Research Article

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p-variation and Chung's LIL of sub-bifractional Brownian motion and applications

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Abstract: Let $S^{H,K} = \{S^{H,K}(t), t \geq 0\}$ be the sub-bifractional Brownian motion, with $H \in (0, 1)$ and $K \in (0, 1]$. We investigate its p-variation and Chung's law of the iterated logarithm. In addition, we give some applications of these properties.

Keywords: sub-bifractional Brownian motion, p-variation, Chung's law of the iterated logarithm

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1 Introduction

El-Nouty and Journé [1] introduced the process $S^{H,K} = \{S^{H,K}(t), t \geq 0\}$ with $H \in (0, 1)$ and $K \in (0, 1]$, named the sub-bifractional Brownian motion (sbfBm) and defined by

$$S^{H,K}(t) = \frac{1}{2^{(2-K)/2}}(B^{H,K}(t) + B^{H,K}(-t)),$$

where $\{B^{H,K}(t), t \in \mathbf{R}\}$ is a bifractional Brownian motion (bfBm) with $H \in (0, 1)$ and $K \in (0, 1]$. Clearly, the sbfBm is a centered Gaussian process such that $S^{H,K}(0) = 0$, with probability 1, and $\mathrm{Var}(S^{H,K}(t)) = (2^K - 2^{2HK-1})t^{2HK}$. Note that $(2H - 1)K - 1 < K - 1 \leq 0$, we have $2HK - 1 < K$. We can prove that $S^{H,K}$ is self-similar with index HK. When $K = 1$, $S^{H,1}$ is the subfractional Brownian motion (sfBm). We can easily obtain that for all $s, t \geq 0$,

$$\mathbf{E}(S^{H,K}(t)S^{H,K}(s)) = (t^{2H} + s^{2H})^K - \frac{1}{2}(t + s)^{2HK} - \frac{1}{2}|t - s|^{2HK} \tag{1.1}$$

and

$$C_1 |t - s|^{2HK} \leq \mathbf{E}[(S^{H,K}(t) - S^{H,K}(s))^2] \leq C_2 |t - s|^{2HK}, \tag{1.2}$$

where

$$C_1 = \min\{2^K - 1, 2^K - 2^{2HK-1}\}, \quad C_2 = \max\{1, 2 - 2^{2HK-1}\}.$$

(See [1]).

El-Nouty and Journé [1] proved that the sbfBm is a quasi-helix in the sense of Kahane, and the upper classes of some of its increments are characterized by an integral test. Kuang [2] investigated the collision local time of two independent sbfBms. Kuang and Li [3] obtained Berry-Esséen bounds and proved the almost sure

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central limit theorem for the quadratic variation in the sbfBm. Finally, Kuang and Xie [4] studied least squares-type estimators for the drift parameters in the sub-bifractional Vasicek processes.

In this article, we investigate p-variation and Chung's law of the iterated logarithm (Chung's LIL) of sbfBm. In addition, we give some applications of these properties.

Throughout this article, some specific constants in Section i are numbered as $c_{i,1}, c_{i,2}, \ldots$.

This study is organized as follows: In Section 2, we study p-variation. Section 3 is devoted to Chung's LIL. Section 4 contains some applications of its properties.

2 p-variation

The variation in Gaussian processes was studied extensively since the works of [5], which proved almost sure convergence to 1 of the quadratic variation $\sum_{j=1}^{2^n}|B(j/2^n) - B((j-1)/2^n)|^2$ of the Brownian motion B on $[0, 1]$. Many new results about the variation in Gaussian processes with stationary increments were obtained (refer [6–9] and references therein). Wang [10] studied the p-variation in bfBm. Shen et al. [11] obtained the power variation in the sfBm.

We will consider p-variation in sbfBm by using the ideas of Wang [10] and Shen et al. [11]. However, the increments of sbfBm are not independent and not stationary, this causes some difficulties to investigate the variation in the process. In order to overcome the difficulties, we develop a stochastic integral representation of sbfBm.

Now, we state our main results in this section as follows.

Theorem 2.1. *Let $T > 0$, $a > 0$, and $v_n = n^a$. Then, for any $p \geq 1$, we have, as $n \to \infty$,*

$$\frac{1}{v_n^{1-pHK}} \sum_{j=1}^{[Tv_n]} \left| S^{H,K}\left(\frac{j}{v_n}\right) - S^{H,K}\left(\frac{j-1}{v_n}\right) \right|^p \longrightarrow T \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})}, \quad a.s., \tag{2.1}$$

where $[x]$ denotes the integer part of $x > 0$, and $\Gamma(x) \coloneqq \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ for $x > 0$, which is a Gamma function.

Corollary 2.2. *Let $T > 0$, $a > 0$, and $v_n = n^a$. Then, for any $p \geq 1$, we have, as $n \to \infty$,*

$$\frac{1}{v_n^{1-pHK}} \sum_{j=1}^{[Tv_n]} \left| \left(S^{H,K}\left(\frac{j}{v_n}\right)\right)^2 - \left(S^{H,K}\left(\frac{j-1}{v_n}\right)\right)^2 \right|^p \longrightarrow T \frac{2^{(3p)/2}\Gamma(\frac{p+1}{2})}{\Gamma\left(\frac{1}{2}\right)} \int_0^T |S^{H,K}(x)|^p \mathrm{d}x, \quad a.s. \tag{2.2}$$

Theorem 2.3. *Let $T > 0$, $a > 0$, and $v_n = n^a$. Then, we have, as $n \to \infty$,*

$$\frac{1}{v_n^{1-HK}} \sum_{j=1}^{[Tv_n]} \sup_{\frac{j-1}{v_n} < t,u < \frac{j}{v_n}} |S^{H,K}(t) - S^{H,K}(u)| \longrightarrow 2\mathbf{E}\left(\sup_{0 \leq t \leq T^{1/(HK)}} S^{H,K}(t) \right), \quad a.s. \tag{2.3}$$

In order to prove Theorems 2.1 and 2.3, we give some technical lemmas. Lemma 2.1 is a Fernique-type inequality for $S^{H,K}$.

Lemma 2.1. *For any $\varepsilon > 0$, there exists a positive constant $c_{2,1} = c_{2,1}(\varepsilon) > 0$ such that*

$$\mathbf{P}\left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq a} |S^{H,K}(t+s) - S^{H,K}(t)| \geq x a^{HK} \right\} \leq c_{2,1}\left(\frac{T}{a} + 1\right) e^{-\frac{x^2}{2(1+\varepsilon)}}, \tag{2.4}$$

for any $T \geq 0$, $a > 0$, and $x \geq x_0 > 0$ with some $x_0 > 0$.

Proof. By (1.2) and the inequality for the normal distribution function $\Phi(x) : 1 - \Phi(x) \le \frac{1}{x} e^{-\frac{x^2}{2}}$ for all $x > 0$, we obtain that

$$\mathbf{P}\{|S^{H,K}(t+h) - S^{H,K}(t)| \ge x h^{HK}\} \le c_{2,2} e^{-\frac{x^2}{2}}$$

for any $t \ge 0$, $h > 0$, and $x \ge x^* > 0$ with some $x^* > 0$. Therefore, by Lemmas 2.1 and 2.2 in [8] (when applied to $\sigma_1(h) = h^{HK}$ and $\sigma_2(\cdot) \equiv 0$), we obtain (2.4) immediately. □

Lemma 2.2 is from [12].

