pedal surfaces and negative pedal surfaces.

In the present paper, we define pedal surfaces, negative pedal surfaces and negative pedal curves of framed curves. The singular locus of the pedal surface is the pedal curve and the pedal point. For the singular locus of the negative pedal surface, if we can establish the Frenet type frame along it and take its pedal curve, then it is the original curve. So we call it the negative pedal curve. We get sufficient conditions when the pedal surface

and the negative pedal surface are framed base surfaces. Finally, we give two examples to show connections among pedal curves, negative pedal curves, pedal surfaces and negative pedal surfaces.

All maps and manifolds considered here are differentiable of class C^∞ .

2 Preliminaries

Let $\mathbb{R}^3$ be the Euclidean 3-space with the inner product $\cdot$, the vector product $\times$ and the norm $\|\cdot\|$. We denote $\Delta = \{(\mathbf{a}, \mathbf{b}) \in S^2 \times S^2 \mid \mathbf{a} \cdot \mathbf{b} = 0\}$.

Definition 2.1 ([3]). I is an interval of $\mathbb{R}$. $(\gamma, \mathbf{v}_1, \mathbf{v}_2): I \rightarrow \mathbb{R}^3 \times \Delta$ is called a framed curve if $\gamma'(t) \cdot \mathbf{v}_1(t) = \gamma'(t) \cdot \mathbf{v}_2(t) = 0$ for all $t \in I$. We call $\gamma: I \rightarrow \mathbb{R}^3$ a framed base curve if there exists $(\mathbf{v}_1, \mathbf{v}_2): I \rightarrow \Delta$ such that $(\gamma, \mathbf{v}_1, \mathbf{v}_2)$ is a framed curve.

For a framed curve $(\gamma, \mathbf{v}_1, \mathbf{v}_2)$, define $\boldsymbol{\mu}(t) = \mathbf{v}_1(t) \times \mathbf{v}_2(t)$. Then $\{\mathbf{v}_1(t), \mathbf{v}_2(t), \boldsymbol{\mu}(t)\}$ is a moving frame along $\gamma(t)$. The Frenet type formulas are

$$\begin{pmatrix} \mathbf{v}_1'(t) \\ \mathbf{v}_2'(t) \\ \boldsymbol{\mu}'(t) \end{pmatrix} = \begin{pmatrix} 0 & l(t) & m(t) \\ -l(t) & 0 & n(t) \\ -m(t) & -n(t) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \gamma'(t) = \alpha(t)\boldsymbol{\mu}(t),$$

where

$$l(t) = \mathbf{v}_1'(t) \cdot \mathbf{v}_2(t), m(t) = \mathbf{v}_1'(t) \cdot \boldsymbol{\mu}(t), n(t) = \mathbf{v}_2'(t) \cdot \boldsymbol{\mu}(t), \alpha(t) = \gamma'(t) \cdot \boldsymbol{\mu}(t).$$

The map $(l, m, n, \alpha): I \rightarrow \mathbb{R}^4$ is called the curvature of the framed curve $(\gamma, \mathbf{v}_1, \mathbf{v}_2)$. It is clear that γ is singular at t_0 if and only if $\alpha(t_0) = 0$. We always suppose singularities of γ are discrete.

By the above definition, we know the linear combination of $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ is orthogonal to $\gamma'(t)$. If $(m(t), n(t)) \neq (0, 0)$ for all $t \in I$, we take

$$\begin{pmatrix} \mathbf{n}_1(t) \\ \mathbf{n}_2(t) \end{pmatrix} = \frac{1}{\sqrt{m^2(t) + n^2(t)}} \begin{pmatrix} m(t) & n(t) \\ -n(t) & m(t) \end{pmatrix} \begin{pmatrix} \mathbf{v}_1(t) \\ \mathbf{v}_2(t) \end{pmatrix}.$$

Then $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ is also a framed curve and $\mathbf{n}_1(t) \times \mathbf{n}_2(t) = \boldsymbol{\mu}(t)$. $\{\mathbf{n}_1(t), \mathbf{n}_2(t), \boldsymbol{\mu}(t)\}$ is called the Frenet type frame along $\gamma(t)$. The Frenet type formulas are

$$\begin{pmatrix} \mathbf{n}_1'(t) \\ \mathbf{n}_2'(t) \\ \boldsymbol{\mu}'(t) \end{pmatrix} = \begin{pmatrix} 0 & L(t) & M(t) \\ -L(t) & 0 & 0 \\ -M(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{n}_1(t) \\ \mathbf{n}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \gamma'(t) = \alpha(t)\boldsymbol{\mu}(t),$$

where

$$L(t) = \frac{m(t)n'(t) - m'(t)n(t) + l(t)(m^2(t) + n^2(t))}{m^2(t) + n^2(t)}, M(t) = \sqrt{m^2(t) + n^2(t)}.$$

The map $(L, M, 0, \alpha): I \rightarrow \mathbb{R}^4$ is the curvature of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ [1, 5]. The $\mathbf{n}_1, \mathbf{n}_2, \boldsymbol{\mu}$ direction are called the principal normal direction, the binormal direction and the tangent direction of γ . In the present paper, we always use $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ to represent a framed curve with the Frenet type frame $\{\mathbf{n}_1, \mathbf{n}_2, \boldsymbol{\mu}\}$ and the curvature $(L, M, 0, \alpha)$. The surface which consists of all tangent lines of γ is called the tangent developable of γ , that is $\mathbf{x}(t, u) = \gamma(t) + u\boldsymbol{\mu}(t)$.

