**Research Article**

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# The hyperbolic CS decomposition of tensors based on the C-product

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**Abstract:** This paper studies the issues about the hyperbolic CS decomposition of tensors under the C-product. The aim of this paper is fourfold. Firstly, we establish the CS decomposition of a complex unitary tensor, including the thin version and the standard version. The corresponding numerical algorithm is also given. Next, we define three kinds of tensors, i.e., the strong unitary tensor, the mode-1 strong unitary tensor, the mode-2 strong unitary tensor, and we give the CS decomposition of the last two kind of tensors aforementioned. Numerical algorithms are also obtained to compute the two types of the CS decompositions. Moreover, we give the definition of another three classes of tensors, called the $\mathcal{J}$-orthogonal tensor, the mode-1 strong $\mathcal{J}$-orthogonal tensor and the mode-2 strong $\mathcal{J}$-orthogonal tensor. The corresponding hyperbolic CS decompositions and numerical algorithms are also established. Finally, we give an application to the computation of the C-eigenvalues of the orthogonal tensor. Numerical examples are given to test our results.

**Keywords:** C-product; hyperbolic CS decomposition; strong unitary tensor; $\mathcal{J}$-orthogonal tensor; C-eigenvalue

**MSC 2020:** 15A18; 15A23; 15A69

## 1 Introduction

In recent years, the researches of tensors or multidimensional arrays have become more popular. A complex tensor can be regarded as a multidimensional array of data, which takes the form $\mathcal{A} = (\mathcal{A}_{i_1\cdots i_p}) \in \mathbb{C}^{n_1\times n_2\times\cdots\times n_p}$. The order of a tensor is the number of dimensions which is also called ways or modes. Therefore, the well-known vectors and matrices are first-order tensors and second-order tensors.

Higher-order tensors have been applied in quite a lot of areas, such as psychometrics [1], chemometrics [2], face recognition [3], image and signal processing [4–9] and so on.

The tensor decomposition has developed very well and found to have good applications in many fields, such as in eigenvalues, genomic signals, data mining, signal processing [10–16]. Kinds of tensor decompositions via diverse tensor products have been investigated to extend the matrices contents. For example, [17–21].

The tensor-tensor products include the Einstein product, T-product, C-product and so on. Lots of excellent works had appeared, such as: Sun et al. [22,23] studied the issue about the generalized inverse of tensors based on the Einstein product and a general product of tensors. Miao et al. [24–26] studied the tensor contents under the T-product. Panigrahy et al. [27] and Behera et al. [28] studied the reverse order law and

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computation concerning the Einstein product and T-product. Liu et al. [29] studied the issue of the dual core generalized inverse. Cong et al. [30] and Sahoo et al. [31] studied the core-EP inverse of tensors. Jin et al. [32] researched some work on the generalized inverse of tensors based on the T-product. Behera et al. [33,34] and Ji et al. [35] got some results on the Drazin inverse of the Einstein product. Cao et al. [36,37] had a further study on the perturbation and inequalities based on the T-product. Che et al. [38] established an efficient algorithm for computing the approximate t-URV. Chen et al. [39] studied the perturbations of the tensor Schur decomposition and Liu et al. [40] studied the weighted generalized tensor functions. Mo et al. [41,42] studied on the Time-varying generalized tensor eigenanalysis and T-eigenvalues of tensors. Wei et al. [43] studied the Neural network models and Shao et al. [44] studied the nonsymmetric algebraic Riccati equations.

Kernfeld et al. [45] defined a new tensor-tensor product, that is, the Cosine Transform Product, referred to as C-product for short. It has been shown that the C-product can be implemented efficiently using DCT (Discrete Cosine Transform). Moreover, the authors indicated that one can use C-product to conveniently specify a discrete image blurring model and the image restoration model. Bentbib et al. [46] explored good applications of the C-product. They proposed new methods for the problem of the third-order tensor completion in combination with the TV regularization procedure and tensor robust principal component analysis by using the C-product. Xu et al. [47] indicated that the advantages of using DCT are: (1) the complex calculation is not involved in the cosine transform based singular value decomposition, so the computational costs can be saved; (2) the intrinsic reflexive boundary condition along the tubes in the third dimension of tensors is employed, so its performance would be better than that by using the periodic boundary condition in DFT (Discrete Fourier Transform).

The CS decomposition is a very useful tool in the matrix analysis. Benítez et al. [48] studied the spectrum and the rank of a linear combination of two orthogonal projectors. By using the CS decomposition, the authors characterized when this linear combination is EP, diagonalizable, idempotent, tripotent, involutive, nilpotent, generalized projector, and hypergeneralized projector. The Moore-Penrose inverse of a linear combination of two orthogonal projectors in a special case was also derived. Calvetti et al. [49] indicated that the Schur form of the real orthogonal matrix can be got from a full CS decomposition. Based on this fact, the authors derived a CS decomposition based on the orthogonal eigenvalue method. An algorithm for an orthogonal similarity transformation of an orthogonal matrix to a condensed product form and an algorithm for full CS decomposition were also described.

Based on these backgrounds, we will study the theory of the hyperbolic CS decomposition of tensors via the C-product in this paper.

This paper is organized as follows. In Section 2, we give the terms and symbols needed to be used in this work. Then, we introduce the C-product of two tensors. In Section 3, we firstly introduce the thin version of the CS decomposition of a complex unitary tensor. Then, a standard CS decomposition of a complex unitary tensor is obtained. In Section 3, we define three kinds of tensors, i.e., the strong unitary tensor, the mode-1 strong unitary tensor and the mode-2 strong unitary tensor. Then, the CS decomposition and corresponding numerical algorithms of the mode-1 strong unitary tensor and the mode-2 strong unitary tensor are constructed. In the next section, we define another three classes of tensors, i.e., the $\mathcal{J}$-orthogonal tensor, the mode-1 strong $\mathcal{J}$-orthogonal tensor and the mode-2 strong $\mathcal{J}$-orthogonal tensor, which the corresponding hyperbolic CS decompositions and numerical algorithms are established. Finally, we give an application to the computation of the C-eigenvalues of the orthogonal tensor. Numerical examples are given to verify our results.

## 2 Preliminaries

In this paper, we denote vectors, matrices, three or higher order tensors like $\mathbf{a}, A, \mathcal{A}$, respectively. Meanwhile, $\mathbf{a}_i$, $A_{ij}$ and $\mathcal{A}_{i_1 i_2 \ldots i_p}$ are the components of the vector $\mathbf{a}$, matrix $A$ and tensor $\mathcal{A}$, respectively. The $n \times n$ identity matrix is denoted by $I_n$. The frontal slice of the tensor $\mathcal{A}$ is $\mathcal{A}(:,:,i)$. For simplicity, we denote the frontal slice as $\mathcal{A}^{(i)}$.

We start this section by introducing the following face-wise product between two tensors.

**Definition 1.** [45] Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_2 \times l \times n_3}$. The **face-wise product** $\mathcal{A} \triangle \mathcal{B} \in \mathbb{C}^{n_1 \times l \times n_3}$ is defined as

$$(\mathcal{A} \triangle \mathcal{B})^{(i)} = \mathcal{A}^{(i)}\mathcal{B}^{(i)}, \quad i = 1, \ldots, n_3.$$

The following example is helpful in understanding this product.

**Example 1.** Let $\mathcal{A} \in \mathbb{C}^{3\times3\times2}$ and $\mathcal{B} \in \mathbb{C}^{3\times2\times2}$ with

$$\mathcal{A}^{(1)} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 0 \\ 0 & 2 & 4 \end{bmatrix}, \ \mathcal{A}^{(2)} = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \ \mathcal{B}^{(1)} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 2 & 1 \end{bmatrix}, \ \mathcal{B}^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 3 \end{bmatrix}.$$

Then, $\mathcal{A} \triangle \mathcal{B} \in \mathbb{C}^{3\times2\times2}$ and

$$(\mathcal{A} \triangle \mathcal{B})^{(1)} = \mathcal{A}^{(1)}\mathcal{B}^{(1)} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 0 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 6 \\ 14 & 2 \\ 12 & 4 \end{bmatrix},$$
$$(\mathcal{A} \triangle \mathcal{B})^{(2)} = \mathcal{A}^{(2)}\mathcal{B}^{(2)} = \begin{bmatrix} 0 & 2 & 2 \\ 3 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 14 \\ 6 & 7 \\ 9 & 9 \end{bmatrix}.$$

□

Now, we will present the C-product of two tensors. Firstly, we give the definition of the operation of **mat**(·).

**Definition 2.** [45] Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$. $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \ldots, \mathcal{A}^{(n_3)}$ are its frontal slices. Then we use $\mathbf{mat}(\mathcal{A})$ to denote the block Toeplitz-plus-Hankel matrix

$$\mathbf{mat}(\mathcal{A}) = \begin{bmatrix} \mathcal{A}^{(1)} & \mathcal{A}^{(2)} & \ldots & \mathcal{A}^{(n_3-1)} & \mathcal{A}^{(n_3)} \\ \mathcal{A}^{(2)} & \mathcal{A}^{(1)} & \ldots & \mathcal{A}^{(n_3-2)} & \mathcal{A}^{(n_3-1)} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathcal{A}^{(n_3-1)} & \mathcal{A}^{(n_3-2)} & \ldots & \mathcal{A}^{(1)} & \mathcal{A}^{(2)} \\ \mathcal{A}^{(n_3)} & \mathcal{A}^{(n_3-1)} & \ldots & \mathcal{A}^{(2)} & \mathcal{A}^{(1)} \end{bmatrix}$$
$$+ \begin{bmatrix} \mathcal{A}^{(2)} & \mathcal{A}^{(3)} & \ldots & \mathcal{A}^{(n_3)} & O \\ \mathcal{A}^{(3)} & \mathcal{A}^{(4)} & \ldots & O & \mathcal{A}^{(n_3)} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathcal{A}^{(n_3)} & O & \ldots & \mathcal{A}^{(4)} & \mathcal{A}^{(3)} \\ O & \mathcal{A}^{(n_3)} & \ldots & \mathcal{A}^{(3)} & \mathcal{A}^{(2)} \end{bmatrix} \in \mathbb{C}^{n_1 n_3 \times n_2 n_3}, \tag{1}$$

where $O$ is the $n_1 \times n_2$ zero matrix.

**Definition 3.** [45] Define **ten**(·) the inverse operation of the **mat**(·), i.e.,

$$\mathbf{ten}[\mathbf{mat}(\mathcal{A})] = \mathcal{A}.$$

Now, we can give the C-product of two tensors.

**Definition 4.** [45] Let $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_2 \times l \times n_3}$. The **cosine transform product**, which is called C-product for short, is defined as

$$\mathcal{A} \star_c \mathcal{B} = \mathbf{ten}[\mathbf{mat}(\mathcal{A}) \cdot \mathbf{mat}(\mathcal{B})].$$

From the above definition, we can see that it is easy to compute $\mathbf{mat}(\mathcal{A})\mathbf{mat}(\mathcal{B})$ by using the technical of the matrices product. In order to compute the C-product, we must deal with the operation "$\mathbf{ten}(\cdot)$", which can be realized by using the following algorithm.

**Algorithm 2.1:** COMPUTE **ten**$(\cdot)$ OF A MATRIX

**Input:** $n_1 n_3 \times n_2 n_3$ block matrix $Z$
**Output:** $n_1 \times n_2 \times n_3$ tensor $\mathcal{A}$
1. Take the bottom left $n_1 \times n_2$ block of $Z$ and $Z = \mathcal{A}^{(n_3)}$
2. for $i = n_3 - 1, \ldots, 1$
    $\mathcal{A}^{(i)} =$ [i-th block of first block column of $Z$]$-\mathcal{A}^{(i+1)}$
   end

Notice that the first column of $\mathbf{mat}(\mathcal{A})$ defined in (1) is

$$\begin{bmatrix} \mathcal{A}^{(1)} + \mathcal{A}^{(2)} \\ \mathcal{A}^{(2)} + \mathcal{A}^{(3)} \\ \vdots \\ \mathcal{A}^{(n_3-1)} + \mathcal{A}^{(n_3)} \\ \mathcal{A}^{(n_3)} \end{bmatrix}.$$

So, it is easy to get all the frontal slices of $\mathcal{A}$ by using the technique of Algorithm 2.1.