Lemma 2.2. *Let $X = \{X(t), t \in \mathbf{R}\}$ be a centered Gaussian process in $\mathbf{R}$ and let $F \subset \mathbf{R}$ be a closed set equipped with the canonical metric defined by*

$$d(s,t) = [\mathbf{E}(X(s) - X(t))^2]^{1/2}. \tag{2.5}$$

Then, there exists a positive constant $c_{2,3}$ such that for all $u > 0$,

$$\mathbf{P}\left\{\sup_{s,t\in F} |X(s) - X(t)| \ge c_{2,3}\left(u + \int_0^D \sqrt{\log N_d(F,\varepsilon)}\,\mathrm{d}\varepsilon\right)\right\} \le \exp\left(-\frac{u^2}{D^2}\right), \tag{2.6}$$

where $N_d(F,\varepsilon)$ denotes the smallest number of open d-balls of radius ε needed to cover F and where $D = \sup\{d(s,t) : s,t \in F\}$ is the diameter of F.

Lemma 2.3. *If $X(t)$ and $Y(t)$ are a.s. bounded, centered Gaussian processes on Λ such that $\mathbf{E}(X^2(t)) = \mathbf{E}(Y^2(t))$ for all $t \in \Lambda$, and*

$$\mathbf{E}[(X(t) - X(s))^2] \le \mathbf{E}[(Y(t) - Y(s))^2], \quad \forall s,t \in \Lambda,$$

then for all real λ,

$$\mathbf{P}\left(\sup_{t\in\Lambda} X(t) > \lambda\right) \le \mathbf{P}\left(\sup_{t\in\Lambda} Y(t) > \lambda\right)$$

and

$$\mathbf{E}\left(\sup_{t\in\Lambda} X(t)\right) \le \mathbf{E}\left(\sup_{t\in\Lambda} Y(t)\right).$$

Proof. It is Slepian's inequality (see, p. 49 in [13]). □

In order to solve the dependence structure of $S^{H,K}$ and to create independence, we will develop the stochastic integral representation of $S^{H,K}$. By Lamperti's transformation [14], we define Gaussian process $Y = \{Y(t), t \in \mathbf{R}\}$ as follows:

$$Y(t) = e^{-HKt} S^{H,K}(e^t), \quad t \in \mathbf{R}. \tag{2.7}$$

The covariance function $r(t) \coloneqq \mathbf{E}(Y(0)Y(t))$ is given by

$$\begin{aligned} r(t) &= e^{-HKt}\left\{(1 + e^{2Ht})^K - \frac{1}{2}(1 + e^t)^{2HK} - \frac{1}{2}|1 - e^t|^{2HK}\right\} \\ &= e^{HKt}\left\{(1 + e^{-2Ht})^K - \frac{1}{2}(1 + e^{-t})^{2HK} - \frac{1}{2}|1 - e^{-t}|^{2HK}\right\} \\ &= r(-t). \end{aligned} \tag{2.8}$$

Hence, $r(t)$ is an even function and, by (2.8) and the Taylor expansion, we verify that $r(t) = O(e^{-\beta t})$ as $t \to \infty$, where $\beta = H(2-K)$. It follows that $r(\cdot) \in L^1(\mathbf{R})$. By (2.8) and the Taylor expansion we obtain

$$r(t) \sim 2^K - 2^{2HK-1} - \frac{1}{2}|t|^{2HK}, \quad t \to 0. \tag{2.9}$$

By Bochner's theorem [15], Y has the stochastic integral representation:

$$Y(t) = \int_{\mathbf{R}} e^{i\lambda t} W(\mathrm{d}\lambda), \quad \forall\, t \in \mathbf{R}, \tag{2.10}$$

where W is a complex Gaussian measure with control measure Δ, whose Fourier transform is $r(\cdot)$. The measure Δ is called the spectral measure of Y.

Since $r(\cdot) \in L^1(\mathbf{R})$, the spectral measure Δ of Y has a continuous density function $f(\lambda)$, which can be represented as the inverse Fourier transform of $r(\cdot)$

$$f(\lambda) = \frac{1}{\pi}\int_0^\infty r(t)\cos(t\lambda)\mathrm{d}t. \tag{2.11}$$

Similar to the proof of (2.10) in [16], we can obtain

$$f(\lambda) \sim c_{2,4}|\lambda|^{-(1+2HK)}, \quad \text{as } \lambda \to \infty, \tag{2.12}$$

where $c_{2,4} > 0$ is an explicit constant depending only on HK.

By (2.7) and (2.10), we obtain

$$S^{H,K}(t) = t^{HK}\int_{\mathbf{R}} e^{i\lambda \log t} W(\mathrm{d}\lambda), \quad \forall t > 0. \tag{2.13}$$

We list two properties of the spectral density $f(\lambda)$ of Y. They follow from (2.12), or from (2.9) and the truncation inequalities in [17], page 209, refer also [18].

Lemma 2.4. *There exist positive constants $c_{2,5}$ and $c_{2,6}$ such that for $u > 1$,*

$$\int_{|\lambda|<u} \lambda^2 f(\lambda)\mathrm{d}\lambda \le c_{2,5}u^{2(1-HK)} \tag{2.14}$$

and

$$\int_{|\lambda|\ge u} f(\lambda)\mathrm{d}\lambda \le c_{2,6}u^{-2HK}. \tag{2.15}$$

Proof of Theorem 2.1. Without loss of generality, we suppose Tv_n is an integer. For integers n and $j \ge 1$, we take $a_{n,j} = (jn)^\beta$, where $\beta > 0$ is a constant. Define two Gaussian processes

$$X_{n,j}^{(1)}(t) = t^{HK}\int_{|\lambda|\in(a_{n,j},\, a_{n,j+1}]} e^{i\lambda \log t} W(\mathrm{d}\lambda)$$

and

$$X_{n,j}^{(2)}(t) = t^{HK}\int_{|\lambda|\notin(a_{n,j}, a_{n,j+1}]} e^{i\lambda \log t} W(\mathrm{d}\lambda).$$

Clearly, by (2.13), we have

$$S^{H,K}(t) = X_{n,j}^{(1)}(t) + X_{n,j}^{(2)}(t), \quad \text{for all } t \ge 0. \tag{2.16}$$

It is important to note that for a fixed n, the Gaussian processes $X_{n,j}^{(1)}(t), j = 1, 2, \dots$, are independent; moreover, for every $j \ge 1$, $X_{n,j}^{(1)}(t)$ and $X_{n,j}^{(2)}(t)$ are also independent. Since

$$\begin{aligned}
&\left|\frac{1}{v_n^{1-pHK}}\sum_{j=1}^{Tv_n}\left|S^{H,K}\left(\frac{j}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p - T\frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})}\right| \\
&\le \frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{pHK}\left|\left|S^{H,K}\left(\frac{j}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p - \left|X_{n,j}^{(1)}\left(\frac{j}{v_n}\right)-X_{n,j}^{(1)}\left(\frac{j-1}{v_n}\right)\right|^p\right| \\
&\quad + \frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{pHK}\left|\mathbf{E}\left|S^{H,K}\left(\frac{j}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p - \mathbf{E}\left|X_{n,j}^{(1)}\left(\frac{j}{v_n}\right)-X_{n,j}^{(1)}\left(\frac{j-1}{v_n}\right)\right|^p\right| \\
&\quad + \frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{pHK}\left|\left|X_{n,j}^{(1)}\left(\frac{j}{v_n}\right)-X_{n,j}^{(1)}\left(\frac{j-1}{v_n}\right)\right|^p - \mathbf{E}\left|X_{n,j}^{(1)}\left(\frac{j}{v_n}\right)-X_{n,j}^{(1)}\left(\frac{j-1}{v_n}\right)\right|^p\right| \\
&\quad + \left|\frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{pHK}\mathbf{E}\left|S^{H,K}\left(\frac{j}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p - T\frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})}\right| \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

In the following, we will show that the terms I_1 and I_3 almost surely converge to zero, I_2 and I_4 converge to zero, as $n \to \infty$, respectively.