In our previous paper, we have defined pedal curves of framed curves by orthogonal projection.

Definition 2.2 ([12]). Let $(\gamma, \mathbf{n}_1, \mathbf{n}_2): I \rightarrow \mathbb{R}^3 \times \Delta$ be a framed curve. $\mathbf{p} \in \mathbb{R}^3$ is a fixed point. The pedal curve $Pe_{\gamma, \mathbf{p}}: I \rightarrow \mathbb{R}^3$ of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$ is

$$Pe_{\gamma, \mathbf{p}}(t) = \mathbf{p} - ((\mathbf{p} - \gamma(t)) \cdot \mathbf{n}_2(t))\mathbf{n}_2(t).$$

We call $\mathbf{p}$ the pedal point.

Definition 2.3 ([27]). U is an open domain in $\mathbb{R}^2$. $(\mathbf{x}, \mathbf{n}, \mathbf{s}): U \rightarrow \mathbb{R}^3 \times \Delta$ is called a framed surface if $\mathbf{x}_u(u, v) \cdot \mathbf{n}(u, v) = \mathbf{x}_v(u, v) \cdot \mathbf{n}(u, v) = 0$ for all $(u, v) \in U$, where $\mathbf{x}_u(u, v) = \frac{\partial \mathbf{x}}{\partial u}(u, v)$ and $\mathbf{x}_v(u, v) = \frac{\partial \mathbf{x}}{\partial v}(u, v)$.

Let $F: J \times \mathbb{R}^3 \rightarrow \mathbb{R}$, $(t, \mathbf{x}) \mapsto F(t, \mathbf{x})$ be a smooth map, where J is an interval of $\mathbb{R}$. Then $F = 0$ means a one-parameter family of surfaces in $\mathbb{R}^3$. If there exists a surface S tangent to each member of the family at some point and any point on S is on one of surfaces in the family, then S is called the envelope of the family of surfaces [28]. One can get the envelope by eliminating the parameter t from $F(t, \mathbf{x}) = \frac{\partial F}{\partial t}(t, \mathbf{x}) = 0$.

3 Pedal and negative pedal surfaces

Definition 3.1. Let $(\gamma, \mathbf{n}_1, \mathbf{n}_2): I \rightarrow \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature $(L, M, 0, \alpha)$ and $\mathbf{p} \in \mathbb{R}^3$.

(a) The pedal surface $\mathcal{P}S_{\gamma, \mathbf{p}}: I \times [0, 2\pi] \rightarrow \mathbb{R}^3$ of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$ is

$$\begin{aligned} \mathcal{P}S_{\gamma, \mathbf{p}}(t, \theta) = \mathbf{p} + & \frac{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t) + \sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2} \cos \theta}{2} \mathbf{n}_1(t) \\ & + \frac{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t) + \sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2} \sin \theta}{2} \mathbf{n}_2(t). \end{aligned}$$

(b) Suppose the point $\mathbf{p}$ satisfies $(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t) \neq 0$ for all $t \in I$. The negative pedal surface $\mathcal{N}\mathcal{P}S_{\gamma, \mathbf{p}}: I \times \mathbb{R} \rightarrow \mathbb{R}^3$ of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$ is

$$\mathcal{N}\mathcal{P}S_{\gamma, \mathbf{p}}(t, \lambda) = 2\gamma(t) - \mathbf{p} + \lambda \mathbf{n}_1(t) - \frac{(\gamma(t) - \mathbf{p}) \cdot (\gamma(t) - \mathbf{p}) + \lambda(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t)}{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t)} \mathbf{n}_2(t).$$

Remark 3.2. If the point $\mathbf{p}$ is on the tangent developable of γ , then the pedal surface $\mathcal{P}S_{\gamma, \mathbf{p}}$ is not a smooth surface.

When the curve γ is regular, the two surfaces defined in Definition 3.1 can be both regarded as envelopes of families of surfaces.

Proposition 3.3. Let $(\gamma, \mathbf{n}_1, \mathbf{n}_2): I \rightarrow \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature $(L, M, 0, \alpha)$ and $\mathbf{p} \in \mathbb{R}^3$, where $\alpha(t) \neq 0$.