Now, we present another way to define the C-product of two tensors by using the face-wise product. Before that, the mode-3 product of a tensor with a matrix is required.

**Definition 5.** [20] The **mode-3 product** of a tensor $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ with a matrix $U \in \mathbb{C}^{J \times n_3}$ is denoted by $\mathcal{A} \times_3 U$. More precise, we have

$$(\mathcal{A} \times_3 U)_{i_1 i_2 j} = \sum_{i_3=1}^{n_3} \mathcal{A}_{i_1 i_2 i_3} U_{j i_3}, \;\; i_1 = 1, \ldots, n_1, \;\; i_2 = 1, \ldots, n_2, \;\; j = 1, \ldots, J.$$

Now, we will introduce how to compute the mode-3 product of a tensor with a matrix. Let the frontal slice of $\mathcal{A} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ be

$$\mathcal{A}^{(1)} = \begin{bmatrix} \mathcal{A}_{111} & \mathcal{A}_{121} & \cdots & \mathcal{A}_{1n_2 1} \\ \mathcal{A}_{211} & \mathcal{A}_{221} & \cdots & \mathcal{A}_{2n_2 1} \\ \vdots & \vdots & & \vdots \\ \mathcal{A}_{n_1 11} & \mathcal{A}_{n_1 21} & \cdots & \mathcal{A}_{n_1 n_2 1} \end{bmatrix}, \ldots, \mathcal{A}^{(n_3)} = \begin{bmatrix} \mathcal{A}_{11n_3} & \mathcal{A}_{12n_3} & \cdots & \mathcal{A}_{1n_2 n_3} \\ \mathcal{A}_{21n_3} & \mathcal{A}_{22n_3} & \cdots & \mathcal{A}_{2n_2 n_3} \\ \vdots & \vdots & & \vdots \\ \mathcal{A}_{n_1 1n_3} & \mathcal{A}_{n_1 2n_3} & \cdots & \mathcal{A}_{n_1 n_2 n_3} \end{bmatrix}.$$

Then, the mode-3 unfolding of $\mathcal{A}$, denoted by $\mathcal{A}_{(3)}$, is

$$\mathcal{A}_{(3)} = \begin{bmatrix} \mathcal{A}_{111} & \mathcal{A}_{211} & \cdots & \mathcal{A}_{n_1 11} & \mathcal{A}_{121} & \mathcal{A}_{221} & \cdots & \mathcal{A}_{n_1 21} & \cdots & \mathcal{A}_{1n_2 1} & \mathcal{A}_{2n_2 1} & \cdots & \mathcal{A}_{n_1 n_2 1} \\ \mathcal{A}_{112} & \mathcal{A}_{212} & \cdots & \mathcal{A}_{n_1 12} & \mathcal{A}_{122} & \mathcal{A}_{222} & \cdots & \mathcal{A}_{n_1 22} & \cdots & \mathcal{A}_{1n_2 2} & \mathcal{A}_{2n_2 2} & \cdots & \mathcal{A}_{n_1 n_2 2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{A}_{11n_3} & \mathcal{A}_{21n_3} & \cdots & \mathcal{A}_{n_1 1n_3} & \mathcal{A}_{12n_3} & \mathcal{A}_{22n_3} & \cdots & \mathcal{A}_{n_1 2n_3} & \cdots & \mathcal{A}_{1n_2 n_3} & \mathcal{A}_{2n_2 n_3} & \cdots & \mathcal{A}_{n_1 n_2 n_3} \end{bmatrix}. \quad (2)$$

Notice that $\mathcal{A} \times_3 U$ can be computed using the following matrix-matrix product. See [20] for details.

$$\mathcal{W} = \mathcal{A} \times_3 U \Leftrightarrow \mathcal{W}_{(3)} = U\mathcal{A}_{(3)}. \tag{3}$$

The following example shows how to compute the mode-3 product of a tensor with a matrix.

**Example 2.** Let $\mathcal{A} \in \mathbb{C}^{3\times3\times2}$ and $U \in \mathbb{C}^{2\times2}$ with

$$\mathcal{A}^{(1)} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ 3 & 3 & 0 \end{bmatrix}, \ \mathcal{A}^{(2)} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \ U = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$

Suppose $\mathcal{W} = \mathcal{A} \times_3 U$. Then,

$$\mathcal{A}_{(3)} = \begin{bmatrix} 1 & 2 & 3 & 0 & 3 & 3 & 2 & 0 & 0 \\ 0 & 3 & 0 & 1 & 2 & 0 & 2 & 0 & 3 \end{bmatrix}.$$

Hence,

$$\begin{aligned} \mathcal{W}_{(3)} = U\mathcal{A}_{(3)} &= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 0 & 3 & 3 & 2 & 0 & 0 \\ 0 & 3 & 0 & 1 & 2 & 0 & 2 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 5 & 3 & 1 & 5 & 3 & 4 & 0 & 3 \\ 2 & 7 & 6 & 1 & 8 & 6 & 6 & 0 & 3 \end{bmatrix}. \end{aligned}$$

Thus,

$$\mathcal{W}^{(1)} = \begin{bmatrix} 1 & 1 & 4 \\ 5 & 5 & 0 \\ 3 & 3 & 3 \end{bmatrix}, \ \mathcal{W}^{(2)} = \begin{bmatrix} 2 & 1 & 6 \\ 7 & 8 & 0 \\ 6 & 6 & 3 \end{bmatrix}.$$

□

Based on the above preparation work, we can get the alternative expression of the C-product. Observe that

$$L(\mathcal{A}) = \mathcal{A} \times_3 M \quad \text{and} \quad L^{-1}(\mathcal{A}) = \mathcal{A} \times_3 M^{-1}. \tag{4}$$

**Lemma 1.** [45] *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_2\times l\times n_3}$. Then,*

$$\mathcal{A} \star_c \mathcal{B} = L^{-1}[L(\mathcal{A}) \bigtriangleup L(\mathcal{B})] = [(\mathcal{A} \times_3 M) \bigtriangleup (\mathcal{B} \times_3 M)] \times_3 M^{-1}, \tag{5}$$

*where $M = W^{-1}C(I+Z)$, $W = \mathrm{diag}(C(:,1))$, $C(:,1)$ is the first column of $C$, the matrix $Z \in \mathbb{C}^{n_3\times n_3}$ is the circulant upshift matrix defined by*

$$Z = \mathrm{diag}(\mathrm{ones}(n_3 - 1, 1), 1),$$

*$C$ is the orthogonal DCT matrix of size $n_3 \times n_3$ and its elements are defined as*

$$C_{ij} = \sqrt{\frac{2-\delta_{ij}}{n_3}} \cos\left(\frac{(i-1)(2j-1)\pi}{2n_3}\right), \quad i, j = 1, \ldots, n_3, \tag{6}$$

*$\delta_{ij}$ is the Kronecker symbol.*

Now, we give an example to show the details to implement the C-product.

**Example 3.** Let $\mathcal{A} \in \mathbb{C}^{3\times3\times2}$ and $\mathcal{B} \in \mathbb{C}^{3\times2\times2}$ with

$$\mathcal{A}^{(1)} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 2 & 2 & 2 \end{bmatrix}, \; \mathcal{A}^{(2)} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 3 & 0 \\ 2 & 2 & 3 \end{bmatrix}, \; \mathcal{B}^{(1)} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}, \; \mathcal{B}^{(2)} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}.$$

In this case, $M$ is a $2 \times 2$ matrix and can be computed by using the Matlab. More precisely, $M$ and $M^{-1}$ are

$$M = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \; M^{-1} = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.5 \end{bmatrix}.$$

By (4), we can compute $L(\mathcal{A}) = \mathcal{A} \times_3 M$ and $L(\mathcal{B}) = \mathcal{B} \times_3 M$, i.e.,

$$L(\mathcal{A})^{(1)} = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 6 & 3 \\ 6 & 6 & 8 \end{bmatrix}, \; L(\mathcal{A})^{(2)} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 2 & 2 & 2 \end{bmatrix},$$
$$L(\mathcal{B})^{(1)} = \begin{bmatrix} 8 & 2 \\ 0 & 2 \\ 6 & 5 \end{bmatrix}, \; L(\mathcal{B})^{(2)} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}.$$

Then, we can get $L(\mathcal{A}) \triangle L(\mathcal{B})$, that is,

$$(L(\mathcal{A}) \triangle L(\mathcal{B}))^{(1)} = \begin{bmatrix} 32 & 32 \\ 18 & 27 \\ 96 & 64 \end{bmatrix}, \; (L(\mathcal{A}) \triangle L(\mathcal{B}))^{(2)} = \begin{bmatrix} 6 & 4 \\ 6 & 3 \\ 8 & 6 \end{bmatrix}.$$

By the last step, we can get the C-product of $\mathcal{A}$ and $\mathcal{B}$, i.e.,

$$(\mathcal{A} \star_c \mathcal{B})^{(1)} = \begin{bmatrix} 6 & 4 \\ 6 & 3 \\ 8 & 6 \end{bmatrix}, \; (\mathcal{A} \star_c \mathcal{B})^{(2)} = \begin{bmatrix} 13 & 14 \\ 6 & 12 \\ 44 & 29 \end{bmatrix}.$$

□

An algorithm of the C-product of $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_2\times l\times n_3}$ is given below.

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**Algorithm 2.2:** COMPUTE THE C-PRODUCT OF TWO TENSORS [45]

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**Input:** $n_1 \times n_2 \times n_3$ tensor $\mathcal{A}$ and $n_2 \times l \times n_3$ tensor $\mathcal{B}$
**Output:** $n_1 \times l \times n_3$ tensor $\mathcal{C}$
1. Compute $M = W^{-1}C(I + Z)$ as in Lemma 1
2. Compute $\widehat{\mathcal{A}} = L(\mathcal{A}) = \mathcal{A} \times_3 M, \widehat{\mathcal{B}} = L(\mathcal{B}) = \mathcal{B} \times_3 M^{-1}$
3. for $i = 1, \ldots, n_3$
   $\widehat{\mathcal{C}}^{(i)} = \widehat{\mathcal{A}}^{(i)}\widehat{\mathcal{B}}^{(i)}$
   end
4. $\mathcal{C} = L^{-1}(\widehat{\mathcal{C}})$

---

By Algorithm 2.2, we can get the following lemma.

**Lemma 2.** *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be tensors with proper sizes. Then, the following statements are true.*
(1) $\mathcal{C} = \mathcal{A} + \mathcal{B} \Longleftrightarrow L(\mathcal{C})^{(i)} = L(\mathcal{A})^{(i)} + L(\mathcal{B})^{(i)}$.
(2) $\mathcal{C} = \mathcal{A} \star_c \mathcal{B} \Longleftrightarrow L(\mathcal{C})^{(i)} = L(\mathcal{A})^{(i)} L(\mathcal{B})^{(i)}$.

Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$. The following lemma shows that $\mathbf{mat}(\mathcal{A})$ can be block diagonalized.

**Lemma 3.** [45] *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$. Then,*

$$(C_{n_3} \otimes I_{n_1})\mathbf{mat}(\mathcal{A})\left(C_{n_3}^{-1} \otimes I_{n_2}\right) = \operatorname{diag}\left(L(\mathcal{A})^{(1)}, L(\mathcal{A})^{(2)}, \dots, L(\mathcal{A})^{(n_3)}\right),$$

*where $\otimes$ is the Kronecker product and $C_{n_3}$ is the $n_3 \times n_3$ orthogonal DCT matrix.*

**Definition 6.** [45] Let $L(\mathcal{I}) = \widehat{\mathcal{I}} \in \mathbb{C}^{n\times n\times n_3}$ be such that $\widehat{\mathcal{I}}^{(i)} = I_n$, $i = 1, 2, \dots, n_3$. Then, $\mathcal{I} = L^{-1}(\widehat{\mathcal{I}})$ is the identity tensor.