First, we prove for $a > 0$, $T > 0$, and $p \ge 1$, as $n \to \infty$,

$$I_1 = \frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{pHK}\left|\left|S^{H,K}\left(\frac{j}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p - \left|X_{n,j}^{(1)}\left(\frac{j}{v_n}\right)-X_{n,j}^{(1)}\left(\frac{j-1}{v_n}\right)\right|^p\right| \longrightarrow 0, \quad \text{a.s.} \tag{2.17}$$

In fact,

$$\begin{aligned}
&v_n^{pHK}\left|\left|S^{H,K}\left(\frac{j}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p - \left|X_{n,j}^{(1)}\left(\frac{j}{v_n}\right)-X_{n,j}^{(1)}\left(\frac{j-1}{v_n}\right)\right|^p\right| \\
&\le c_{2,7}\left(\sup_{0\le t\le T}|S^{H,K}(t)|^{p-1} + \sup_{0\le t\le T}|X_{n,j}^{(1)}(t)|^{p-1}\right)v_n^{(p-1)HK}Y_{n,j},
\end{aligned} \tag{2.18}$$

where

$$Y_{n,j} = v_n^{HK}\sup_{\frac{j-1}{v_n}<t,u<\frac{j}{v_n}}|X_{n,j}^{(2)}(t) - X_{n,j}^{(2)}(u)|, \tag{2.19}$$

and where we use the fact

$$||x|^p - |y|^p| \le p2^{p-1}||x|^{p-1} + |y|^{p-1}||x-y|.$$

We have

$$\begin{aligned}
&v_n^{pHK}\left|\mathbf{E}\left|S^{H,K}\left(\frac{j}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p - \mathbf{E}\left|X_{n,j}^{(1)}\left(\frac{j}{v_n}\right)-X_{n,j}^{(1)}\left(\frac{j-1}{v_n}\right)\right|^p\right| \\
&\le c_{2,8}\left(\mathbf{E}\sup_{0\le t\le T}|S^{H,K}(t)|^{2(p-1)} + \mathbf{E}\sup_{0\le t\le T}|X_{n,j}^{(1)}(t)|^{2(p-1)}\right)^{\frac{1}{2}} v_n^{(p-1)HK}(\mathbf{E}Y_{n,j}^2)^{\frac{1}{2}}.
\end{aligned} \tag{2.20}$$

For $Y_{n,j}$, by Lemmas 2.2 and 2.4, elementary calculus can show that there exists n_0 such that for any $n \ge n_0$, for every $1 \le j \le Tv_n$ and for any $t > 0$,

$$\mathbf{P}(Y_{n,j} > t) \le c_{2,9}n^{a+\beta}\exp(-c_{2,10}n^{2\beta HK-2a}t^2). \tag{2.21}$$

Thus, for any $\varepsilon > 0$, we obtain

$$\mathbf{P}\left(\frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{(p-1)HK} Y_{n,j} > \varepsilon\right) \le \mathbf{P}\left(\max_{1\le j\le Tv_n} Y_{n,j} > \frac{\varepsilon}{T} v_n^{-(p-1)HK}\right)$$
$$\le \sum_{j=1}^{Tv_n} \mathbf{P}\left(Y_{n,j} > \frac{\varepsilon}{T} v_n^{-(p-1)HK}\right)$$
$$\le c_{2,11} n^{2a+\beta} \exp(-c_{2,12} n^{2\beta HK-2a-2a(p-1)HK}).$$

Taking $\beta > 0$ large enough such that $2\beta HK - 2a - 2a(p-1)HK > 0$, by the Borel-Cantelli lemma, we have

$$\lim_{n\to\infty} \frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{(p-1)HK} Y_{n,j} = 0, \quad \text{a.s.} \tag{2.22}$$

Combining (2.18) and (2.22), we prove that (2.17) holds.

Second, we prove for $a > 0$, $T > 0$, and $p \ge 1$, as $n \to \infty$,

$$I_2 = \frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{pHK} \left|\mathbf{E}\left|S^{H,K}\left(\frac{j}{v_n}\right) - S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p - \mathbf{E}\left|X_{n,j}^{(1)}\left(\frac{j}{v_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{v_n}\right)\right|^p\right| \longrightarrow 0. \tag{2.23}$$

In fact, for any $1 \le j \le Tv_n$ and $r > 2$, by (2.21) and Hölder's inequality, we obtain

$$\begin{aligned}(\mathbf{E}(Y_{n,j})^2)^{\frac{1}{2}} &\le (\mathbf{E}(Y_{n,j})^r)^{\frac{1}{r}}\\ &= \left(\int_0^\infty \mathbf{P}((Y_{n,j})^r > s)\mathrm{d}s\right)^{\frac{1}{r}}\\ &\le \left(c_{2,9} n^{a+\beta}\int_0^\infty \exp\left(-c_{2,10} n^{2\beta HK-2a} s^{\frac{2}{r}}\right)\mathrm{d}s\right)^{\frac{1}{r}}\\ &= c_{2,13} n^{-\beta HK+a+\frac{a+\beta}{r}},\end{aligned} \tag{2.24}$$

where we use by letting $s^{\frac{2}{r}} = t$, then

$$\begin{aligned}\int_0^\infty \exp\left(-c_{2,10} n^{2\beta HK-2a} s^{\frac{2}{r}}\right)\mathrm{d}s &= \frac{r}{2}\int_0^\infty t^{\frac{r}{2}-1}\exp(-c_{2,10} n^{2\beta HK-2a} t)\mathrm{d}t\\ &= \frac{r}{2}\int_0^\infty \left(\frac{y}{c_{2,10} n^{2\beta HK-2a}}\right)^{\frac{r}{2}-1} \cdot \frac{e^{-y}\mathrm{d}y}{c_{2,10} n^{2\beta HK-2a}}\\ &= \frac{\frac{r}{2}\Gamma(\frac{r}{2}) n^{(-\beta HK+a)r}}{(c_{2,10})^{\frac{r}{2}}}.\end{aligned}$$

Hence,

$$\frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{(p-1)HK}(\mathbf{E}(Y_{n,j})^2)^{\frac{1}{2}} \le c_{2,14} n^{a(p-1)HK-\beta HK+a+\frac{a+\beta}{r}}.$$

Taking first $\beta > 0$ large enough and then taking $r > 2$ large enough such that $a(p-1)HK - \beta HK + a + \frac{a+\beta}{r} < 0$. Therefore, we obtain

$$\lim_{n\to\infty} \frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{(p-1)HK}(\mathbf{E}(Y_{n,j})^2)^{\frac{1}{2}} = 0. \tag{2.25}$$

Similar to (2.24), by using (2.4), for any $p > 1$, we have that

$$\mathbf{E}\left(\sup_{0\le t\le T} |S^{H,K}(t)|^{2(p-1)}\right) \le c_{2,15}. \tag{2.26}$$

Since, for any $j \ge 1$, $t \ge 0$, $h > 0$, we have

$$\mathbf{E}(X_{n,j}^{(1)}(t+h) - X_{n,j}^{(1)}(t))^2 \le \mathbf{E}(S^{H,K}(t+h) - S^{H,K}(t))^2.$$

Then, (2.4) remains true for $X_{n,j}^{(1)}$. Thus, similar to (2.26), we obtain for any $1 \le j \le T\nu_n$,

$$\mathbf{E}\left(\sup_{0\le t\le T} |X_{n,j}^{(1)}(t)|^{2(p-1)}\right) \le c_{2,16}.$$

Hence, combining (2.20) and (2.25), we have that (2.23) holds.