- (a) The pedal surface $\mathcal{P}S_{\gamma, \mathbf{p}}$ of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$ is the envelope of a family of spheres.
 (b) If the point $\mathbf{p}$ satisfies $(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t) \neq 0$ for all $t \in I$, then the negative pedal surface $\mathcal{N}\mathcal{P}S_{\gamma, \mathbf{p}}$ of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$ is the envelope of a family of planes.

Proof. (a) Define $F: I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ by $F(t, \mathbf{x}) = (\gamma(t) - \mathbf{x}) \cdot (\mathbf{x} - \mathbf{p})$. It is clear that $F = 0$ is a family of spheres and

$$\frac{\partial F}{\partial t}(t, \mathbf{x}) = \alpha(t) \boldsymbol{\mu}(t) \cdot (\mathbf{x} - \mathbf{p}).$$

Let $F = \frac{\partial F}{\partial t} = 0$, then there exists $\theta \in [0, 2\pi]$ such that

$$\begin{aligned} \mathbf{x} - \mathbf{p} = & \frac{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t) + \sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2} \cos \theta}{2} \mathbf{n}_1(t) \\ & + \frac{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t) + \sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2} \sin \theta}{2} \mathbf{n}_2(t). \end{aligned}$$

(b) Define $G: I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ by $G(t, \mathbf{x}) = (\gamma(t) - \mathbf{x}) \cdot (\gamma(t) - \mathbf{p})$. It is clear that $G = 0$ is a family of planes and

$$\frac{\partial G}{\partial t}(t, \mathbf{x}) = \alpha(t)\boldsymbol{\mu}(t) \cdot (2\gamma(t) - \mathbf{p} - \mathbf{x}).$$

Let $G = \frac{\partial G}{\partial t} = 0$, then there exists $\lambda \in \mathbb{R}$ such that

$$\mathbf{x} = 2\gamma(t) - \mathbf{p} + \lambda \mathbf{n}_1(t) - \frac{(\gamma(t) - \mathbf{p}) \cdot (\gamma(t) - \mathbf{p}) + \lambda(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t)}{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t)} \mathbf{n}_2(t).$$

□

Definition 3.4. Let $(\gamma, \mathbf{n}_1, \mathbf{n}_2): I \rightarrow \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature $(L, M, 0, \alpha)$ and $\mathbf{p} \in \mathbb{R}^3$ satisfy $(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t) \neq 0$ for all $t \in I$. The negative pedal curve of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$ is

$$\begin{aligned} \mathcal{NP}e_{\gamma, \mathbf{p}}(t) \\ = 2\gamma(t) - \mathbf{p} - \frac{2\alpha(t)}{M(t)} \mathbf{n}_1(t) - \frac{1}{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t)} \left((\gamma(t) - \mathbf{p}) \cdot (\gamma(t) - \mathbf{p}) - \frac{2\alpha(t)}{M(t)} (\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t) \right) \mathbf{n}_2(t). \end{aligned}$$

Now we consider singularities of the pedal surface and negative pedal surface.

Theorem 3.5. Let $(\gamma, \mathbf{n}_1, \mathbf{n}_2): I \rightarrow \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature $(L, M, 0, \alpha)$ and $\mathbf{p} \in \mathbb{R}^3$.

- (a) Suppose that the point $\mathbf{p}$ is not on the tangent developable of γ . $\mathcal{PS}_{\gamma, \mathbf{p}}$ is the pedal surface of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$. The singular locus of $\mathcal{PS}_{\gamma, \mathbf{p}}$ coincides with the pedal curve $\mathcal{Pe}_{\gamma, \mathbf{p}}$ and the point $\mathbf{p}$.
- (b) Suppose that the point $\mathbf{p}$ satisfies $(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t) \neq 0$ for all $t \in I$. $\mathcal{NP}S_{\gamma, \mathbf{p}}$ is the negative pedal surface of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$. The singular locus of $\mathcal{NP}S_{\gamma, \mathbf{p}}$ coincides with the negative pedal curve $\mathcal{NP}e_{\gamma, \mathbf{p}}$.

Proof. (a) By calculation, we have

$$\begin{aligned} 2 \frac{\partial \mathcal{PS}_{\gamma, \mathbf{p}}}{\partial t}(t, \theta) \\ = M(t) \left((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t) + \sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2 \cos \theta} \right) \boldsymbol{\mu}(t) \\ + M(t) \frac{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))((\gamma(t) - \mathbf{p}) \cdot \boldsymbol{\mu}(t))}{\sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2}} (\mathbf{n}_1(t) \cos \theta + \mathbf{n}_2(t) \sin \theta) \end{aligned} \quad (1)$$

$$\begin{aligned} + L(t) \sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2} (-\mathbf{n}_1(t) \sin \theta + \mathbf{n}_2(t) \cos \theta) \\ + M(t)((\gamma(t) - \mathbf{p}) \cdot \boldsymbol{\mu}(t)) \mathbf{n}_1(t), \\ \frac{\partial \mathcal{PS}_{\gamma, \mathbf{p}}}{\partial \theta}(t, \theta) = \frac{1}{2} \sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2} (-\mathbf{n}_1(t) \sin \theta + \mathbf{n}_2(t) \cos \theta) \end{aligned} \quad (2)$$