**Definition 7.** [45] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n\times n\times n_3}$. If

$$\mathcal{A}\star_c\mathcal{B} = \mathcal{I} \quad \text{and} \quad \mathcal{B}\star_c\mathcal{A} = \mathcal{I},$$

then $\mathcal{A}$ is said to be **invertible** and $\mathcal{B}$ is the **inverse** of $\mathcal{A}$, which is denoted by $\mathcal{A}^{-1}$.

We can check that the inverse of a tensor, if exists, is unique. The conjugate transpose of tensors can be defined as follows.

**Definition 8.** [45] If $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$, then the **conjugate transpose** of $\mathcal{A}$, which is denoted by $\mathcal{A}^H$, is such that

$$L(\mathcal{A}^H)^{(i)} = (L(\mathcal{A})^{(i)})^H, \quad i = 1, 2, \dots, n_3.$$

**Lemma 4.** [45] *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_2\times l\times n_3}$. It holds that*

$$(\mathcal{A}\star_c\mathcal{B})^H = \mathcal{B}^H\star_c\mathcal{A}^H.$$

**Definition 9.** [45] The tensor $\mathcal{Q} \in \mathbb{C}^{n\times n\times n_3}$ is said **unitary** if $\mathcal{Q}^H\star_c\mathcal{Q} = \mathcal{Q}\star_c\mathcal{Q}^H = \mathcal{I}$. The tensor $\mathcal{Q} \in \mathbb{C}^{n_1\times n_2\times n_3}$ is said **partially unitary** if $\mathcal{Q}^H\star_c\mathcal{Q} = \mathcal{I}$.

**Definition 10.** [50] Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$. Then, $\mathcal{A}$ is called an **F-diagonal/F-upper/F-lower** tensor if all frontal slices $\mathcal{A}^{(i)}$, $i = 1, 2, \dots, n_3$, of $\mathcal{A}$ are diagonal/upper triangular/lower triangular matrices.

The following lemma is helpful in establishing the main result of next sections.

**Lemma 5.** [50] *Let $\mathcal{A} \in \mathbb{C}^{n_1\times n_2\times n_3}$. Then, $\mathcal{A}$ is an F-diagonal/F-upper/F-lower tensor if and only if $L(\mathcal{A})$ is an F-diagonal/F-upper/F-lower tensor.*

The next lemma explains the operation of two block tensors.

**Lemma 6.** [51] *Let $\mathcal{A} \in \mathbb{C}^{m_1\times n_1\times p}$, $\mathcal{B} \in \mathbb{C}^{m_1\times n_2\times p}$, $\mathcal{C} \in \mathbb{C}^{m_2\times n_1\times p}$, $\mathcal{D} \in \mathbb{C}^{m_2\times n_2\times p}$, $\mathcal{E} \in \mathbb{C}^{n_1\times m_1\times p}$, $\mathcal{F} \in \mathbb{C}^{n_1\times m_2\times p}$, $\mathcal{G} \in \mathbb{C}^{n_2\times m_1\times p}$ and $\mathcal{H} \in \mathbb{C}^{n_2\times m_2\times p}$. Then,*

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \star_c \begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{G} & \mathcal{H} \end{bmatrix} = \begin{bmatrix} \mathcal{A}\star_c\mathcal{E} + \mathcal{B}\star_c\mathcal{G} & \mathcal{A}\star_c\mathcal{F} + \mathcal{B}\star_c\mathcal{H} \\ \mathcal{C}\star_c\mathcal{E} + \mathcal{D}\star_c\mathcal{G} & \mathcal{C}\star_c\mathcal{F} + \mathcal{D}\star_c\mathcal{H} \end{bmatrix}.$$

In the following, we show that the $L(\cdot)$ operation of the block tensor has the good character.

**Lemma 7.** *Let $\mathcal{A} \in \mathbb{C}^{m_1\times n_1\times p}$, $\mathcal{B} \in \mathbb{C}^{m_1\times n_2\times p}$, $\mathcal{C} \in \mathbb{C}^{m_2\times n_1\times p}$ and $\mathcal{D} \in \mathbb{C}^{m_2\times n_2\times p}$. Then,*

$$L\left(\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}\right) = \begin{bmatrix} L(\mathcal{A}) & L(\mathcal{B}) \\ L(\mathcal{C}) & L(\mathcal{D}) \end{bmatrix}.$$

*Proof.* Let $M \in \mathbb{C}^{p\times p}$. By the definition of $L(\cdot)$, one has

$$L\left(\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}\right) = \begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}\times_3 M \Longleftrightarrow L\left(\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}\right)_{(3)} = M\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}_{(3)}$$

$$\Longleftrightarrow L\left(\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}\right)_{(3)} = \begin{bmatrix}M\mathcal{A}_{(3)} & M\mathcal{C}_{(3)} & M\mathcal{B}_{(3)} & M\mathcal{D}_{(3)}\end{bmatrix}$$

$$\Longleftrightarrow L\left(\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}\right) = \begin{bmatrix}\mathcal{A}\times_3 M & \mathcal{B}\times_3 M\\ \mathcal{C}\times_3 M & \mathcal{D}\times_3 M\end{bmatrix}.$$

Hence, we claim that

$$L\left(\begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{bmatrix}\right) = \begin{bmatrix}L(\mathcal{A}) & L(\mathcal{B})\\ L(\mathcal{C}) & L(\mathcal{D})\end{bmatrix}.$$

□

## 3 The CS decomposition of the unitary tensor

In this section, we will study the CS decomposition of the unitary tensor based on C-product. Firstly, we give a thin version of the CS decomposition of the unitary tensor.

**Theorem 1.** *Let $\mathcal{W}_1 \in \mathbb{C}^{m_1\times n_1\times p}$ with $m_1 \geq n_1$, $\mathcal{W}_2 \in \mathbb{C}^{m_2\times n_1\times p}$ with $m_2 \geq n_1$ and*

$$\mathcal{W} = \begin{bmatrix}\mathcal{W}_1\\ \mathcal{W}_2\end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times n_1\times p}$$

*be a partially unitary tensor. Then, there exist unitary tensors $\mathcal{U}_1 \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{U}_2 \in \mathbb{C}^{m_2\times m_2\times p}$ and $\mathcal{V} \in \mathbb{C}^{n_1\times n_1\times p}$ such that*

$$\mathcal{W} = \begin{bmatrix}\mathcal{U}_1 & \mathcal{O}\\ \mathcal{O} & \mathcal{U}_2\end{bmatrix}\star_c\begin{bmatrix}\mathcal{C}\\ \mathcal{S}\end{bmatrix}\star_c\mathcal{V}^H, \tag{7}$$

*where $\mathcal{C} \in \mathbb{C}^{m_1\times n_1\times p}$, $\mathcal{S} \in \mathbb{C}^{m_2\times n_1\times p}$ are F-diagonal tensors and*

$$\mathcal{C}^H\star_c\mathcal{C} + \mathcal{S}^H\star_c\mathcal{S} = \mathcal{I}. \tag{8}$$

*Proof.* Since $\mathcal{W}$ is a partially unitary tensor, then $\mathcal{W}^H\star_c\mathcal{W} = \mathcal{I}$. By Lemma 2, we have

$$L(\mathcal{W}^H)^{(i)}L(\mathcal{W})^{(i)} = I_{n_1}, i = 1, \ldots, p.$$

Hence, $L(\mathcal{W})^{(i)}, i = 1, \ldots, p$, are partially unitary matrices. Notice that

$$\mathcal{W} = \begin{bmatrix}\mathcal{W}_1\\ \mathcal{W}_2\end{bmatrix},$$

by using Lemma 7, we have

$$L(\mathcal{W})^{(i)} = \begin{bmatrix}L(\mathcal{W}_1)^{(i)}\\ L(\mathcal{W}_2)^{(i)}\end{bmatrix}, i = 1, \ldots, p.$$

Now, we can get the CS decomposition of $L(\mathcal{W})^{(i)}$ by using [52, Theorem 2.5.2], that is

$$L(\mathcal{W})^{(i)} = \begin{bmatrix}L(\mathcal{U}_1)^{(i)} & O\\ O & L(\mathcal{U}_2)^{(i)}\end{bmatrix}\begin{bmatrix}L(\mathcal{C})^{(i)}\\ L(\mathcal{S})^{(i)}\end{bmatrix}L(\mathcal{V}^H)^{(i)}, \tag{9}$$

where $L(\mathcal{U}_1)^{(i)} \in \mathbb{C}^{m_1\times m_1}$, $L(\mathcal{U}_2)^{(i)} \in \mathbb{C}^{m_2\times m_2}$, $L(\mathcal{V})^{(i)} \in \mathbb{C}^{n_1\times n_1}$ and

$$L(\mathcal{C})^{(i)} = \mathrm{diag}(c_{i1}, c_{i2}, \dots, c_{in_1}) \in \mathbb{C}^{m_1\times n_1},$$

$$L(\mathcal{S})^{(i)} = \mathrm{diag}(s_{i1}, s_{i2}, \dots, s_{in_1}) \in \mathbb{C}^{m_2\times n_1},$$

with

$$\left(L(\mathcal{C})^{(i)}\right)^H L(\mathcal{C})^{(i)} + \left(L(\mathcal{S})^{(i)}\right)^H L(\mathcal{S})^{(i)} = I_{n_1}, i = 1, \dots, p. \tag{10}$$

By Definition 1 and using (9) and (10), we have

$$L(\mathcal{W}) = \begin{bmatrix} L(\mathcal{U}_1) & O \\ O & L(\mathcal{U}_2) \end{bmatrix} \triangle \begin{bmatrix} L(\mathcal{C}) \\ L(\mathcal{S}) \end{bmatrix} \triangle L(\mathcal{V}^H) \tag{11}$$

with

$$\left(L(\mathcal{C})\right)^H \triangle L(\mathcal{C}) + \left(L(\mathcal{S})\right)^H \triangle L(\mathcal{S}) = L(\mathcal{I}). \tag{12}$$

Implementing the operation "$L^{-1}(\cdot)$" on both sides of the equalities (11) and (12), we have

$$\mathcal{W} = \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \begin{bmatrix} \mathcal{C} \\ \mathcal{S} \end{bmatrix} \star_c \mathcal{V}^H,$$

with

$$\mathcal{C}^H \star_c \mathcal{C} + \mathcal{S}^H \star_c \mathcal{S} = \mathcal{I}.$$

Since $L(\mathcal{C})$ and $L(\mathcal{S})$ are F-diagonal tensors, by Lemma 5, we have $\mathcal{C}$ and $\mathcal{S}$ are F-diagonal tensors. □

In the following, we will establish a more general version of the CS decomposition of tensors.