Third, we prove that for $a > 0$, $T > 0$, and $p \ge 1$, as $n \to \infty$,

$$I_3 = \frac{1}{\nu_n}\sum_{j=1}^{T\nu_n} \nu_n^{pHK}\left|\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p - \mathbf{E}\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p\right| \longrightarrow 0, \quad \text{a.s.} \tag{2.27}$$

In fact, since for a fixed n, the processes $\{X_{n,j}^{(1)}(t), t \ge 0\}, j = 1, 2, \dots, T\nu_n$ are independent and so are $\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p, j = 1, 2, \dots, T\nu_n$. For any $\varepsilon > 0$ and $r > 1$, by Markov inequality and the moment inequality of partial sums of independent random variables, we have

$$\begin{aligned}
&\mathbf{P}\left(\sum_{j=1}^{T\nu_n} \nu_n^{pHK}\left|\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p - \mathbf{E}\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p\right| > \varepsilon\nu_n\right)\\
&\le \frac{c_{2,17}}{\nu_n^r}\mathbf{E}\left[\left|\sum_{j=1}^{T\nu_n} \nu_n^{pHK}\left|\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p - \mathbf{E}\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p\right|\right|^r\right]\\
&\le \frac{c_{2,18}}{\nu_n^r}\nu_n^{\frac{r}{2}-1}\sum_{j=1}^{T\nu_n}\mathbf{E}\left[\nu_n^{prHK}\left|\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p - \mathbf{E}\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p\right|^r\right]\\
&\le \frac{c_{2,19}}{\nu_n^{\frac{r}{2}+1}}\sum_{j=1}^{T\nu_n}\mathbf{E}\left[\left|\nu_n^{pHK}\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p\right|^r\right].
\end{aligned} \tag{2.28}$$

Since

$$\sigma_{n,j}^2 := \mathbf{E}\left(X_{n,j}^{(1)}\left(\frac{j+t}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j+u}{\nu_n}\right)\right)^2 \le \mathbf{E}\left(S^{H,K}\left(\frac{j+t}{\nu_n}\right) - S^{H,K}\left(\frac{j+u}{\nu_n}\right)\right)^2$$

and

$$\begin{aligned}
&\frac{\mathbf{E}\left(S^{H,K}\left(\frac{j+t}{\nu_n}\right) - S^{H,K}\left(\frac{j+u}{\nu_n}\right)\right)^2}{\frac{1}{\nu_n^{2HK}}}\\
&= (2^K - 2^{2HK-1})(j+t)^{2HK} + (2^K - 2^{2HK-1})(j+u)^{2HK} - 2((j+t)^{2H} + (j+u)^{2H})^K\\
&\quad + (2j+t+u)^{2HK} + |t-u|^{2HK}\\
&\longrightarrow |t-u|^{2HK}, \quad \text{as } j \to \infty,
\end{aligned} \tag{2.29}$$

for any $t, u \in [0, 1]$ and $n \ge 1$.

Hence, for $t = 1, u = 0$, there exists $j_0 \ge 1$ such that for any $j \ge j_0$,

$$\mathbf{E}\left[\left|\nu_n^{pHK}\left|X_{n,j}^{(1)}\left(\frac{j}{\nu_n}\right) - X_{n,j}^{(1)}\left(\frac{j-1}{\nu_n}\right)\right|^p\right|^r\right] \le c_{2,20}\nu_n^{prHK}\sigma_{n,j}^{pr} \le c_{2,21},$$

where we have used the fact: let X be a random variable following an $N(0,\sigma^2)$, then for any $\gamma>0$,

$$\mathbf{E}(|X|^{\gamma}) = \frac{2^{\frac{\gamma}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\sigma^{\gamma}. \tag{2.30}$$

Therefore,

$$\mathbf{P}\left(\sum_{j=1}^{Tv_n} v_n^{pHK}\left|\left|X_{n,j}^{(1)}\left(\frac{j}{v_n}\right)-X_{n,j}^{(1)}\left(\frac{j-1}{v_n}\right)\right|^p-\mathbf{E}\left|X_{n,j}^{(1)}\left(\frac{j}{v_n}\right)-X_{n,j}^{(1)}\left(\frac{j-1}{v_n}\right)\right|^p\right|>\varepsilon v_n\right)\le c_{2,22}n^{-\frac{ar}{2}}.$$

Taking $r>1$ large enough such that $\frac{ar}{2}>1$ and by Borel-Cantelli lemma, we obtain (2.27) holds.

Finally, we prove that for $a>0$, $T>0$, and $p\ge 1$, as $n\to\infty$,

$$I_4 = \frac{1}{v_n}\sum_{j=1}^{Tv_n} v_n^{pHK}\mathbf{E}\left|S^{H,K}\left(\frac{j}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p - T\frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})}\longrightarrow 0. \tag{2.31}$$

In fact, by (2.29) and (2.30), we have for large j,

$$v_n^{pHK}\mathbf{E}\left|S^{H,K}\left(\frac{j}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p\longrightarrow\frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{1}{2})}.$$

Hence, (2.31) holds. Thus, the proof of Theorem 2.1 is complete. □

Proof of Corollary 2.2. By Theorem 2.1, following the same lines as the proof of Theorem 1.2 in [19], we can easily prove the corollary, and omit the details. □

Proof of Theorem 2.3. For simplicity, we assume that Tv_n is an integer. For $a>0$, we denote

$$\xi_{n,j} = \xi_{n,j}(S^{H,K},a) = \sup_{\frac{j-1}{v_n}<t,u<\frac{j}{v_n}}|S^{H,K}(t)-S^{H,K}(u)|,$$
$$\eta_{n,j} = \eta_{n,j}(S^{H,K},a) = v_n^{HK}\xi_{n,j}.$$

We first prove that for every $a>0$,

$$\lim_{n\to\infty}\frac{1}{v_n^{1-HK}}\sum_{j=1}^{Tv_n}|\xi_{n,j}-\mathbf{E}\xi_{n,j}| = 0, \quad \text{a.s.} \tag{2.32}$$

Denote $\zeta_{n,j}=\xi_{n,j}(X_{n,j}^{(1)},a)$, $Y_{n,j}=v_n^{HK}\xi_{n,j}(X_{n,j}^{(2)},a)$ ($Y_{n,j}$ is actually defined by (2.19)). In order to show (2.32), it is enough to prove that

$$\lim_{n\to\infty}\frac{1}{v_n^{1-HK}}\sum_{j=1}^{Tv_n}|\zeta_{n,j}-\mathbf{E}\zeta_{n,j}| = 0, \quad \text{a.s.}, \tag{2.33}$$

$$\lim_{n\to\infty}\frac{1}{v_n}\sum_{j=1}^{Tv_n}|Y_{n,j}| = 0, \quad \text{a.s.}, \tag{2.34}$$

and

$$\lim_{n\to\infty}\frac{1}{v_n^{1-HK}}\sum_{j=1}^{Tv_n}|\mathbf{E}\zeta_{n,j}-\mathbf{E}\xi_{n,j}| = 0, \tag{2.35}$$

By equalities (2.22) and (2.23), we can obtain equalities (2.34) and (2.35), respectively. We are preparing to prove (2.33).

In fact, since for a fixed $n, X_{n,j}^{(1)}, j = 1, 2, \dots, Tv_n$, are independent; so are $v_n^{HK}\zeta_{n,j}, j = 1, 2, \dots, Tv_n$, similar to (2.28), for any $\varepsilon > 0$ and $r > 1$, we have

$$\mathbf{P}\left(\sum_{j=1}^{Tv_n} |v_n^{HK}\zeta_{n,j} - \mathbf{E}(v_n^{HK}\zeta_{n,j})| > \varepsilon v_n\right) \le \frac{c_{2,23}}{v_n^{\frac{r}{2}+1}} \sum_{j=1}^{Tv_n} \mathbf{E}[(v_n^{HK}\zeta_{n,j})^r]. \tag{2.36}$$

By Lemma 2.1, we obtain for every $t > t_0$ with some $t_0 > 0$ and $1 \le j \le Tv_n$,

$$\mathbf{P}(\eta_{n,j} > t) = \mathbf{P}\left(\sup_{\frac{j-1}{v_n} < t,u < \frac{j}{v_n}} |S^{H,K}(t) - S^{H,K}(u)| \ge tv_n^{-HK}\right) \le c_{2,24} v_n e^{-c_{2,25}t^2}.$$