and

$$\begin{aligned} \frac{\partial \mathcal{PS}_{\gamma, \mathbf{p}}}{\partial t}(t, \theta) \times \frac{\partial \mathcal{PS}_{\gamma, \mathbf{p}}}{\partial \theta}(t, \theta) \\ = \frac{1}{4} M(t) \left((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t) + \sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2 \cos \theta} \right) \\ \left(-\sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2} (\mathbf{n}_1(t) \cos \theta + \mathbf{n}_2(t) \sin \theta) \right. \\ \left. + ((\gamma(t) - \mathbf{p}) \cdot \boldsymbol{\mu}(t)) \boldsymbol{\mu}(t) \right). \end{aligned}$$

So (t_0, θ_0) is a singularity of $\mathcal{PS}_{\gamma,p}$ if and only if

$$(\gamma(t_0) - \mathbf{p}) \cdot \mathbf{n}_1(t_0) + \sqrt{((\gamma(t_0) - \mathbf{p}) \cdot \mathbf{n}_1(t_0))^2 + ((\gamma(t_0) - \mathbf{p}) \cdot \mathbf{n}_2(t_0))^2} \cos \theta_0 = 0.$$

This means that

$$\mathcal{PS}_{\gamma,p}(t_0, \theta_0) = \mathbf{p} + ((\gamma(t_0) - \mathbf{p}) \cdot \mathbf{n}_2(t_0))\mathbf{n}_2(t_0)$$

or

$$\mathcal{PS}_{\gamma,p}(t_0, \theta_0) = \mathbf{p}.$$

(b) By calculation, we have

$$\begin{aligned} \frac{\partial \mathcal{NPS}_{\gamma,p}}{\partial t}(t, \lambda) &= L(t) \frac{(\gamma(t) - \mathbf{p}) \cdot (\gamma(t) - \mathbf{p}) + \lambda(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t)}{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t)} \mathbf{n}_1(t) + (2\alpha(t) + \lambda M(t))\boldsymbol{\mu}(t) \\ &\quad - \frac{1}{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2} ((2\alpha(t) + \lambda M(t))((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))((\gamma(t) - \mathbf{p}) \cdot \boldsymbol{\mu}(t)) \\ &\quad + L(t)((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))((\gamma(t) - \mathbf{p}) \cdot (\gamma(t) - \mathbf{p}) + \lambda(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))) \mathbf{n}_2(t), \end{aligned} \quad (3)$$

$$\frac{\partial \mathcal{NPS}_{\gamma,p}}{\partial \lambda}(t, \lambda) = \mathbf{n}_1(t) - \frac{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t)}{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t)} \mathbf{n}_2(t) \quad (4)$$

and

$$\frac{\partial \mathcal{NPS}_{\gamma,p}}{\partial t}(t, \lambda) \times \frac{\partial \mathcal{NPS}_{\gamma,p}}{\partial \lambda}(t, \lambda) = \frac{2\alpha(t) + \lambda M(t)}{(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t)} (\gamma(t) - \mathbf{p}).$$

So (t_0, λ_0) is a singularity of $\mathcal{NPS}_{\gamma,p}$ if and only if $2\alpha(t_0) + \lambda_0 M(t_0) = 0$. This means that

$$\mathcal{NPS}_{\gamma,p}(t_0, \lambda_0) = 2\gamma(t_0) - \mathbf{p} - \frac{2\alpha(t_0)}{M(t_0)} \mathbf{n}_1(t_0) - \frac{(\gamma(t_0) - \mathbf{p}) \cdot (\gamma(t_0) - \mathbf{p}) - \frac{2\alpha(t_0)}{M(t_0)} (\gamma(t_0) - \mathbf{p}) \cdot \mathbf{n}_1(t_0)}{(\gamma(t_0) - \mathbf{p}) \cdot \mathbf{n}_2(t_0)} \mathbf{n}_2(t_0).$$

□

The pedal surface and negative pedal surface may have singular points. We give conditions when they are framed base surfaces. For framed base surfaces, please see [27].

Proposition 3.6. Let $(\gamma, \mathbf{n}_1, \mathbf{n}_2): I \rightarrow \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature $(L, M, 0, \alpha)$ and $\mathbf{p} \in \mathbb{R}^3$.

- (a) If $\mathbf{p}$ is not on the tangent developable of γ , then the pedal surface $\mathcal{PS}_{\gamma,p}$ is a framed base surface.
- (b) Suppose that the point $\mathbf{p}$ satisfies $(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t) \neq 0$ for all $t \in I$. Then the negative pedal surface $\mathcal{NPS}_{\gamma,p}$ is a framed base surface.