**Theorem 2.** *Let $\mathcal{W}_{11} \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{W}_{22} \in \mathbb{C}^{m_2\times m_2\times p}$ and*

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p},\ m_1 \le m_2$$

*be a unitary tensor. Then, there exist unitary tensors $\mathcal{U}_1 \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{U}_2 \in \mathbb{C}^{m_2\times m_2\times p}$, $\mathcal{V}_1 \in \mathbb{C}^{m_1\times m_1\times p}$ and $\mathcal{V}_2 \in \mathbb{C}^{m_2\times m_2\times p}$ such that*

$$\mathcal{W} = \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \left[\begin{array}{c|cc} \mathcal{C} & \mathcal{S} & \mathcal{O} \\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \begin{bmatrix} \mathcal{V}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{V}_2 \end{bmatrix}^H, \tag{13}$$

*where $\mathcal{C} \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{S} \in \mathbb{C}^{m_1\times m_1\times p}$ are F-diagonal tensors and*

$$\mathcal{C}^H \star_c \mathcal{C} + \mathcal{S}^H \star_c \mathcal{S} = \mathcal{I}. \tag{14}$$

*Proof.* Since $\mathcal{W}$ is a unitary tensor, we have $\mathcal{W}^H \star_c \mathcal{W} = \mathcal{W} \star_c \mathcal{W}^H = \mathcal{I}$. Then, by Lemma 2, we have

$$L(\mathcal{W}^H)^{(i)} L(\mathcal{W})^{(i)} = L(\mathcal{W})^{(i)} L(\mathcal{W}^H)^{(i)} = I_{m_1+m_2}, i = 1, \dots, p,$$

which implies that $L(\mathcal{W})^{(i)}, i = 1, \dots, p$, are unitary matrices. Since

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix},$$

by Lemma 7, we get

$$L(\mathcal{W})^{(i)} = \begin{bmatrix} L(\mathcal{W}_{11})^{(i)} & L(\mathcal{W}_{12})^{(i)} \\ L(\mathcal{W}_{21})^{(i)} & L(\mathcal{W}_{22})^{(i)} \end{bmatrix}, i = 1, \dots, p.$$

By using the CS decomposition [53] of $L(\mathcal{W})^{(i)}$, we have

$$L(\mathcal{W})^{(i)} = \begin{bmatrix} L(\mathcal{U}_1)^{(i)} & O \\ O & L(\mathcal{U}_2)^{(i)} \end{bmatrix} \left[\begin{array}{c|cc} L(\mathcal{C})^{(i)} & L(\mathcal{S})^{(i)} & O \\ \hline -L(\mathcal{S})^{(i)} & L(\mathcal{C})^{(i)} & O \\ O & O & L(\mathcal{I})^{(i)} \end{array}\right] \begin{bmatrix} L(\mathcal{V}_1)^{(i)} & O \\ O & L(\mathcal{V}_2)^{(i)} \end{bmatrix}^H, \tag{15}$$

where $L(\mathcal{U}_1)^{(i)} \in \mathbb{C}^{m_1\times m_1}$, $L(\mathcal{U}_2)^{(i)} \in \mathbb{C}^{m_2\times m_2}$, $L(\mathcal{V}_1)^{(i)} \in \mathbb{C}^{m_1\times m_1}$, $L(\mathcal{V}_2)^{(i)} \in \mathbb{C}^{m_2\times m_2}$ and

$$L(\mathcal{C})^{(i)} = \operatorname{diag}(c_{i1}, c_{i2}, \ldots, c_{im_1}) \in \mathbb{C}^{m_1\times m_1},$$

$$L(\mathcal{S})^{(i)} = \operatorname{diag}(s_{i1}, s_{i2}, \ldots, s_{im_1}) \in \mathbb{C}^{m_1\times m_1},$$

with

$$\left(L(\mathcal{C})^{(i)}\right)^H L(\mathcal{C})^{(i)} + \left(L(\mathcal{S})^{(i)}\right)^H L(\mathcal{S})^{(i)} = I_{m_1}, i = 1, \ldots, p. \tag{16}$$

By Definition 1, we have

$$L(\mathcal{W}) = \begin{bmatrix} L(\mathcal{U}_1) & O \\ O & L(\mathcal{U}_2) \end{bmatrix} \triangle \left[\begin{array}{c|cc} L(\mathcal{C}) & -L(\mathcal{S}) & O \\ \hline -L(\mathcal{S}) & L(\mathcal{C}) & O \\ O & O & L(\mathcal{I}) \end{array}\right] \triangle \begin{bmatrix} L(\mathcal{V}_1) & O \\ O & L(\mathcal{V}_2) \end{bmatrix}^H \tag{17}$$

with

$$\left(L(\mathcal{C})\right)^H \triangle L(\mathcal{C}) + \left(L(\mathcal{S})\right)^H \triangle L(\mathcal{S}) = L(\mathcal{I}). \tag{18}$$

Utilizing the operation "$L^{-1}(\cdot)$" on both sides of the equalities (17) and (18), we have

$$\mathcal{W} = \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \left[\begin{array}{c|cc} \mathcal{C} & \mathcal{S} & \mathcal{O} \\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \begin{bmatrix} \mathcal{V}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{V}_2 \end{bmatrix}^H,$$

with

$$\mathcal{C}^H \star_c \mathcal{C} + \mathcal{S}^H \star_c \mathcal{S} = \mathcal{I}.$$

By Lemma 5, we get $\mathcal{C}$ and $\mathcal{S}$ are F-diagonal tensors due to $L(\mathcal{C})$ and $L(\mathcal{S})$ being F-diagonal tensors. □

In the following, we will build an efficient algorithm to compute the CS decomposition of a unitary tensor by Theorem 2.

**Algorithm 3.1:** Compute the CS decomposition of a unitary tensor

**Input:** $(m_1 + m_2) \times (m_1 + m_2) \times p$ unitary tensor $\mathcal{W}$
**Output:** $\mathcal{U}_1 \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{U}_2 \in \mathbb{C}^{m_2\times m_2\times p}$, $\mathcal{V}_1 \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{V}_2 \in \mathbb{C}^{m_2\times m_2\times p}$, $\mathcal{C} \in \mathbb{C}^{m_1\times m_1\times p}$ and $\mathcal{S} \in \mathbb{C}^{m_1\times m_1\times p}$

1. Compute $\widehat{\mathcal{W}} = L(\mathcal{W}) = \mathcal{W} \times_3 M$, where $M$ is defined in (5)
2. for $i = 1, \ldots, p$
   $\left[\widehat{\mathcal{U}_1}^{(i)}, \widehat{\mathcal{U}_2}^{(i)}, \widehat{\mathcal{V}}_1^{(i)}, \widehat{\mathcal{V}}_2^{(i)}, \widehat{\mathcal{C}}^{(i)}, \widehat{\mathcal{S}}^{(i)}\right] = \operatorname{csd}(\widehat{\mathcal{W}}^{(i)})$
   end
3. $\mathcal{U}_1 = L^{-1}(\widehat{\mathcal{U}_1})$, $\mathcal{U}_2 = L^{-1}(\widehat{\mathcal{U}_2})$, $\mathcal{V}_1 = L^{-1}(\widehat{\mathcal{V}}_1)$, $\mathcal{V}_2 = L^{-1}(\widehat{\mathcal{V}}_2)$, $\mathcal{C} = L^{-1}(\widehat{\mathcal{C}})$, $\mathcal{S} = L^{-1}(\widehat{\mathcal{S}})$

**Example 4.** Let $\mathcal{W}_{11} \in \mathbb{C}^{4\times4\times2}$, $\mathcal{W}_{22} \in \mathbb{C}^{6\times6\times2}$ and $\mathcal{W} = \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} \in \mathbb{C}^{10\times10\times2}$ with

$$\mathcal{W}^{(1)} = \begin{bmatrix} -0.3027 & -0.3743 & 0.2779 & -0.2215 & 0.3083 & -0.1542 & -0.4304 & -0.1624 & -0.5042 & -0.2393 \\ -0.3447 & -0.3034 & 0.3176 & 0.5575 & -0.3394 & 0.1318 & -0.2568 & 0.2835 & 0.1014 & 0.2972 \\ -0.3050 & 0.0483 & 0.2434 & -0.2416 & 0.2551 & -0.1318 & 0.0886 & 0.5412 & 0.5124 & -0.3760 \\ -0.2851 & 0.4978 & -0.1574 & 0.5621 & 0.2790 & -0.1894 & 0.1385 & 0.1395 & -0.3759 & -0.1911 \\ -0.2608 & -0.2768 & -0.4709 & 0.2590 & 0.1477 & 0.3053 & -0.1028 & -0.3940 & 0.3791 & -0.3768 \\ -0.3014 & 0.6169 & 0.0246 & -0.2510 & 0.0014 & 0.3029 & -0.5232 & -0.1531 & 0.1373 & 0.2392 \\ -0.2986 & 0.0714 & 0.4863 & 0.0514 & 0.0712 & -0.1829 & 0.4514 & -0.5976 & 0.2002 & 0.1646 \\ -0.3477 & 0.0083 & 0.0250 & -0.2481 & -0.3741 & 0.5556 & 0.4230 & 0.0955 & -0.3537 & -0.2347 \\ -0.3539 & -0.2345 & -0.3958 & -0.1973 & 0.3855 & 0 & 0.2236 & 0.1787 & -0.0542 & 0.6250 \\ -0.3479 & 0.0180 & -0.3488 & -0.1807 & -0.5757 & -0.6152 & -0.0696 & -0.0645 & 0.0309 & -0.0671 \end{bmatrix},$$

$$\mathcal{W}^{(2)} = \begin{bmatrix} -0.0122 & -0.0544 & -0.2396 & 0.1050 & -0.0260 & 0.1035 & 0.3180 & -0.0026 & 0.2706 & 0.4656 \\ 0.0358 & 0.0575 & -0.0896 & -0.2251 & 0.4898 & 0.0739 & 0.0414 & -0.2913 & -0.0307 & -0.3987 \\ -0.0350 & -0.0530 & -0.1058 & 0.3458 & -0.2566 & 0.1341 & 0.0966 & -0.2178 & -0.5945 & 0.1192 \\ -0.0338 & -0.3478 & 0.0776 & -0.3527 & -0.3688 & -0.0882 & 0.0365 & -0.3182 & 0.3337 & -0.0589 \\ -0.0128 & 0.1435 & 0.1838 & 0.0963 & -0.2405 & -0.0367 & -0.2637 & 0.2248 & -0.0169 & 0.2571 \\ 0.0105 & -0.1996 & 0.1704 & 0.1723 & 0.1220 & -0.5052 & 0.1060 & 0.0832 & -0.1625 & -0.0215 \\ 0.0476 & 0.0874 & -0.6697 & -0.1163 & 0.0322 & 0.0029 & -0.3094 & 0.3066 & -0.1967 & -0.1566 \\ -0.0052 & 0.1740 & -0.0162 & 0.2149 & 0.2823 & -0.2862 & 0.0337 & 0.1679 & 0.4246 & 0.0891 \\ 0.0069 & -0.0672 & 0.2727 & -0.1791 & -0.2487 & 0.0297 & -0.2126 & 0.2010 & 0.0011 & -0.3846 \\ 0.0099 & 0.2727 & 0.2803 & -0.1206 & 0.2403 & 0.5181 & 0.0267 & -0.1242 & -0.0747 & 0.1492 \end{bmatrix}.$$