Hence, by (2.16) and (2.21), we obtain

$$\mathbf{P}(v_n^{HK}\zeta_{n,j} > t) \le \mathbf{P}\left(\eta_{n,j} > \frac{t}{2}\right) + \mathbf{P}\left(Y_{n,j} > \frac{t}{2}\right) \le c_{2,26} n^{\alpha+\beta} e^{-c_{2,27}t^2}.$$

Therefore, for every $1 \le j \le Tv_n$,

$$\begin{aligned}
\mathbf{E}(v_n^{HK}\zeta_{n,j})^r &= \int_0^\infty \mathbf{P}\left(v_n^{HK}\zeta_{n,j} > t^{\frac{1}{r}}\right) \mathrm{d}t \\
&\le t_0 + \int_{t_0}^\infty \mathbf{P}\left(v_n^{HK}\zeta_{n,j} > t^{\frac{1}{r}}\right) \mathrm{d}t \\
&\le t_0 + c_{2,28} n^{\alpha+\beta} \int_0^\infty \exp(-c_{2,29} t^{2/r}) \mathrm{d}t \le c_{2,30} n^{\alpha+\beta}.
\end{aligned}$$

Hence, by (2.36), we obtain

$$\mathbf{P}\left(\sum_{j=1}^{Tv_n} |v_n^{HK}\zeta_{n,j} - \mathbf{E}(v_n^{HK}\zeta_{n,j})| > \varepsilon v_n\right) \le \frac{c_{2,31} n^{\alpha+\beta}}{v_n^{\frac{r}{2}}} = c_{2,32} n^{-\frac{\alpha r}{2}+\alpha+\beta}.$$

Taking $r > 1$ large enough such that $-\frac{\alpha r}{2} + \alpha + \beta < -1$, by the Borel-Cantelli lemma, we have

$$\frac{1}{v_n^{1-HK}} \sum_{j=1}^{Tv_n} |\zeta_{n,j} - \mathbf{E}\zeta_{n,j}| \longrightarrow 0, \quad \text{a.s.},$$

as $n \to \infty$. Thus, (2.33) holds.

In order to finish the proof of Theorem 2.3, by the self-similarity of $S^{H,K}$, we only need to show

$$\lim_{n\to\infty} \frac{1}{v_n} \sum_{j=1}^{Tv_n} \mathbf{E}(\eta_{n,j}) = 2T\mathbf{E}\left(\sup_{0\le t\le 1} S^{H,K}(t)\right). \tag{2.37}$$

By (2.29), there exists $j_0 > 0$ such that for every $t, u \in [0, 1]$ and all $j \ge j_0$,

$$\lim_{n\to\infty} \frac{\mathbf{E}\left(S^{H,K}\left(\frac{j+t}{v_n}\right) - S^{H,K}\left(\frac{j+u}{v_n}\right)\right)^2}{\frac{|t-u|^{2HK}}{v_n^{2HK}}} = 1.$$

Hence, by Lemma 2.3, for every $j \ge j_0$,

$$\lim_{n\to\infty} \mathbf{E}\left[\sup_{0<t<1}\left(S^{H,K}\left(\frac{j-1+t}{v_n}\right) - S^{H,K}\left(\frac{j-1}{v_n}\right)\right)\right] = \lim_{n\to\infty} \mathbf{E}\left[\sup_{0<t<1}\left(S^{H,K}\left(\frac{t}{v_n}\right)\right)\right].$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\mathbf{E}(\xi_{n,j}) &= \lim_{n\to\infty}\mathbf{E}\left(\sup_{0<t,u<1}\left|\left(S^{H,K}\left(\frac{j-1+t}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right)-\left(S^{H,K}\left(\frac{j-1+u}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right)\right|\right)\\ &= \lim_{n\to\infty}\mathbf{E}\left(\sup_{0<t,u<1}\left[\left(S^{H,K}\left(\frac{j-1+t}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right)-\left(S^{H,K}\left(\frac{j-1+u}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right)\right]\right)\\ &= \lim_{n\to\infty}2\mathbf{E}\left(\sup_{0<t<1}\left[S^{H,K}\left(\frac{j-1+t}{v_n}\right)-S^{H,K}\left(\frac{j-1}{v_n}\right)\right]\right)\\ &= \lim_{n\to\infty}2\mathbf{E}\left[\sup_{0<t<1}\left(S^{H,K}\left(\frac{t}{v_n}\right)\right)\right].\end{aligned}$$

Hence, by the self-similarity of $S^{H,K}$, for any $j \geq j_0$,

$$\begin{aligned}\lim_{n\to\infty}\mathbf{E}(\eta_{n,j}) &= \lim_{n\to\infty}\mathbf{E}(v_n^{HK}\xi_{n,j})\\ &= 2\lim_{n\to\infty}\left(v_n^{HK}\mathbf{E}\left[\sup_{0<t<1}\left(S^{H,K}\left(\frac{t}{v_n}\right)\right)\right]\right)\\ &= 2\mathbf{E}\left[\sup_{0<t<1}(S^{H,K}(t))\right].\end{aligned}$$

By (2.26), we have

$$\begin{aligned}\max_{1\leq j\leq j_0}\mathbf{E}(\eta_{n,j}) &\leq \mathbf{E}\left(v_n^{HK}\sup_{0<t,u<\frac{j_0}{v_n}}|S^{H,K}(t)-S^{H,K}(u)|\right)\\ &= 2\mathbf{E}\left(v_n^{HK}\sup_{0<t<\frac{j_0}{v_n}}S^{H,K}(t)\right)\\ &= 2j_0^{HK}\mathbf{E}\left(\sup_{0<t<1}S^{H,K}(t)\right)\\ &\leq c_{2,33}.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{1}{v_n}\sum_{j=1}^{Tv_n}\mathbf{E}(\eta_{n,j}) = \lim_{n\to\infty}\frac{1}{v_n}\sum_{j=1}^{j_0}\mathbf{E}(\eta_{n,j}) + \lim_{n\to\infty}\frac{1}{v_n}\sum_{j=j_0+1}^{Tv_n}\mathbf{E}(\eta_{n,j}) = 2T\mathbf{E}\left[\sup_{0<t<1}(S^{H,K}(t))\right].$$

The proof of Theorem 2.3 is completed. □

3 Chung's LIL

In [16,18,20,21] the authors established Chung's LIL for fBm and other strongly locally nondeterministic Gaussian processes with stationary increments. Luan [22] obtained Chung's LIL for sbfBm. In this section, we prove the Chung's LIL for sbfBm $S^{H,K}$ in $\mathbf{R}$.

Theorem 3.1. *Let $S^{H,K} = \{S^{H,K}(t), t \geq 0\}$ be the sbfBms in $\mathbf{R}$, with $H \in (0,1)$ and $K \in (0,1]$. Then, there exists a positive and finite constant $c_{3,1}$ such that*

$$\liminf_{r\to 0}\frac{\max_{t\in[0,r]}|S^{H,K}(t)|}{r^{HK}/(\log\log(1/r))^{HK}} = c_{3,1}, \quad \text{a.s.} \tag{3.1}$$

In order to prove Theorem 3.1, we need several lemmas. Lemma 3.1 shows that the sbfBms $S^{H,K}$ has strong local nondeterminism. Lemma 3.2 gives estimates on the small ball probability of $S^{H,K}$.