Proof. (a) Take

$$\begin{aligned} \mathbf{n}(t, \theta) &= \frac{1}{\|\gamma(t) - \mathbf{p}\|} \left(((\gamma(t) - \mathbf{p}) \cdot \boldsymbol{\mu}(t))\boldsymbol{\mu}(t) \right. \\ &\quad \left. - \sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2} (\mathbf{n}_1(t) \cos \theta + \mathbf{n}_2(t) \sin \theta) \right) \end{aligned}$$

and

$$\mathbf{s}(t, \theta) = \mathbf{n}_1(t) \sin \theta - \mathbf{n}_2(t) \cos \theta.$$

By Equations (1) and (2), we can check that

$$\frac{\partial \mathcal{P}S_{\gamma,p}}{\partial t}(t, \theta) \cdot \mathbf{n}(t, \theta) = \frac{\partial \mathcal{P}S_{\gamma,p}}{\partial \theta}(t, \theta) \cdot \mathbf{n}(t, \theta) = 0.$$

So $(\mathcal{P}S_{\gamma,p}, \mathbf{n}, \mathbf{s})$ is a framed surface.

(b) Take

$$\mathbf{N}(t, \lambda) = \frac{\gamma(t) - \mathbf{p}}{\|\gamma(t) - \mathbf{p}\|}$$

and

$$\mathbf{S}(t, \lambda) = \frac{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))\mathbf{n}_1(t) - ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))\mathbf{n}_2(t)}{\sqrt{((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_1(t))^2 + ((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t))^2}}.$$

By Equations (3) and (4), we can check that

$$\frac{\partial \mathcal{N} \mathcal{P}S_{\gamma,p}}{\partial t}(t, \lambda) \cdot \mathbf{N}(t, \lambda) = \frac{\partial \mathcal{N} \mathcal{P}S_{\gamma,p}}{\partial \lambda}(t, \lambda) \cdot \mathbf{N}(t, \lambda) = 0.$$

So $(\mathcal{N} \mathcal{P}S_{\gamma,p}, \mathbf{N}, \mathbf{S})$ is a framed surface. $\square$

Following propositions claim a fact that the negative pedal curve is also a framed base curve and the relation between the negative pedal curve and the pedal curve.

Proposition 3.7. Let $(\gamma, \mathbf{n}_1, \mathbf{n}_2): I \rightarrow \mathbb{R}^3 \times \Delta$ be a framed curve with the curvature $(L, M, 0, \alpha)$ and $\mathbf{p} \in \mathbb{R}^3$ satisfy $(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t) > 0$ for all $t \in I$. $\mathcal{N} \mathcal{P}e_{\gamma,p}$ is the negative pedal curve of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$. Let

$$\begin{aligned} \bar{\mathbf{n}}_1(t) &= \left(\frac{\sqrt{((\gamma - \mathbf{p}) \cdot \mathbf{n}_1)^2 + ((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)^2}}{\|\gamma - \mathbf{p}\|} \mu - \frac{(\gamma - \mathbf{p}) \cdot \mu}{\|\gamma - \mathbf{p}\|} \frac{((\gamma - \mathbf{p}) \cdot \mathbf{n}_1)\mathbf{n}_1 + ((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)\mathbf{n}_2}{\sqrt{((\gamma - \mathbf{p}) \cdot \mathbf{n}_1)^2 + ((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)^2}} \right)(t), \\ \bar{\mathbf{n}}_2(t) &= \frac{\gamma - \mathbf{p}}{\|\gamma - \mathbf{p}\|}(t). \end{aligned}$$

Then $(\mathcal{N} \mathcal{P}e_{\gamma,p}, \bar{\mathbf{n}}_1, \bar{\mathbf{n}}_2)$ is a framed curve with the curvature $(\bar{L}, \bar{M}, 0, \bar{\alpha})$, where

$$\begin{aligned} \bar{L}(t) &= -\frac{\alpha \sqrt{((\gamma - \mathbf{p}) \cdot \mathbf{n}_1)^2 + ((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)^2}}{(\gamma - \mathbf{p}) \cdot (\gamma - \mathbf{p})}(t), \\ \bar{M}(t) &= \frac{M \|\gamma - \mathbf{p}\| (\gamma - \mathbf{p}) \cdot \mathbf{n}_2}{((\gamma - \mathbf{p}) \cdot \mathbf{n}_1)^2 + ((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)^2}(t), \\ \bar{\alpha}(t) &= \left(g \sqrt{((\gamma - \mathbf{p}) \cdot \mathbf{n}_1)^2 + ((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)^2} \right)(t), \\ g(t) &= \left(-\frac{2\alpha' M - 2\alpha M'}{M^2 (\gamma - \mathbf{p}) \cdot \mathbf{n}_2} + \frac{L(\gamma - \mathbf{p}) \cdot (\gamma - \mathbf{p})}{((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)^2} - \frac{2\alpha L(\gamma - \mathbf{p}) \cdot \mathbf{n}_1}{M((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)^2} \right)(t). \end{aligned}$$