A simple computation gives

$$L(\mathcal{W})^{(1)} = \begin{bmatrix} -0.3272 & -0.4832 & -0.2014 & -0.0116 & 0.2564 & 0.0528 & 0.2055 & -0.1675 & 0.0369 & 0.6918 \\ -0.2730 & -0.1885 & 0.1383 & 0.1074 & 0.6401 & 0.2795 & -0.1740 & -0.2990 & 0.0400 & -0.5002 \\ -0.3749 & -0.0578 & 0.0317 & 0.4500 & -0.2581 & 0.1364 & 0.2818 & 0.1056 & -0.6766 & -0.1377 \\ -0.3528 & -0.1977 & -0.0023 & -0.1432 & -0.4586 & -0.3657 & 0.2115 & -0.4968 & 0.2916 & -0.3088 \\ -0.2863 & 0.0102 & -0.1033 & 0.4516 & -0.3334 & 0.2319 & -0.6302 & 0.0557 & 0.3454 & 0.1373 \\ -0.2804 & 0.2178 & 0.3654 & 0.0935 & 0.2455 & -0.7076 & -0.3112 & 0.0133 & -0.1877 & 0.1962 \\ -0.2034 & 0.2462 & -0.8531 & -0.1811 & 0.1355 & -0.1772 & -0.1674 & 0.0156 & -0.1932 & -0.1486 \\ -0.3581 & 0.3563 & -0.0074 & 0.1817 & 0.1904 & -0.0168 & 0.4903 & 0.4314 & 0.4955 & -0.0565 \\ -0.3401 & -0.3689 & 0.1495 & -0.5556 & -0.1119 & 0.0595 & -0.2016 & 0.5806 & -0.0520 & -0.1442 \\ -0.3281 & 0.5634 & 0.2119 & -0.4220 & -0.0951 & 0.4209 & -0.0161 & -0.3129 & -0.1186 & 0.2314 \end{bmatrix},$$

$$L(\mathcal{W})^{(2)} = \begin{bmatrix} -0.3027 & -0.3743 & 0.2779 & -0.2215 & 0.3083 & -0.1542 & -0.4304 & -0.1624 & -0.5042 & -0.2393 \\ -0.3447 & -0.3034 & 0.3176 & 0.5575 & -0.3394 & 0.1318 & -0.2568 & 0.2835 & 0.1014 & 0.2972 \\ -0.3050 & 0.0483 & 0.2434 & -0.2416 & 0.2551 & -0.1318 & 0.0886 & 0.5412 & 0.5124 & -0.3760 \\ -0.2851 & 0.4978 & -0.1574 & 0.5621 & 0.2790 & -0.1894 & 0.1385 & 0.1395 & -0.3759 & -0.1911 \\ -0.2608 & -0.2768 & -0.4709 & 0.2590 & 0.1477 & 0.3053 & -0.1028 & -0.3940 & 0.3791 & -0.3768 \\ -0.3014 & 0.6169 & 0.0246 & -0.2510 & 0.0014 & 0.3029 & -0.5232 & -0.1531 & 0.1373 & 0.2392 \\ -0.2986 & 0.0714 & 0.4863 & 0.0514 & 0.0712 & -0.1829 & 0.4514 & -0.5976 & 0.2002 & 0.1646 \\ -0.3477 & 0.0083 & 0.0250 & -0.2481 & -0.3741 & 0.5556 & 0.4230 & 0.0955 & -0.3537 & -0.2347 \\ -0.3539 & -0.2345 & -0.3958 & -0.1973 & 0.3855 & -0.0000 & 0.2236 & 0.1787 & -0.0542 & 0.6250 \\ -0.3479 & 0.0180 & -0.3488 & -0.1807 & -0.5757 & -0.6152 & -0.0696 & -0.0645 & 0.0309 & -0.0671 \end{bmatrix}.$$

The first step, we will compute the singular value decomposition of $L(\mathcal{W}_{11})^{(1)}$. That is

$$L(\mathcal{W}_{11})^{(1)} = \begin{bmatrix} -0.6441 & -0.4914 & 0.5771 & -0.1035 \\ -0.3967 & 0.1466 & -0.4581 & -0.7818 \\ -0.5040 & 0.7902 & 0.1654 & 0.3070 \\ -0.4169 & -0.3356 & -0.6556 & 0.5327 \end{bmatrix} \times \begin{bmatrix} 0.8492 & 0 & 0 & 0 \\ 0 & 0.5081 & 0 & 0 \\ 0 & 0 & 0.2427 & 0 \\ 0 & 0 & 0 & 0.1236 \end{bmatrix}$$
$$\times \begin{bmatrix} 0.7714 & -0.1123 & 0.4348 & -0.4509 \\ 0.5859 & 0.4536 & -0.2985 & 0.6015 \\ 0.0705 & 0.2855 & -0.7122 & -0.6374 \\ -0.2381 & 0.8367 & 0.4633 & -0.1691 \end{bmatrix}^H.$$

Then,

$$L(\mathcal{C})^{(1)} = \operatorname{diag}(c_1, c_2, c_3, c_4) = \operatorname{diag}(0.8492, 0.5081, 0.2427, 0.1236),$$

$$L(\mathcal{S})^{(1)} = \operatorname{diag}\left(\sqrt{1-c_1^2}, \sqrt{1-c_2^2}, \sqrt{1-c_3^2}, \sqrt{1-c_4^2}\right) = \operatorname{diag}(0.5281, 0.8613, 0.9701, 0.9923).$$

Similarly, by computing the singular value decomposition of $L(\mathcal{W}_{11})^{(2)}$, one has

$$L(\mathcal{C})^{(2)} = \operatorname{diag}(0.9445, 0.8620, 0.4675, 0.0986),$$

$$L(\mathcal{S})^{(2)} = \operatorname{diag}(0.3285, 0.5070, 0.8840, 0.9923).$$

Hence, there exist orthogonal tensors $\mathcal{U}$ and $\mathcal{V}$ such that

$$L\left(\mathcal{U}^H \star_c \mathcal{W} \star_c \mathcal{V}\right)^{(1)}$$
$$= \begin{bmatrix} 0.8492 & 0 & 0 & 0 & 0.5281 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5081 & 0 & 0 & 0 & 0.8613 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2427 & 0 & 0 & 0 & 0.9701 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1236 & 0 & 0 & 0 & 0.9923 & 0 & 0 \\ -0.5281 & 0 & 0 & 0 & 0.8492 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.8613 & 0 & 0 & 0 & 0.5081 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9701 & 0 & 0 & 0 & 0.2427 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9923 & 0 & 0 & 0 & 0.1236 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$L(\mathcal{U}^H \star_c \mathcal{W} \star_c \mathcal{V})^{(2)} = \begin{bmatrix} 0.9445 & 0 & 0 & 0 & 0.3285 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8620 & 0 & 0 & 0 & 0.5070 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4675 & 0 & 0 & 0 & 0.8840 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0986 & 0 & 0 & 0 & 0.9951 & 0 & 0 \\ -0.3285 & 0 & 0 & 0 & 0.9445 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5070 & 0 & 0 & 0 & 0.8620 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.8840 & 0 & 0 & 0 & 0.4675 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9951 & 0 & 0 & 0 & 0.0986 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then,

$$(\mathcal{U}^H \star_c \mathcal{W} \star_c \mathcal{V})^{(1)} = \begin{bmatrix} 0.9445 & 0 & 0 & 0 & 0.3285 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8620 & 0 & 0 & 0 & 0.5070 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4675 & 0 & 0 & 0 & 0.8840 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0986 & 0 & 0 & 0 & 0.9951 & 0 & 0 \\ -0.3285 & 0 & 0 & 0 & 0.9445 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5070 & 0 & 0 & 0 & 0.8620 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.8840 & 0 & 0 & 0 & 0.4675 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9951 & 0 & 0 & 0 & 0.0986 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$(\mathcal{U}^H \star_c \mathcal{W} \star_c \mathcal{V})^{(2)} = \begin{bmatrix} -0.0477 & 0 & 0 & 0 & 0.0998 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1769 & 0 & 0 & 0 & 0.1771 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1124 & 0 & 0 & 0 & 0.0430 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0125 & 0 & 0 & 0 & -0.0014 & 0 & 0 \\ -0.0998 & 0 & 0 & 0 & -0.047 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1771 & 0 & 0 & 0 & -0.1769 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0430 & 0 & 0 & 0 & -0.1124 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0014 & 0 & 0 & 0 & 0.0125 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, one can check that $\mathcal{C}^H \star_c \mathcal{C} + \mathcal{S}^H \star_c \mathcal{S} = \mathcal{I}$. □

# 4 The CS decomposition of a strong unitary tensor

In this section, we will study the CS decomposition of a strong unitary tensor. Firstly, let us give the definition of a related tensor.

**Definition 11.** Let $\mathcal{Q} \in \mathbb{C}^{n\times n\times n_3}$. If

$$\mathcal{Q}^H \star_c \mathcal{Q} = \mathcal{Q} \star_c \mathcal{Q}^H = \mathcal{D},$$

where $\mathcal{D}$ is an F-diagonal tensor with $\mathcal{D}_{iij} > 0$, $i = 1, \ldots, n$, $j = 1, \ldots, n_3$, then $\mathcal{Q}$ is call a strong unitary tensor.

Two more general versions of the strong unitary tensor is given as follows.

**Definition 12.** Let $\mathcal{Q} \in \mathbb{C}^{n\times n\times n_3}$. If

$$\mathcal{Q} \star_c \mathcal{Q}^H = \mathcal{D},$$

where $\mathcal{D}$ is an F-diagonal tensor with $\mathcal{D}_{iij} > 0$, $i = 1, \ldots, n$, $j = 1, \ldots, n_3$, then $\mathcal{Q}$ is call a mode-1 strong unitary tensor. If

$$\mathcal{Q}^H \star_c \mathcal{Q} = \mathcal{D},$$

where $\mathcal{D}$ is an F-diagonal tensor with $\mathcal{D}_{iij} > 0$, $i = 1, \ldots, n$, $j = 1, \ldots, n_3$, then $\mathcal{Q}$ is call a mode-2 strong unitary tensor.

Now, we will do some researches on the CS decomposition of the mode-1 (mode-2) strong unitary tensor.

**Theorem 3.** *Let $\mathcal{W}_{11} \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{W}_{22} \in \mathbb{C}^{m_2\times m_2\times p}$ and*

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}, \ m_1 \le m_2$$

*be a mode-1 strong unitary tensor. Then, there exist a mode-1 strong unitary tensors $\mathcal{U} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$ and unitary tensors $\mathcal{V}_1 \in \mathbb{C}^{m_1\times m_1\times p}$ and $\mathcal{V}_2 \in \mathbb{C}^{m_2\times m_2\times p}$ such that*

$$\mathcal{W} = \mathcal{U} \star_c \left[\begin{array}{c|cc} \mathcal{C} & \mathcal{S} & \mathcal{O} \\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \begin{bmatrix} \mathcal{V}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{V}_2 \end{bmatrix}^H,$$

*where $\mathcal{C} \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{S} \in \mathbb{C}^{m_1\times m_1\times p}$ are F-diagonal tensors and*

$$\mathcal{C}^H \star_c \mathcal{C} + \mathcal{S}^H \star_c \mathcal{S} = \mathcal{I}.$$

*Proof.* Since $\mathcal{W}$ is a mode-1 strong unitary tensor, we have $\mathcal{W} \star_c \mathcal{W}^H = \mathcal{D}$, a F-diagonal tensor with $D_{iij} > 0$. Define $\mathcal{P} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$ as an F-diagonal tensor with $\mathcal{P}_{iij} = \sqrt{\mathcal{D}_{iij}^{-1}}$, $i = 1, \ldots, m_1 + m_2$, $j = 1, \ldots, p$. Then, we have

$$\mathcal{P} \star_c \mathcal{W} \star_c \mathcal{W}^H \star_c \mathcal{P} = \mathcal{I}.$$

Let $\mathcal{W}_0 = \mathcal{P} \star_c \mathcal{W}$. Then, $\mathcal{W}_0$ is a unitary tensor. By Theorem 2, we have

$$\mathcal{W}_0 = \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \left[\begin{array}{c|cc} \mathcal{C} & \mathcal{S} & \mathcal{O} \\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \begin{bmatrix} \mathcal{V}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{V}_2 \end{bmatrix}^H,$$

where $\mathcal{C} \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{S} \in \mathbb{C}^{m_1\times m_1\times p}$ are F-diagonal tensors and

$$\mathcal{C}^H \star_c \mathcal{C} + \mathcal{S}^H \star_c \mathcal{S} = \mathcal{I}.$$

Hence,

$$\mathcal{W} = \mathcal{P}^{-1} \star_c \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \left[\begin{array}{c|cc} \mathcal{C} & \mathcal{S} & \mathcal{O} \\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \begin{bmatrix} \mathcal{V}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{V}_2 \end{bmatrix}^H.$$

Let $\mathcal{U} = \mathcal{P}^{-1}\star_c \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix}$. It is easy to see $\mathcal{U}\star_c\mathcal{U}^H = (\mathcal{P}^{-1})^2$, which means $\mathcal{U}$ is a mode-1 strong unitary tensors. □

**Theorem 4.** *Let $\mathcal{W}_{11} \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{W}_{22} \in \mathbb{C}^{m_2\times m_2\times p}$ and*

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p},\ m_1 \le m_2$$

*be a mode-2 strong unitary tensor. Then, there exist unitary tensors $\mathcal{U}_1 \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{U}_2 \in \mathbb{C}^{m_2\times m_2\times p}$, and mode-1 strong unitary tensors $\mathcal{V} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$ such that*

$$\mathcal{W} = \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \left[\begin{array}{c|cc} \mathcal{C} & -\mathcal{S} & \mathcal{O} \\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \mathcal{V}^H, \tag{19}$$

*where $\mathcal{C} \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{S} \in \mathbb{C}^{m_1\times m_1\times p}$ are F-diagonal tensors and*

$$\mathcal{C}^H\star_c\mathcal{C} + \mathcal{S}^H\star_c\mathcal{S} = \mathcal{I}. \tag{20}$$

*Proof.* The proof is similar as Theorem 3. □

In the following, we will give an algorithm to compute the CS decomposition of a mode-1 strong unitary tensor.