Lemma 3.1. *For all constants $0 < a < b$, $S^{H,K}$ is strongly locally φ-nondeterministic on $I = [a, b]$ with $\varphi(r) = r^{2HK}$. That is, there exist positive constants $c_{3,2}$ and r_0 such that for all $t \in I$ and all $0 < r \leq \min\{t, r_0\}$,*

$$\mathrm{Var}\{S^{H,K}(t)|S^{H,K}(s) : s \in I, r \leq |s - t| \leq r_0\} \geq c_{3,2}\varphi(r). \tag{3.2}$$

Proof. See the proof of Proposition 2.1 in [2], the proof follows the same line as Proposition 2.1 in [16]. □

Lemma 3.2. *There exist positive constants $c_{3,3}$ and $c_{3,4}$ such that for all $t_0 \in [0, 1]$ and $x \in (0, 1)$,*

$$\exp\left(-\frac{c_{3,3}}{x^{1/(HK)}}\right) \leq \mathbf{P}\left\{\max_{t\in[0,1]} |S^{H,K}(t) - S^{H,K}(t_0)| \leq x\right\} \leq \exp\left(-\frac{c_{3,4}}{x^{1/(HK)}}\right). \tag{3.3}$$

Proof. By Lemma 3.1 and (1.2), we know that $S^{H,K}$ satisfies conditions $(C1)$ and $(C2)$ of [23]. Hence, this lemma holds by Theorem 3.1 of [23]. □

The following Lemma 3.3 is from [16], which provides a zero-one law for ergodic self-similar processes.

Lemma 3.3. *Let $X = \{X_t, t \in \mathbf{R}\}$ be a separable, self-similar process with index k. We assume that $X_0 = 0$ and that X is ergodic. Then, for any increasing function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$, we have $\mathbf{P}(E_{k,\psi}) = 0$ or 1, where*

$$E_{k,\psi} = \left\{\omega : \text{ there exists } \delta > 0 \text{ such that } \sup_{0\leq s\leq t} |X_s| \geq t^k\psi(t) \text{ for all } 0 < t \leq \delta\right\}.$$

By a result of [24] on ergodicity and mixing properties of stationary Gaussian processes, we see that $S^{H,K}$ is mixing. Hence, we can obtain the following lemma.

Lemma 3.4. *There exists a constant $c_{3,5} \in [0, \infty]$ such that*

$$\liminf_{t\to 0_+} \frac{(\log\log(1/t))^{HK}}{t^{HK}} \max_{0\leq s\leq t} |S^{H,K}(s)| = c_{3,5}, \quad a.s. \tag{3.4}$$

Proof. We take $\psi_c(t) = c(\log\log(1/t))^{-HK}$ and define $c_{3,5} = \sup\{c \geq 0 : \mathbf{P}(E_{k,\psi_c}) = 1\}$. Then, (3.4) holds from Lemma 3.3. □

Theorem 3.1 will be established if we prove $c_{3,5} \in (0, \infty)$ from Lemma 3.4. This is where Lemmas 3.2 and 2.2 are needed.

Now, we proceed to prove Theorem 3.1.

Proof of Theorem 3.1. We prove the lower bound first. For any integer $n \geq 1$, let $r_n = e^{-n}$. Let $0 < \gamma < c_{3,4}$ be a constant and consider the event

$$A_n = \left\{\max_{0\leq s\leq r_n} |S^{H,K}(s)| \leq \gamma^{HK} r_n^{HK}/(\log\log(1/r_n))^{HK}\right\}. \tag{3.5}$$

Then, by the self-similarity of $S^{H,K}$ and Lemma 3.2,

$$\begin{aligned}\mathbf{P}\{A_n\} &= \mathbf{P}\left\{\max_{0\leq s\leq r_n} |S^{H,K}(s)| \leq \gamma^{HK} r_n^{HK}/(\log\log(1/r_n))^{HK}\right\} \\ &= \mathbf{P}\left\{\max_{0\leq s\leq 1} |S^{H,K}(r_n s)| \leq \gamma^{HK} r_n^{HK}/(\log\log(1/r_n))^{HK}\right\}\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&= \mathbf{P}\left\{ r_n^{HK} \max_{0\le s\le 1} |S^{H,K}(s)| \le \gamma^{HK} r_n^{HK}/(\log\log(1/r_n))^{HK} \right\} \\
&= \mathbf{P}\left\{ \max_{0\le s\le 1} |S^{H,K}(s)| \le \gamma^{HK}/(\log\log(1/r_n))^{HK} \right\} \\
&\le \exp\left(-\frac{c_{3,4}}{\gamma} \log n \right) \\
&= n^{-c_{3,4}/\gamma}.
\end{aligned} \tag{3.6}
$$

Since $\sum_{n=1}^{\infty} \mathbf{P}\{A_n\} < \infty$, by the Borel-Cantelli lemma, we obtain

$$
\liminf_{n\to\infty} \frac{\max_{0\le s\le r_n} |S^{H,K}(s)|}{r_n^{HK}/(\log\log(1/r_n))^{HK}} \ge c_{3,4} \quad \text{a.s.} \tag{3.7}
$$

By (3.7) and a standard monotonicity argument, we have

$$
\liminf_{r\to 0} \frac{\max_{0\le s\le r} |S^{H,K}(s)|}{r^{HK}/(\log\log(1/r))^{HK}} \ge c_{3,6} \quad \text{a.s.} \tag{3.8}
$$

We will prove the upper bound by the following stochastic integral representation of $S^{H,K}$. For every $t > 0$, by (2.13), we have

$$
S^{H,K}(t) = t^{HK} \int_{\mathbf{R}} e^{i\lambda \log t} W(\mathrm{d}\lambda).
$$

For every integer $n \ge 1$, we take

$$
t_n = n^{-n} \quad \text{and} \quad d_n = n^{\beta}, \tag{3.9}
$$

where $\beta > 0$ is a constant whose value will be determined later. It is sufficient to prove that there exists a finite constant $c_{3,7}$ such that

$$
\liminf_{n\to\infty} \frac{\max_{0\le s\le t_n} |S^{H,K}(s)|}{t_n^{HK}/(\log\log(1/t_n))^{HK}} \le c_{3,7} \quad \text{a.s.} \tag{3.10}
$$

Define two Gaussian processes, X_n^1 and X_n^2, by

$$
X_n^1(t) \coloneqq t^{HK} \int_{|\lambda|\in(d_{n-1},d_n]} e^{i\lambda \log t} W(\mathrm{d}\lambda) \tag{3.11}
$$

and

$$
X_n^2(t) \coloneqq t^{HK} \int_{|\lambda|\notin(d_{n-1},d_n]} e^{i\lambda \log t} W(\mathrm{d}\lambda), \tag{3.12}
$$

respectively. Clearly, $S^{H,K}(t) = X_n^1(t) + X_n^2(t)$ for all $t \ge 0$. It is important to note that the Gaussian processes $X_n^1 (n = 1, 2, \ldots)$ are independent; moreover, for every $n \ge 1$, the processes X_n^1 and X_n^2 are also independent.

Let $h(r) = r^{HK}(\log\log(1/r))^{-HK}$. We make the following two claims:

(i) There is a constant $\gamma > 0$ such that

$$
\sum_{n=1}^{\infty} \mathbf{P}\left\{ \max_{s\in[0,t_n]} |X_n^1(s)| \le \gamma^{HK} h(t_n) \right\} = \infty. \tag{3.13}
$$

(ii) For every $\varepsilon > 0$,

$$
\sum_{n=1}^{\infty} \mathbf{P}\left\{ \max_{s\in[0,t_n]} |X_n^2(s)| > \varepsilon h(t_n) \right\} < \infty. \tag{3.14}
$$

Since the events in (3.13) are independent, we see that (3.10) follows from (3.13), (3.14), and a standard Borel-Cantelli argument.