Proof. We have

$$\mathcal{N} \mathcal{P}e'_{\gamma,p}(t) = (g((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)\mathbf{n}_1 - ((\gamma - \mathbf{p}) \cdot \mathbf{n}_1)\mathbf{n}_2)(t).$$

It is obvious that $\mathcal{N} \mathcal{P}e'_{\gamma,p}(t) \cdot \bar{\mathbf{n}}_1(t) = \mathcal{N} \mathcal{P}e'_{\gamma,p}(t) \cdot \bar{\mathbf{n}}_2(t) = 0$. So $(\mathcal{N} \mathcal{P}e_{\gamma,p}, \bar{\mathbf{n}}_1, \bar{\mathbf{n}}_2)$ is a framed curve. Let

$$\bar{\mu}(t) = \bar{\mathbf{n}}_1(t) \times \bar{\mathbf{n}}_2(t) = \frac{-((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)\mathbf{n}_1 + ((\gamma - \mathbf{p}) \cdot \mathbf{n}_1)\mathbf{n}_2}{\sqrt{((\gamma - \mathbf{p}) \cdot \mathbf{n}_1)^2 + ((\gamma - \mathbf{p}) \cdot \mathbf{n}_2)^2}}(t).$$

Note that

$$\bar{L}(t) = \bar{\mathbf{n}}'_1(t) \cdot \bar{\mathbf{n}}_2(t), \quad \bar{M}(t) = \bar{\mathbf{n}}'_1(t) \cdot \bar{\boldsymbol{\mu}}(t), \quad \bar{\alpha}(t) = \mathcal{NPe}'_{\gamma,p}(t) \cdot \bar{\boldsymbol{\mu}}(t)$$

and

$$\bar{\mathbf{n}}'_2(t) \cdot \bar{\boldsymbol{\mu}}(t) = 0.$$

We complete the proof. $\square$

Proposition 3.8. *Let $(\gamma, \mathbf{n}_1, \mathbf{n}_2): I \rightarrow \mathbb{R}^3 \times \Delta$ be a framed curve and $\mathbf{p} \in \mathbb{R}^3$ satisfy $(\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t) \neq 0$ for all $t \in I$. $\mathcal{NPe}_{\gamma,p}$ is the negative pedal curve of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$. Then the pedal curve of $\mathcal{NPe}_{\gamma,p}$ relative to $\mathbf{p}$ coincides with γ .*

Proof. By Proposition 3.7, we know the binormal direction of $\mathcal{NPe}_{\gamma,p}$ is

$$\bar{\mathbf{n}}_2(t) = \text{sign}((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2(t)) \frac{\gamma(t) - \mathbf{p}}{\|\gamma(t) - \mathbf{p}\|},$$

where sign is the signature function. Note that

$$\begin{aligned} & (\mathcal{NPe}_{\gamma,p}(t) - \mathbf{p}) \cdot \bar{\mathbf{n}}_2(t) \\ &= \left(\left(2\gamma - 2\mathbf{p} - \frac{2\alpha}{M} \mathbf{n}_1 - \frac{(\gamma - \mathbf{p}) \cdot (\gamma - \mathbf{p}) - \frac{2\alpha}{M} (\gamma - \mathbf{p}) \cdot \mathbf{n}_1}{(\gamma - \mathbf{p}) \cdot \mathbf{n}_2} \mathbf{n}_2 \right) \cdot \bar{\mathbf{n}}_2 \right)(t) \\ &= \text{sign}((\gamma(t) - \mathbf{p}) \cdot \mathbf{n}_2) \|\gamma(t) - \mathbf{p}\|. \end{aligned}$$

So the pedal curve of $\mathcal{NPe}_{\gamma,p}$ relative to $\mathbf{p}$ is

$$\begin{aligned} \mathcal{Pe}_{\mathcal{NPe}_{\gamma,p},p}(t) &= \mathbf{p} - ((\mathbf{p} - \mathcal{NPe}_{\gamma,p}(t)) \cdot \bar{\mathbf{n}}_2(t)) \bar{\mathbf{n}}_2(t) \\ &= \mathbf{p} + \|\gamma(t) - \mathbf{p}\| \frac{\gamma(t) - \mathbf{p}}{\|\gamma(t) - \mathbf{p}\|} \\ &= \gamma(t). \end{aligned}$$

$\square$

4 Examples

Here, we give two examples to show pedal and negative pedal surfaces of framed curves.

Example 4.1. Let $(\gamma, \mathbf{n}_1, \mathbf{n}_2): (0, +\infty) \rightarrow \mathbb{R}^3 \times \Delta$ be

$$\gamma(t) = (\cos t, \sin t, t), \quad \mathbf{n}_1(t) = (\cos t, \sin t, 0), \quad \mathbf{n}_2(t) = \frac{\sqrt{2}}{2}(-\sin t, \cos t, -1)$$

with the curvature $L(t) = \frac{\sqrt{2}}{2}$, $M(t) = \frac{\sqrt{2}}{2}$, $\alpha(t) = \sqrt{2}$.