**Algorithm 4.1:** COMPUTE THE CS DECOMPOSITION OF A MODE-1 STRONG UNITARY TENSOR

**Input:** $(m_1 + m_2) \times (m_1 + m_2) \times p$ mode-1 strong unitary $\mathcal{W}$
**Output:** $\mathcal{U} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$, $\mathcal{V}_1 \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{V}_2 \in \mathbb{C}^{m_2\times m_2\times p}$, $\mathcal{C} \in \mathbb{C}^{m_1\times m_1\times p}$ and $\mathcal{S} \in \mathbb{C}^{m_1\times m_1\times p}$

1. Compute $\mathcal{D} = \mathcal{W}\star_c\mathcal{W}^H$
2. Construct an F-diagonal tensor $\mathcal{P} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$ with $\mathcal{P}_{iij} = \sqrt{\mathcal{D}_{iij}^{-1}}$, $i = 1, \ldots, m_1 + m_2$, $j = 1, \ldots, p$
3. Compute $\mathcal{W}_0 = \mathcal{P}\star_c\mathcal{W}$
4. Compute $\widehat{\mathcal{W}_0} = L(\mathcal{W}_0) = \mathcal{W}_0 \times_3 M$, where $M$ is defined in (5)
5. for $i = 1, \ldots, p$
   $\left[\widehat{\mathcal{U}_1}^{(i)}, \widehat{\mathcal{U}_2}^{(i)}, \widehat{\mathcal{V}}_1^{(i)}, \widehat{\mathcal{V}}_2^{(i)}, \widehat{\mathcal{C}}^{(i)}, \widehat{\mathcal{S}}^{(i)}\right] = \mathrm{csd}(\widehat{\mathcal{W}_0}^{(i)})$
   end
6. $\mathcal{U}_1 = L^{-1}(\widehat{\mathcal{U}_1})$, $\mathcal{U}_2 = L^{-1}(\widehat{\mathcal{U}_2})$, $\mathcal{V}_1 = L^{-1}(\widehat{\mathcal{V}}_1)$, $\mathcal{V}_2 = L^{-1}(\widehat{\mathcal{V}}_2)$, $\mathcal{C} = L^{-1}(\widehat{\mathcal{C}})$, $\mathcal{S} = L^{-1}(\widehat{\mathcal{S}})$
7. $\mathcal{U} = \mathcal{P}^{-1}\star_c\mathrm{diag}(\mathcal{U}_1, \mathcal{U}_2)$

**Example 5.** Let $\mathcal{W} \in \mathbb{C}^{3\times3\times3}$ be a mode-1 strong unitary tensor with

$$\mathcal{W}^{(1)} = \begin{bmatrix} \frac{10}{3} & 0 & \frac{2}{3} \\ 0 & \frac{7}{3} + \frac{4}{3}i & -\frac{4}{3} + 2i \\ \frac{4}{3} & -\frac{4}{3} + 2i & 2 + \frac{4}{3}i \end{bmatrix},$$

$$\mathcal{W}^{(2)}=\begin{bmatrix}-\frac{2}{3}+\frac{1}{2}i & 0 & \frac{1}{2}-i\\ 0 & -i & 1-\frac{3}{2}i\\ 1-\frac{1}{2}i & 1-\frac{3}{2}i & -1\end{bmatrix},\ \mathcal{W}^{(3)}=\begin{bmatrix}-\frac{1}{6}-\frac{1}{2}i & 0 & \frac{1}{6}+i\\ 0 & -\frac{2}{3}+\frac{1}{3}i & -\frac{1}{3}+\frac{1}{2}i\\ \frac{1}{3}+\frac{1}{2}i & -\frac{1}{3}+\frac{1}{2}i & -\frac{2}{3}i\end{bmatrix}.$$

A simple computation gives

$$L(\mathcal{W})^{(1)}=\begin{bmatrix}0&0&2\\0&1&0\\4&0&0\end{bmatrix},\ L(\mathcal{W})^{(2)}=\begin{bmatrix}2+i&0&1-2i\\0&3&0\\2-i&0&1+2i\end{bmatrix},\ L(\mathcal{W})^{(3)}=\begin{bmatrix}5&0&0\\0&3+2i&-2+3i\\0&-2+3i&3+2i\end{bmatrix}.$$

Then,

$$L\left(\mathcal{W}\star_c\mathcal{W}^H\right)^{(1)}=\begin{bmatrix}4&0&0\\0&1&0\\0&0&16\end{bmatrix},\ L\left(\mathcal{W}\star_c\mathcal{W}^H\right)^{(2)}=\begin{bmatrix}10&0&0\\0&9&0\\0&0&10\end{bmatrix},\ L\left(\mathcal{W}\star_c\mathcal{W}^H\right)^{(3)}=\begin{bmatrix}25&0&0\\0&26&0\\0&0&26\end{bmatrix},$$

which means $\mathcal{W}$ is a mode-1 strong unitary tensor. By simple computations, one has

$$L(\mathcal{P})^{(1)}=\begin{bmatrix}\frac{1}{2}&0&0\\0&1&0\\0&0&\frac{1}{4}\end{bmatrix},\ L(\mathcal{P})^{(2)}=\begin{bmatrix}\frac{1}{\sqrt{10}}&0&0\\0&\frac{1}{3}&0\\0&0&\frac{1}{\sqrt{10}}\end{bmatrix},\ L(\mathcal{P})^{(3)}=\begin{bmatrix}\frac{1}{5}&0&0\\0&\frac{1}{\sqrt{26}}&0\\0&0&\frac{1}{\sqrt{26}}\end{bmatrix}.$$

Then,

$$L(\mathcal{W}_0)^{(1)}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\ L(\mathcal{W}_0)^{(2)}=\begin{bmatrix}\frac{2}{\sqrt{10}}+\frac{1}{\sqrt{10}}i & 0 & \frac{1}{\sqrt{10}}-\frac{2}{\sqrt{10}}i\\ 0&1&0\\ \frac{2}{\sqrt{10}}-\frac{1}{\sqrt{10}}i & 0 & \frac{1}{\sqrt{10}}2+\frac{2}{\sqrt{10}}i\end{bmatrix},$$

$$L(\mathcal{W}_0)^{(3)}=\begin{bmatrix}1&0&0\\0&\frac{3}{\sqrt{26}}+\frac{2}{\sqrt{26}}i & -\frac{2}{\sqrt{26}}+\frac{3}{\sqrt{26}}i\\ 0&-\frac{2}{\sqrt{26}}+\frac{3}{\sqrt{26}}i & \frac{3}{\sqrt{26}}+\frac{2}{\sqrt{26}}i\end{bmatrix}.$$

Moreover,

$$\mathcal{W}_0^{(1)}=\begin{bmatrix}\frac{2}{3}&0&\frac{1}{3}\\ 0&\frac{1}{3}+\frac{2}{\sqrt{26}}+\frac{4}{3\sqrt{26}}i & -\frac{4}{3\sqrt{26}}+\frac{2}{\sqrt{26}}i\\ \frac{1}{3}&-\frac{4}{3\sqrt{26}}+\frac{2}{\sqrt{26}}i & \frac{2}{\sqrt{26}}+\frac{4}{3\sqrt{26}}i\end{bmatrix},$$

$$\mathcal{W}_0^{(2)}=\begin{bmatrix}-\frac{1}{2}+\frac{1}{\sqrt{10}}+\frac{1}{2\sqrt{10}}i & 0 & \frac{1}{2\sqrt{10}}-\frac{1}{\sqrt{10}}i\\ 0 & \frac{1}{2}-\frac{3}{2\sqrt{26}}-\frac{1}{\sqrt{26}}i & \frac{1}{\sqrt{26}}-\frac{3}{2\sqrt{26}}i\\ \frac{1}{\sqrt{10}}-\frac{1}{2\sqrt{10}}i & \frac{1}{\sqrt{26}}-\frac{3}{2\sqrt{26}}i & \frac{1}{2\sqrt{10}}-\frac{3}{2\sqrt{26}}+\left(\frac{1}{\sqrt{10}}-\frac{1}{\sqrt{26}}\right)i\end{bmatrix},$$

$$\mathcal{W}_0^{(3)} = \begin{bmatrix} \frac{1}{6} - \frac{1}{\sqrt{10}} - \frac{1}{2\sqrt{10}}i & 0 & \frac{1}{3} - \frac{1}{2\sqrt{10}} + \frac{1}{\sqrt{10}}i \\ 0 & -\frac{1}{6} + \frac{1}{2\sqrt{26}} + \frac{1}{3\sqrt{26}}i & -\frac{1}{3\sqrt{26}} + \frac{1}{2\sqrt{26}}i \\ \frac{1}{3} - \frac{1}{\sqrt{10}} + \frac{1}{2\sqrt{10}}i & -\frac{1}{3\sqrt{26}} + \frac{1}{2\sqrt{26}}i & \frac{1}{2\sqrt{26}} - \frac{1}{2\sqrt{10}} + \left(\frac{1}{3\sqrt{26}} - \frac{1}{\sqrt{10}}\right)i \end{bmatrix}.$$

Next, it is easy to execute Algorithm 3.1 to get the CS decomposition of the mode-1 strong unitary tensor. □

Similar as Algorithm 4.1, we can get an algorithm to compute the CS decomposition of a mode-2 strong unitary tensor.

## 5 The hyperbolic CS decomposition of tensors

In this section, we firstly give some definitions of the $\mathcal{J}$-orthogonal tensors. Then, we establish the hyperbolic CS decomposition of a $\mathcal{J}$-orthogonal tensor, mode-1 strong $\mathcal{J}$-orthogonal tensor and orthogonal tensor. Then, there exist unitary tensors $\mathcal{J}$-orthogonal tensor, respectively.

**Definition 13.** [54] Let $Q \in \mathbb{R}^{n\times n}$. If

$$Q^H JQ = QJQ^H = J,$$

where $J = \operatorname{diag}(I_p, -I_q),\ \ p + q = n$, then $Q$ is called a $J$-orthogonal matrix.

**Definition 14.** Let $\mathcal{Q} \in \mathbb{R}^{n\times n\times n_3}$. If

$$\mathcal{Q}^H \star_c \mathcal{J} \star_c \mathcal{Q} = \mathcal{Q} \star_c \mathcal{J} \star_c \mathcal{Q}^H = \mathcal{J},$$

where $\mathcal{J} \in \mathbb{R}^{n\times n\times n_3}$ is an F-diagonal tensor with

$$L(\mathcal{J})^{(i)} = \operatorname{diag}\left(I_k^{(i)}, -I_{n-k}^{(i)}\right),\ 0 \le k \le n,\ i = 1, \dots, n_3,$$

then $\mathcal{Q}$ is called a $\mathcal{J}$-orthogonal tensor.

The definition of a $\mathcal{J}$-orthogonal tensor can be generalized as follows.