It remains to verify the claims (i) and (ii) above. By Lemma 3.2 and Anderson's inequality [25], we obtain

$$\begin{aligned}\mathbf{P}\Big\{\max_{s\in[0,t_n]}|X_n^1(s)|\le \gamma^{HK}h(t_n)\Big\} &\ge \mathbf{P}\Big\{\max_{s\in[0,t_n]}|S^{H,K}(s)|\le \gamma^{HK}h(t_n)\Big\}\\ &= \mathbf{P}\Big\{\max_{s\in[0,1]}|S^{H,K}(t_ns)|\le \gamma^{HK}h(t_n)\Big\}\\ &= \mathbf{P}\Big\{t_n^{HK}\max_{s\in[0,1]}|S^{H,K}(s)|\le \gamma^{HK}h(t_n)\Big\}\\ &= \mathbf{P}\Big\{\max_{s\in[0,1]}|S^{H,K}(s)|\le \gamma^{HK}/(\log\log(1/t_n))^{HK}\Big\}\\ &\ge \exp\Big(-\frac{c_{3,3}}{\gamma}\log(n\log n)\Big)\\ &= (n\log n)^{-c_{3,3}/\gamma}.\end{aligned}\tag{3.15}$$

Thus, (i) holds for $\gamma \ge c_{3,3}$.

In order to prove (ii), we divide $[0,t_n]$ into p_n+1 non-overlapping subintervals $J_{n,j}=[a_{n,j-1},a_{n,j}]$, $j=0,1,\dots,p_n$, and then apply Lemma 2.2 to X_n^2 on each of $J_{n,j}$. Let $\beta>0$ be the constant in (3.9) and take $J_{n,0}=[0,t_nn^{-\beta}]$. After $J_{n,0}$ has been defined, we take $a_{n,j+1}=a_{n,j}(1+n^{-\beta})$. It can be verified that the number of such subintervals of $[0,t_n]$ satisfies the following bound:

$$p_n+1\le cn^{\beta}\log n. \tag{3.16}$$

Moreover, for every $j\ge 1$, if $s,t\in J_{n,j}$ and $s<t$, then we have $t/s-1\le n^{-\beta}$ and this yields

$$t-s\le sn^{-\beta}\quad\text{and}\quad \log\Big(\frac{t}{s}\Big)\le n^{-\beta}. \tag{3.17}$$

(1.2) implies that the canonical metric d for the process X_n^2 satisfies

$$d(s,t)\le c\,|s-t|^{HK}\quad \text{for all } s,t>0 \tag{3.18}$$

and $d(0,s)\le ct_n^{HK}n^{-\beta HK}$ for every $s\in J_{n,0}$. It follows that $D_0 \coloneqq \sup\{d(s,t);\, s,t\in J_{n,0}\}\le ct_n^{HK}n^{-\beta HK}$, and

$$N_d(J_{n,0},\varepsilon)\le \frac{t_nn^{-\beta}}{(\varepsilon/c)^{1/(HK)}}. \tag{3.19}$$

Some simple calculations yield

$$\begin{aligned}\int_0^{D_0}\sqrt{\log N_d(J_{n,0},\varepsilon)}\,\mathrm{d}\varepsilon &\le \int_0^{ct_n^{HK}n^{-\beta HK}}\sqrt{\log\Big(\frac{t_nn^{-\beta}}{(\varepsilon/c)^{1/(HK)}}\Big)}\,\mathrm{d}\varepsilon\\ &= \int_0^1 ct_n^{HK}n^{-\beta HK}\sqrt{\log\Big(\frac{1}{u}\Big)^{1/(HK)}}\,\mathrm{d}u\\ &= c\sqrt{1/(HK)}\,t_n^{HK}n^{-\beta HK}\int_0^1\sqrt{\log\Big(\frac{1}{u}\Big)}\,\mathrm{d}u\\ &= c_{3,8}t_n^{HK}n^{-\beta HK}.\end{aligned}\tag{3.20}$$

It follows from Lemma 2.2 and (3.20) that

$$\begin{aligned}\mathbf{P}\left\{\max_{s\in J_{n,0}}|X_n^2(s)|>\varepsilon h(t_n)\right\}&=\mathbf{P}\left\{\max_{s\in J_{n,0}}|X_n^2(s)-X_n^2(0)|>\varepsilon h(t_n)\right\}\\&\le\mathbf{P}\left\{\max_{s,t\in J_{n,0}}|X_n^2(s)-X_n^2(t)|>c_{3,9}2u\right\}\\&\le\mathbf{P}\left\{\max_{s,t\in J_{n,0}}|X_n^2(s)-X_n^2(t)|>c_{3,9}\left(u+\int_0^{D_0}\sqrt{\log N_d(J_{n,0},\varepsilon)}\,\mathrm{d}\varepsilon\right)\right\}\\&\le\exp\left(-\frac{u^2}{(D_0)^2}\right)\\&\le\exp\left(-c\frac{t_n^{2HK}(\log(n\log n))^{-2HK}}{t_n^{2HK}n^{-2\beta HK}}\right)\\&=\exp\left(-c\frac{n^{2\beta HK}}{(\log(n\log n))^{2HK}}\right),\end{aligned}\tag{3.21}$$

where $u=\frac{\varepsilon}{2c_{3,9}}h(t_n)$, which is larger than $\int_0^{D_0}\sqrt{\log N_d(J_{n,0},\varepsilon)}\,\mathrm{d}\varepsilon$.

For every $1\le j\le p_n$, we estimate the d-diameter of $J_{n,j}$. It follows from (3.12) that for any $s,t\in J_{n,j}$ with $s<t$,

$$\begin{aligned}\mathbf{E}(X_n^2(s)-X_n^2(t))^2&=\int_{|\lambda|\le d_{n-1}}|t^{HK}e^{i\lambda\log t}-s^{HK}e^{i\lambda\log s}|^2\,f(\lambda)\mathrm{d}\lambda\\&\quad+\int_{|\lambda|>d_n}|t^{HK}e^{i\lambda\log t}-s^{HK}e^{i\lambda\log s}|^2\,f(\lambda)\mathrm{d}\lambda\\&:=T_1+T_2.\end{aligned}\tag{3.22}$$

For T_2, we have, for all $s,t\in J_{n,j}$,

$$T_2\le 4t_n^{2HK}\int_{|\lambda|>d_n}f(\lambda)\mathrm{d}\lambda\le c_{3,10}t_n^{2HK}n^{-2\beta HK},\tag{3.23}$$

where the last inequality follows from (2.15).

For T_1, we use the elementary inequalities $1-\cos x\le x^2$ for every $x\in\mathbf{R}$ and $x^\alpha-1\le(x-1)^\alpha$ for $x>1$ and $0<\alpha<1$ to derive that, for all $s,t\in J_{n,j}$ with $s<t$,

$$\begin{aligned}T_1&=\int_{|\lambda|\le d_{n-1}}|t^{HK}e^{i\lambda\log t}-s^{HK}e^{i\lambda\log s}|^2\,f(\lambda)\mathrm{d}\lambda\\&=\int_{|\lambda|\le d_{n-1}}\left[(t^{HK}-s^{HK})^2+2t^{HK}s^{HK}\left(1-\cos\left(\lambda\log\frac{t}{s}\right)\right)\right]f(\lambda)\mathrm{d}\lambda\\&\le s^{2HK}\left(\left(\frac{t}{s}\right)^{HK}-1\right)^2\int_{\mathbf{R}}f(\lambda)\mathrm{d}\lambda+2t^{2HK}\int_{|\lambda|\le d_{n-1}}\left(1-\cos\left(\lambda\log\frac{t}{s}\right)\right)f(\lambda)\mathrm{d}\lambda\\&\le s^{2HK}\left(\frac{t}{s}-1\right)^{2HK}+2t^{2HK}\left(\log\frac{t}{s}\right)^2\int_{|\lambda|\le d_{n-1}}\lambda^2f(\lambda)\mathrm{d}\lambda\\&\le t_n^{2HK}n^{-2\beta HK}+2t_n^{2HK}n^{-2\beta}c_1(n-1)^{2\beta(1-HK)}\\&\le c_{3,11}t_n^{2HK}n^{-2\beta HK},\end{aligned}\tag{3.24}$$

where in deriving the last but one inequality, we have used (3.17) and (2.14), respectively.