Let $\mathbf{p} = \mathbf{0}$. The pedal curve, negative pedal curve, pedal surface and negative pedal surface of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$ are

$$\begin{aligned} \mathcal{Pe}_{\gamma,p}(t) &= -\frac{t}{2}(-\sin t, \cos t, -1), \\ \mathcal{NPe}_{\gamma,p}(t) &= (-2 \cos t, -2 \sin t, 2t) + \frac{t^2 - 3}{t}(-\sin t, \cos t, -1), \\ \mathcal{PS}_{\gamma,p}(t, \theta) &= \frac{\sqrt{2}}{4} \left(\sqrt{2} + \sqrt{2 + t^2} \cos \theta \right) (\cos t, \sin t, 0) + \frac{-t + \sqrt{2 + t^2} \sin \theta}{4} (-\sin t, \cos t, -1) \end{aligned}$$

and

$$\mathcal{NP}S_{\gamma,p}(t, \lambda) = (2 \cos t, 2 \sin t, 2t) + \lambda(\cos t, \sin t, 0) + \frac{t^2 + 1 + \lambda}{t}(-\sin t, \cos t, -1),$$

respectively (Figures 2 and 3).

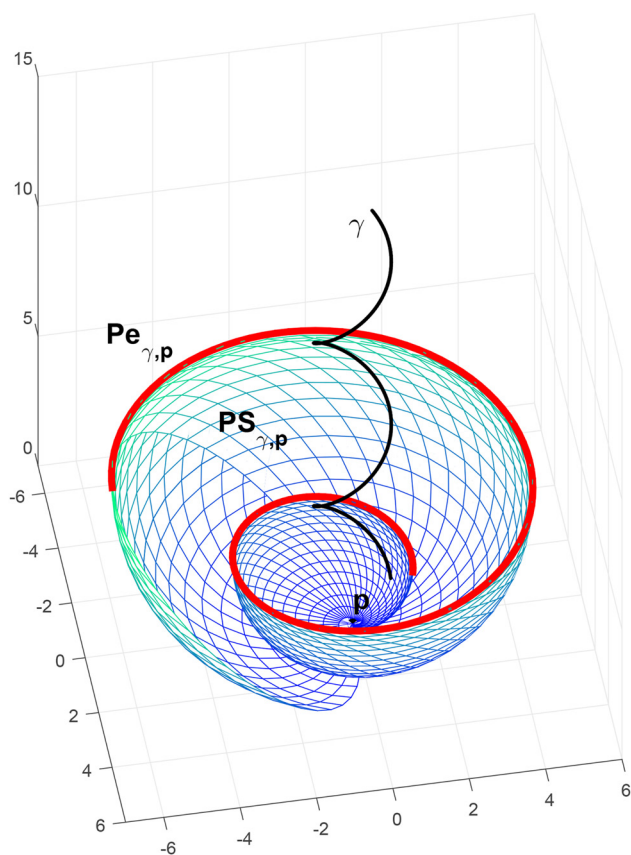


Figure 2: γ (black), its pedal curve (red) and pedal surface (mesh).

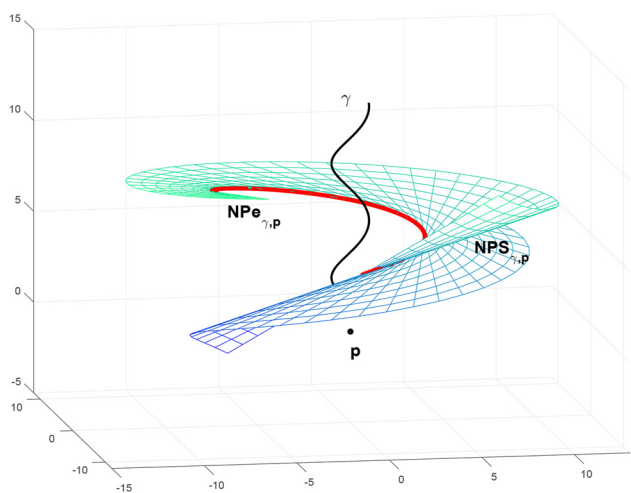


Figure 3: γ (black), its negative pedal curve (red) and negative pedal surface (mesh).

Example 4.2. Let $(\gamma, \mathbf{n}_1, \mathbf{n}_2): (-\sqrt{2}, \sqrt{2}) \rightarrow \mathbb{R}^3 \times \Delta$ be

$$\gamma(t) = (3(t \sin t + \cos t), 3(t \cos t - \sin t), 2t^2),$$

$$\mathbf{n}_1(t) = (-\sin t, -\cos t, 0),$$

$$\mathbf{n}_2(t) = \frac{1}{5}(-4 \cos t, 4 \sin t, 3)$$

with the curvature $L(t) = \frac{4}{5}$, $M(t) = \frac{3}{5}$, $\alpha(t) = -5t$.