**Definition 15.** Let $\mathcal{Q} \in \mathbb{R}^{n\times n\times n_3}$. If

$$\mathcal{Q}^H \star_c \mathcal{J} \star_c \mathcal{Q} = \mathcal{Q} \star_c \mathcal{J} \star_c \mathcal{Q}^H = \mathcal{J}_0,$$

where $\mathcal{J} \in \mathbb{R}^{n\times n\times n_3}$ and $\mathcal{J}_0 \in \mathbb{R}^{n\times n\times n_3}$ are F-diagonal tensors with

$$L(\mathcal{J})^{(i)} = \operatorname{diag}(I_k^{(i)}, -I_{n-k}^{(i)}) \ \text{ and } \ L(\mathcal{J}_0)^{(i)} = \operatorname{diag}(\alpha_1^{(i)}, \dots, \alpha_k^{(i)}, -\beta_{k+1}^{(i)}, \dots, -\beta_n^{(i)}),$$

where $\alpha_1^{(i)}, \dots, \alpha_k^{(i)} > 0, \beta_{k+1}^{(i)}, \dots, \beta_n^{(i)} > 0, 0 \le k \le n, i = 1, \dots, n_3$, then $\mathcal{Q}$ is called a strong $\mathcal{J}$-orthogonal tensor.

Moreover, the definition of the strong $\mathcal{J}$-orthogonal tensors can be extended to the mode-1 strong $\mathcal{J}$-orthogonal tensor and mode-2 strong $\mathcal{J}$-orthogonal tensor.

**Definition 16.** Let $\mathcal{Q} \in \mathbb{R}^{n\times n\times n_3}$. If

$$\mathcal{Q} \star_c \mathcal{J} \star_c \mathcal{Q}^H = \mathcal{J}_0,$$

where $\mathcal{J} \in \mathbb{R}^{n\times n\times n_3}$ and $\mathcal{J}_0 \in \mathbb{R}^{n\times n\times n_3}$ are F-diagonal tensors with

$$L(\mathcal{J})^{(i)} = \operatorname{diag}(I_k^{(i)}, -I_{n-k}^{(i)}) \ \text{ and } \ L(\mathcal{J}_0)^{(i)} = \operatorname{diag}(\alpha_1^{(i)}, \dots, \alpha_k^{(i)}, -\beta_{k+1}^{(i)}, \dots, -\beta_n^{(i)}),$$

where $\alpha_1^{(i)},\dots,\alpha_k^{(i)}>0, \beta_{k+1}^{(i)},\dots,\beta_n^{(i)}>0, 0\le k\le n, i=1,\dots,n_3$, then $\mathcal{Q}$ is called a mode-1 strong $\mathcal{J}$-orthogonal tensor.

**Definition 17.** Let $\mathcal{Q}\in\mathbb{R}^{n\times n\times n_3}$. If

$$\mathcal{Q}^H\star_c\mathcal{J}\star_c\mathcal{Q}=\mathcal{J}_0,$$

where $\mathcal{J}\in\mathbb{R}^{n\times n\times n_3}$ and $\mathcal{J}_0\in\mathbb{R}^{n\times n\times n_3}$ are F-diagonal tensors with

$$L(\mathcal{J})^{(i)}=\operatorname{diag}(I_k^{(i)},-I_{n-k}^{(i)})\quad\text{and}\quad L(\mathcal{J}_0)^{(i)}=\operatorname{diag}(\alpha_1^{(i)},\dots,\alpha_k^{(i)},-\beta_{k+1}^{(i)},\dots,-\beta_n^{(i)}),$$

where $\alpha_1^{(i)},\dots,\alpha_k^{(i)}>0, \beta_{k+1}^{(i)},\dots,\beta_n^{(i)}>0, 0\le k\le n, i=1,\dots,n_3$, then $\mathcal{Q}$ is called a mode-2 strong $\mathcal{J}$-orthogonal tensor.

The following result is the hyperbolic CS decomposition of a $\mathcal{J}$-orthogonal tensor.

**Theorem 5.** *Let $\mathcal{W}_{11}\in\mathbb{R}^{m_1\times m_1\times p}$, $\mathcal{W}_{22}\in\mathbb{R}^{m_2\times m_2\times p}$ and*

$$\mathcal{W}=\begin{bmatrix}\mathcal{W}_{11} & \mathcal{W}_{12}\\ \mathcal{W}_{21} & \mathcal{W}_{22}\end{bmatrix}\in\mathbb{R}^{(m_1+m_2)\times(m_1+m_2)\times p},\ m_1\le m_2$$

*be a $\mathcal{J}$-orthogonal tensor. Then, there exist unitary tensors $\mathcal{U}_1\in\mathbb{R}^{m_1\times m_1\times p}$, $\mathcal{U}_2\in\mathbb{R}^{m_2\times m_2\times p}$, $\mathcal{V}_1\in\mathbb{R}^{m_1\times m_1\times p}$ and $\mathcal{V}_2\in\mathbb{R}^{m_2\times m_2\times p}$ such that*

$$\mathcal{W}_0=\begin{bmatrix}\mathcal{U}_1 & \mathcal{O}\\ \mathcal{O} & \mathcal{U}_2\end{bmatrix}\star_c\left[\begin{array}{c|cc}\mathcal{C} & -\mathcal{S} & \mathcal{O}\\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O}\\ \mathcal{O} & \mathcal{O} & \mathcal{I}\end{array}\right]\star_c\begin{bmatrix}\mathcal{V}_1 & \mathcal{O}\\ \mathcal{O} & \mathcal{V}_2\end{bmatrix}^H, \tag{21}$$

*where $\mathcal{C}\in\mathbb{R}^{m_1\times m_1\times p}$, $\mathcal{S}\in\mathbb{R}^{m_1\times m_1\times p}$ are F-diagonal tensors and*

$$\mathcal{C}^2-\mathcal{S}^2=\mathcal{I}. \tag{22}$$

*Proof.* Since $\mathcal{W}$ is a $\mathcal{J}$-orthogonal tensor, then $\mathcal{W}^H\star_c\mathcal{J}\star_c\mathcal{W}=\mathcal{J}$. According to Lemma 2, we get

$$L(\mathcal{W}^H)^{(i)}L(\mathcal{J}^H)^{(i)}L(\mathcal{W})^{(i)}=L(\mathcal{J}^H)^{(i)}, i=1,\dots,p,$$

which means $L(\mathcal{W})^{(i)}, i=1,\dots,p$ are $\mathcal{J}$-orthogonal matrices. Because of

$$\mathcal{W}=\begin{bmatrix}\mathcal{W}_{11} & \mathcal{W}_{12}\\ \mathcal{W}_{21} & \mathcal{W}_{22}\end{bmatrix},$$

by Lemma 7, we get

$$L(\mathcal{W})^{(i)}=\begin{bmatrix}L(\mathcal{W}_{11})^{(i)} & L(\mathcal{W}_{12})^{(i)}\\ L(\mathcal{W}_{21})^{(i)} & L(\mathcal{W}_{22})^{(i)}\end{bmatrix}, i=1,\dots,p.$$

Using Theorem 3.2 of [54], we have

$$L(\mathcal{W})^{(i)}=\begin{bmatrix}L(\mathcal{U}_1)^{(i)} & O\\ O & L(\mathcal{U}_2)^{(i)}\end{bmatrix}\left[\begin{array}{c|cc}L(\mathcal{C})^{(i)} & -L(\mathcal{S})^{(i)} & O\\ \hline -L(\mathcal{S})^{(i)} & L(\mathcal{C})^{(i)} & O\\ O & O & L(\mathcal{I})^{(i)}\end{array}\right]\begin{bmatrix}L(\mathcal{V}_1)^{(i)} & O\\ O & L(\mathcal{V}_2)^{(i)}\end{bmatrix}^H,$$

where $L(\mathcal{U}_1)^{(i)} \in \mathbb{R}^{m_1\times m_1}$, $L(\mathcal{U}_2)^{(i)} \in \mathbb{R}^{m_2\times m_2}$, $L(\mathcal{V}_1)^{(i)} \in \mathbb{R}^{m_1\times m_1}$, $L(\mathcal{V}_2)^{(i)} \in \mathbb{R}^{m_2\times m_2}$ and

$$L(\mathcal{C})^{(i)} = \operatorname{diag}(c_{i1}, c_{i2}, \dots, c_{im_1}) \in \mathbb{R}^{m_1\times m_1},$$

$$L(\mathcal{S})^{(i)} = \operatorname{diag}(s_{i1}, s_{i2}, \dots, s_{im_1}) \in \mathbb{R}^{m_1\times m_1},$$

with

$$\left(L(\mathcal{C})^{(i)}\right)^2 - \left(L(\mathcal{S})^{(i)}\right)^2 = I_{m_1}, i = 1, \dots, p.$$

By Definition 1, we have

$$L(\mathcal{W}) = \begin{bmatrix} L(\mathcal{U}_1) & O \\ O & L(\mathcal{U}_2) \end{bmatrix} \triangle \left[\begin{array}{c|cc} L(\mathcal{C}) & -L(\mathcal{S}) & O \\ \hline -L(\mathcal{S}) & L(\mathcal{C}) & O \\ O & O & L(\mathcal{I}) \end{array}\right] \triangle \begin{bmatrix} L(\mathcal{V}_1) & O \\ O & L(\mathcal{V}_2) \end{bmatrix}^H \tag{23}$$

with

$$\left(L(\mathcal{C})\right)^H \triangle L(\mathcal{C}) - \left(L(\mathcal{S})\right)^H \triangle L(\mathcal{S}) = L(\mathcal{I}). \tag{24}$$

Implementing the operation "$L^{-1}(\cdot)$" on both sides of the equalities (23) and (24), we have

$$\mathcal{W} = \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \left[\begin{array}{c|cc} \mathcal{C} & -\mathcal{S} & \mathcal{O} \\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \begin{bmatrix} \mathcal{V}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{V}_2 \end{bmatrix}^H,$$

with

$$\mathcal{C}^2 - \mathcal{S}^2 = \mathcal{I}.$$

Similar as Theorem 1, we get that $\mathcal{C}$ and $\mathcal{S}$ are F-diagonal tensors. □

Let

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} \in \mathbb{R}^{(m_1+m_2)\times(m_1+m_2)\times p}.$$

If $\mathcal{W}_{11}$ is invertible, denote

$$\operatorname{exc}(\mathcal{W}) = \begin{bmatrix} \mathcal{W}_{11}^{-1} & -\mathcal{W}_{11}^{-1}\mathcal{W}_{12} \\ \mathcal{W}_{21}\mathcal{W}_{11}^{-1} & \mathcal{W}_{22} - \mathcal{W}_{21}\mathcal{W}_{11}^{-1}\mathcal{W}_{12} \end{bmatrix}. \tag{25}$$

It is easy to check $\operatorname{exc}(\operatorname{exc}(\mathcal{W})) = \mathcal{W}$. Moreover, by [54, Theorem 2.2], one has that if $\mathcal{W}$ is a $\mathcal{J}$-orthogonal tensor, then $\operatorname{exc}(\mathcal{W})$ is an orthogonal tensor. Conversely, if $\mathcal{W}$ is an orthogonal tensor and $\mathcal{W}_{11}$ is invertible, then $\operatorname{exc}(\mathcal{W})$ is a $\mathcal{J}$-orthogonal tensor.

Now, we can set up an algorithm for the hyperbolic CS decomposition of the $\mathcal{J}$-orthogonal tensor based on Theorem 5.