It follows from (3.22), (3.23), and (3.24) that the d-diameter of $J_{n,j}$ satisfies

$$D_j\le c_{3,12}t_n^{HK}n^{-\beta HK}.\tag{3.25}$$

Hence, similar to (3.21), we use Lemma 2.2 and (3.25) to deduce

$$\mathbf{P}\left\{\max_{s\in J_{n,j}}|X_n^2(s)| > \varepsilon h(t_n)\right\} \le \exp\left(-c\frac{n^{2\beta HK}}{(\log(n\log n))^{2HK}}\right). \tag{3.26}$$

By (3.16), (3.21), and (3.26), we deduce that for every $\varepsilon > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbf{P}\left\{\max_{s\in[0,t_n]}|X_n^2(s)| > \varepsilon h(t_n)\right\} &\le \sum_{n=1}^{\infty}\sum_{j=0}^{p_n}\mathbf{P}\left\{\max_{s\in J_{n,j}}|X_n^2(s)| > \varepsilon h(t_n)\right\}\\ &\le c\sum_{n=1}^{\infty}(n^{\beta}\log n)\exp\left(-c\frac{n^{2\beta HK}}{(\log(n\log n))^{2HK}}\right)\\ &< \infty.\end{aligned} \tag{3.27}$$

This proves (3.14) and hence the theorem. □

By the decomposition of sbfBm and Chung's LIL for the sfBm, we give simple proof of Theorem 3.1.

Lemma 3.5. *Let $S^{H,K}$ be an sbfBm, and assume that $\{W_t, t \ge 0\}$ is a standard Brownian motion independent of $S^{H,K}$. Let X^K be the process defined by*

$$X_t^K = \int_0^{\infty}(1 - e^{-\theta t})\theta^{-\frac{1+K}{2}}\mathrm{d}W_\theta. \tag{3.28}$$

Then, the processes $\left\{\sqrt{\frac{K}{\Gamma(1-K)}}X_{t^{2H}}^K + S^{H,K}(t), t \ge 0\right\}$ and $\{S_t^{HK}, t \ge 0\}$ have the same distribution, where $\{S_t^{HK}, t \ge 0\}$ is an sfBm with Hurst parameter HK.

Proof. See the proof of Lemma 2.1 in [26]. For the convenience of readers, we give the proof. By (3.28), we know that X^K is a centered Gaussian process with covariance

$$\begin{aligned}\mathbf{E}(X_t^K X_s^K) &= \int_0^{\infty}(1 - e^{-\theta t})(1 - e^{-\theta s})\theta^{-1-K}\mathrm{d}\theta\\ &= \int_0^{\infty}(1 - e^{-\theta t})\theta^{-1-K}\mathrm{d}\theta - \int_0^{\infty}(1 - e^{-\theta t})e^{-\theta s}\theta^{-1-K}\mathrm{d}\theta\\ &= \int_0^{\infty}\left(\int_0^{t}\theta e^{-\theta u}\mathrm{d}u\right)\theta^{-1-K}\mathrm{d}\theta - \int_0^{\infty}\left(\int_0^{t}\theta e^{-\theta u}\mathrm{d}u\right)e^{-\theta s}\theta^{-1-K}\mathrm{d}\theta\\ &= \int_0^{t}\left(\int_0^{\infty}\theta^{-K}e^{-\theta u}\mathrm{d}\theta\right)\mathrm{d}u - \int_0^{t}\left(\int_0^{\infty}\theta^{-K}e^{-\theta(u+s)}\mathrm{d}\theta\right)\mathrm{d}u\\ &= \frac{\Gamma(1-K)}{K}[t^K + s^K - (t+s)^K].\end{aligned} \tag{3.29}$$

Let $Y_t = \sqrt{\frac{K}{\Gamma(1-K)}}X_{t^{2H}}^K + S^{H,K}(t)$. Then, from (1.1) and (3.29), we have, for $s, t \ge 0$,

$$\begin{aligned}\mathbf{E}(Y_sY_t) &= \frac{K}{\Gamma(1-K)}\mathbf{E}(X_{s^{2H}}^K X_{t^{2H}}^K) + \mathbf{E}(S^{H,K}(s)S^{H,K}(t))\\ &= t^{2HK} + s^{2HK} - (t^{2H} + s^{2H})^K + (t^{2H} + s^{2H})^K - \frac{1}{2}(t+s)^{2HK} - \frac{1}{2}|t-s|^{2HK}\\ &= t^{2HK} + s^{2HK} - \frac{1}{2}(t+s)^{2HK} - \frac{1}{2}|t-s|^{2HK},\end{aligned}$$

which completes the proof. □

Lemma 3.5 implies that

$$\{S^{H,K}(t), t \geq 0\} \overset{d}{=} \left\{S_t^{HK} - \sqrt{\frac{K}{\Gamma(1-K)}} X_{t^{2H}}^K, t \geq 0\right\}, \tag{3.30}$$

where $\overset{d}{=}$ means equality of all finite dimensional distributions.

Simple proof of Theorem 3.1. [22] established Chung's LIL for sfBm S^H. Namely, there exists a positive and finite constant $c_{3,13}$ such that

$$\liminf_{r\to 0} \frac{\max_{t\in[0,r]}|S_t^H|}{r^H/(\log\log(1/r))^H} = c_{3,13}, \quad \text{a.s.}$$

The decomposition (3.30) allows us deduce Chung's LIL for the sbfBm, from the same result for the sfBm with Hurst parameter HK, with the same constant.

4 Applications

In this section, we give some applications of the results in this article. For estimating the self-similar index HK of a sbfBm. We introduce an estimator for the index HK of $S^{H,K}$ given by

$$\hat{HK}_n(p) = \frac{1}{p\log n}\log\frac{S_n(p)}{S_{n^2}(p)},$$

where

$$S_{v_n}(p) = \frac{1}{v_n}\sum_{j=1}^{[v_n]}\left|S^{H,K}\left(\frac{j}{v_n}\right) - S^{H,K}\left(\frac{j-1}{v_n}\right)\right|^p.$$

Theorem 4.1. *For any $p \geq 1$, we have $\hat{HK}_n(p) \longrightarrow HK$ almost surely as $n \to \infty$.*

Proof. By Theorem 2.1, we have, as $n \to \infty$,

$$\begin{aligned}
\hat{HK}_n(p) &= \frac{1}{p\log n}\log\frac{S_n(p)}{S_{n^2}(p)} \\
&= \frac{1}{p\log n}\log\left(\frac{n^{pHK}S_n(p)}{n^{2pHK}S_{n^2}(p)}n^{pHK}\right) \\
&= \frac{1}{p\log n}\log\left(\frac{n^{pHK}S_n(p)}{n^{2pHK}S_{n^2}(p)}\right) + HK \\
&\longrightarrow HK, \quad \text{a.s.}
\end{aligned}$$

Thus, we finish the proof. □

Remark 4.1. We cannot obtain the estimators of H and K, respectively, similar to the estimators of H and K for bfBm in [10]. Because the limit in (2.1) has no relation to K.

In the following, we give some applications of the decomposition of sbfBm.

Recall that a continuous process $\{X_t, t \in [0,T]\}$ admits α-variation (resp. α-strong variation) if the limit in probability

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\left|X_{\frac{(i+1)t}{n}} - X_{\frac{it}{n}}\right|^\alpha$$

(resp.

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t |X_{s+\varepsilon} - X_s|^\alpha \mathrm{d}s),$$

exists for every $t \in [0, T]$.

Then, we have:

Theorem 4.2. *The α-variation (resp. α-strong variation) of sbfBm is $C_{HK}t$, where $C_{HK} = \mathbf{E}(|\xi|^{HK})$ and ξ is a standard normal random variable.*

Proof. The results follow easily from (3.29) and the variation in X^k is 0, since X^k is absolutely continuous. (Refer also the proofs of Proposition 4 in [27] and Proposition 3.6(a) in [28]). □

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