Let $\mathbf{p} = \mathbf{0}$. The pedal curve, negative pedal curve, pedal surface and negative pedal surface of $(\gamma, \mathbf{n}_1, \mathbf{n}_2)$ relative to $\mathbf{p}$ are

$$\mathcal{P}e_{\gamma, \mathbf{p}}(t) = \frac{6t^2 - 12}{25}(-4 \cos t, 4 \sin t, 3),$$

$$\mathcal{N}\mathcal{P}e_{\gamma, \mathbf{p}}(t) = \left(-\frac{32}{3}t \sin t + 6 \cos t, -\frac{32}{3}t \cos t - 6 \sin t, 4t^2\right) + \frac{4t^4 - 41t^2 + 9}{6t^2 - 12}(-4 \cos t, 4 \sin t, 3),$$

$$\begin{aligned} \mathcal{P}S_{\gamma, \mathbf{p}}(t, \theta) &= \frac{-15t + \sqrt{36t^4 + 81t^2 + 144} \cos \theta}{10}(-\sin t, -\cos t, 0) \\ &\quad + \frac{6t^2 - 12 + \sqrt{36t^4 + 81t^2 + 144} \sin \theta}{50}(-4 \cos t, 4 \sin t, 3) \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}\mathcal{P}S_{\gamma, \mathbf{p}}(t, \lambda) &= (6(t \sin t + \cos t), 6(t \cos t - \sin t), 4t^2) - \lambda(\sin t, \cos t, 0) \\ &\quad - \frac{4t^4 + 9t^2 + 9 - 3t\lambda}{6t^2 - 12}(-4 \cos t, 4 \sin t, 3), \end{aligned}$$

respectively (Figures 4 and 5).

From above examples, we can see the singular locus of the pedal surface and the negative pedal surface. This is consistent with Theorem 3.5.

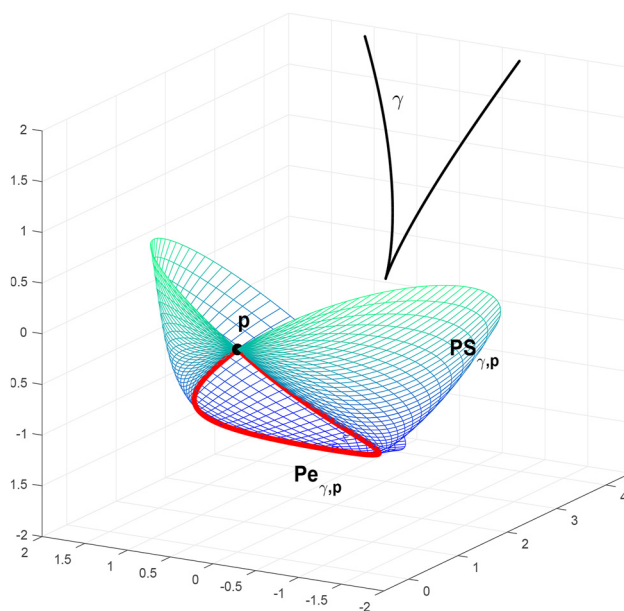


Figure 4: γ (black), its pedal curve (red) and pedal surface (mesh).

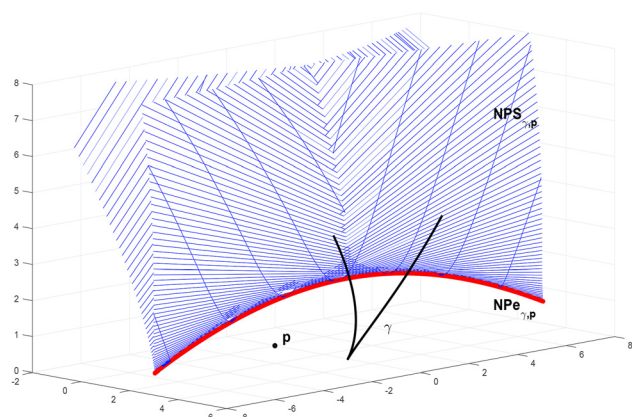


Figure 5: γ (black), its negative pedal curve (red) and negative pedal surface (mesh).

5 Conclusions

We define pedal and negative pedal surfaces of framed curves in the Euclidean 3-space. Their singular loci are pedal and negative pedal curves, respectively. The two surfaces are envelopes of families of spheres and planes, respectively. Moreover, we also give sufficient conditions when they can be framed base surfaces. As for negative pedal curves, we give the sufficient condition when they can be framed base curves and discuss their connection with pedal curves. The theoretical value of our research is the correspondence between curves and surfaces, which is different from the correspondence between manifolds with the same dimension in most previous studies.

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