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**Algorithm 5.1:** COMPUTE THE HYPERBOLIC CS DECOMPOSITION OF A $\mathcal{J}$-ORTHOGONAL TENSOR

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**Input:** $(m_1 + m_2) \times (m_1 + m_2) \times p$ $\mathcal{J}$-orthogonal tensor $\mathcal{W}$
**Output:** $\mathcal{U}_1 \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{U}_2 \in \mathbb{C}^{m_2\times m_2\times p}$, $\mathcal{V}_1 \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{V}_2 \in \mathbb{C}^{m_2\times m_2\times p}$, $\mathcal{C} \in \mathbb{C}^{m_1\times m_1\times p}$ and $\mathcal{S} \in \mathbb{C}^{m_1\times m_1\times p}$

1. Compute $\mathcal{T} = \operatorname{exc}(\mathcal{W})$, where $\operatorname{exc}(\cdot)$ was defined in (25)
2. Compute $\widehat{\mathcal{T}} = L(\mathcal{T}) = \mathcal{T} \times_3 M$, where $M$ is defined in (5)
3. for $i = 1, \dots, p$
   $\left[\widehat{\mathcal{U}_1}^{(i)}, \widehat{\mathcal{U}_2}^{(i)}, \widehat{\mathcal{V}}_1^{(i)}, \widehat{\mathcal{V}}_2^{(i)}, \widehat{\mathcal{C}}_0^{(i)}, \widehat{\mathcal{S}}_0^{(i)}\right] = \operatorname{csd}(\widehat{\mathcal{T}}^{(i)})$
   end
4. $\mathcal{U}_1 = L^{-1}(\widehat{\mathcal{U}_1})$, $\mathcal{U}_2 = L^{-1}(\widehat{\mathcal{U}_2})$, $\mathcal{V}_1 = L^{-1}(\widehat{\mathcal{V}}_1)$, $\mathcal{V}_2 = L^{-1}(\widehat{\mathcal{V}}_2)$, $\mathcal{C}_0 = L^{-1}(\widehat{\mathcal{C}}_0)$, $\mathcal{S}_0 = L^{-1}(\widehat{\mathcal{S}}_0)$
5. $\mathcal{C} = \mathcal{C}_0^{-1} = \mathcal{C}_0 + \mathcal{S}_0 \star_c \mathcal{C}_0^{-1} \star_c \mathcal{S}_0$; $\mathcal{S} = \mathcal{C}_0^{-1} \star_c \mathcal{S}_0$

---

**Example 6.** Let $\mathcal{W} \in \mathbb{R}^{6\times6\times2}$ be a $\mathcal{J}$-orthogonal tensor with

$$\mathcal{W}^{(1)} = \begin{bmatrix} 1.4951 & -3.3435 & 1.3816 & -3.1084 & 0.4438 & 0.8044 \\ -4.0321 & 3.7284 & -1.6996 & 4.8518 & 0.4706 & -1.5841 \\ 1.1742 & -0.3878 & -0.3225 & -1.0894 & -0.9100 & 0.6406 \\ -2.3137 & 3.0639 & -1.0316 & 3.4840 & -0.5632 & -1.4903 \\ -0.7685 & 1.1109 & -0.0678 & 1.6195 & -0.1573 & 0.4154 \\ -3.1892 & 3.6481 & -2.1505 & 4.2724 & 0.4984 & -1.1630 \end{bmatrix},$$

$$\mathcal{W}^{(2)} = \begin{bmatrix} 5.2366 & -10.8788 & -9.4134 & 0.9070 & 7.6567 & 6.9803 \\ -2.9576 & 7.0364 & 7.3158 & -1.8476 & -5.8428 & -4.6961 \\ -4.8489 & 8.9312 & 6.0322 & 0.9117 & -5.3293 & -5.4867 \\ -1.4043 & 3.4248 & 4.0822 & -1.0095 & -2.8848 & -2.1003 \\ -4.6152 & 9.2967 & 7.2715 & -0.4296 & -5.9769 & -5.9621 \\ -1.6583 & 4.3817 & 5.3849 & -1.7032 & -3.9386 & -3.5379 \end{bmatrix}.$$

A simple computation gives

$$L(\mathcal{W})^{(1)} = \begin{bmatrix} 11.9683 & -25.1012 & -17.4453 & -1.2945 & 15.7572 & 14.7650 \\ -9.9473 & 17.8012 & 12.9320 & 1.1565 & -11.2151 & -10.9762 \\ -8.5235 & 17.4747 & 11.7419 & 0.7340 & -11.5687 & -10.3328 \\ -5.1222 & 9.9135 & 7.1328 & 1.4650 & -6.3328 & -5.6910 \\ -9.9989 & 19.7044 & 14.4752 & 0.7603 & -12.1111 & -11.5088 \\ -6.5058 & 12.4115 & 8.6194 & 0.8661 & -7.3789 & -8.2388 \end{bmatrix},$$

$$L(\mathcal{W})^{(2)} = \begin{bmatrix} 1.4951 & -3.3435 & 1.3816 & -3.1084 & 0.4438 & 0.8044 \\ -4.0321 & 3.7284 & -1.6996 & 4.8518 & 0.4706 & -1.5841 \\ 1.1742 & -0.3878 & -0.3225 & -1.0894 & -0.9100 & 0.6406 \\ -2.3137 & 3.0639 & -1.0316 & 3.4840 & -0.5632 & -1.4903 \\ -0.7685 & 1.1109 & -0.0678 & 1.6195 & -0.1573 & 0.4154 \\ -3.1892 & 3.6481 & -2.1505 & 4.2724 & 0.4984 & -1.1630 \end{bmatrix}.$$

By using the formula

$$\operatorname{exc}(\mathcal{W}) = \begin{bmatrix} \mathcal{W}_{11}^{-1} & -\mathcal{W}_{11}^{-1}\mathcal{W}_{12} \\ \mathcal{W}_{21}\mathcal{W}_{11}^{-1} & \mathcal{W}_{22} - \mathcal{W}_{21}\mathcal{W}_{11}^{-1}\mathcal{W}_{12} \end{bmatrix},$$

where $\mathcal{W}_{11} \in \mathbb{R}^{2\times2\times2}$, $\mathcal{W}_{12} \in \mathbb{R}^{2\times4\times2}$, $\mathcal{W}_{21} \in \mathbb{R}^{4\times2\times2}$ and $\mathcal{W}_{22} \in \mathbb{R}^{4\times4\times2}$, one has

$$L(\mathcal{T})^{(1)} = \operatorname{exc}(L(\mathcal{W})^{(1)}) = \begin{bmatrix} -0.4859 & -0.6851 & 0.3838 & 0.1634 & -0.0277 & -0.3461 \\ -0.2715 & -0.3267 & -0.5120 & 0.0263 & 0.6145 & 0.4232 \\ -0.6031 & 0.1312 & -0.4764 & -0.1985 & -0.5937 & 0.0124 \\ -0.2028 & 0.2709 & 0.0911 & 0.8892 & -0.0987 & 0.2772 \\ -0.4917 & 0.4136 & 0.5489 & -0.3545 & 0.2750 & 0.2907 \\ -0.2088 & 0.4028 & -0.2322 & 0.1300 & 0.4286 & -0.7346 \end{bmatrix},$$

$$L(\mathcal{T})^{(2)} = \operatorname{exc}(L(\mathcal{W})^{(2)}) = \begin{bmatrix} -0.4715 & -0.4228 & -0.0672 & 0.5859 & 0.4082 & -0.2906 \\ -0.5099 & -0.1891 & 0.3831 & -0.6677 & 0.3153 & 0.1106 \\ -0.3559 & -0.4232 & -0.5500 & -0.1425 & -0.5529 & 0.2565 \\ -0.4714 & 0.3990 & 0.2979 & 0.0827 & -0.5418 & -0.4790 \\ -0.2041 & 0.1149 & 0.4095 & 0.4275 & -0.1208 & 0.7617 \\ -0.3565 & 0.6588 & -0.5383 & -0.0318 & 0.3466 & 0.1673 \end{bmatrix}.$$

It is easy to check that $L(\mathcal{T})^{(1)}$ and $L(\mathcal{T})^{(2)}$ are orthogonal. Next, using the method of Algorithm 3.1, we can get that there exist orthogonal tensors $\mathcal{U}$ and $\mathcal{V}$ such that

$$L\left(\mathcal{U}^H \star_c \mathcal{T} \star_c \mathcal{V}\right)^{(1)} = \begin{bmatrix} 0.9407 & 0 & 0.3391 & 0 & 0 & 0 \\ 0 & 0.0290 & 0 & 0.9996 & 0 & 0 \\ -0.3391 & 0 & 0.9407 & 0 & 0 & 0 \\ 0 & -0.9996 & 0 & 0.0290 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$L\left(\mathcal{U}^H \star_c \mathcal{T} \star_c \mathcal{V}\right)^{(2)} = \begin{bmatrix} 0.8204 & 0 & 0.5717 & 0 & 0 & 0 \\ 0 & 0.1541 & 0 & 0.9880 & 0 & 0 \\ -0.5717 & 0 & 0.8204 & 0 & 0 & 0 \\ 0 & -0.9880 & 0 & 0.1541 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the last step, we compute

$$L(\mathcal{C})^{(1)} = (L(\mathcal{C}_0)^{(1)})^{-1} = \begin{bmatrix} 1.0630 & 0 \\ 0 & 34.4828 \end{bmatrix}, \quad L(\mathcal{C})^{(2)} = (L(\mathcal{C}_0)^{(2)})^{-1} = \begin{bmatrix} 1.2189 & 0 \\ 0 & 6.4893 \end{bmatrix},$$

$$L(\mathcal{S})^{(1)} = (L(\mathcal{C}_0)^{(1)})^{-1} L(\mathcal{S}_0)^{(1)} = \begin{bmatrix} 0.3605 & 0 \\ 0 & 34.4690 \end{bmatrix},$$

$$L(\mathcal{S})^{(2)} = (L(\mathcal{C}_0)^{(2)})^{-1} L(\mathcal{S}_0)^{(2)} = \begin{bmatrix} 0.6969 & 0 \\ 0 & 6.4114 \end{bmatrix}.$$

Therefore,

$$L\left(\mathcal{U}^H \star_c \mathcal{W} \star_c \mathcal{V}\right)^{(1)} = \begin{bmatrix} 1.0630 & 0 & -0.3605 & 0 & 0 & 0 \\ 0 & 34.4828 & 0 & -34.4690 & 0 & 0 \\ -0.3605 & 0 & 1.0630 & 0 & 0 & 0 \\ 0 & -34.4690 & 0 & 34.4828 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$L(\mathcal{U}^H\star_c\mathcal{W}\star_c\mathcal{V})^{(2)} = \begin{bmatrix} 1.2189 & 0 & -0.6969 & 0 & 0 & 0 \\ 0 & 6.4893 & 0 & -6.4114 & 0 & 0 \\ -0.6969 & 0 & 1.2189 & 0 & 0 & 0 \\ 0 & -6.4114 & 0 & 6.4893 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Hence,

$$(\mathcal{U}^H\star_c\mathcal{W}\star_c\mathcal{V})^{(1)} = \begin{bmatrix} 1.2189 & 0 & -0.6969 & 0 & 0 & 0 \\ 0 & 6.4893 & 0 & -6.4114 & 0 & 0 \\ -0.6969 & 0 & 1.2189 & 0 & 0 & 0 \\ 0 & -6.4114 & 0 & 6.4893 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(\mathcal{U}^H\star_c\mathcal{W}\star_c\mathcal{V})^{(2)} = \begin{bmatrix} -0.0780 & 0 & 0.1682 & 0 & 0 & 0 \\ 0 & 13.9967 & 0 & 14.0288 & 0 & 0 \\ 0.1682 & 0 & -0.0780 & 0 & 0 & 0 \\ 0 & -14.0288 & 0 & 172410 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

□

In the following, we give the hyperbolic CS decomposition of a mode-1 strong $\mathcal{J}$-orthogonal tensor.

**Theorem 6.** *Let $\mathcal{W}_{11} \in \mathbb{R}^{m_1\times m_1\times p}$, $\mathcal{W}_{22} \in \mathbb{R}^{m_2\times m_2\times p}$ and*

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} \in \mathbb{R}^{(m_1+m_2)\times(m_1+m_2)\times p}, \ m_1 \le m_2$$

*be a mode-1 strong $\mathcal{J}$-orthogonal tensor. Then, there exist a mode-1 strong unitary tensor $\mathcal{U} \in \mathbb{R}^{(m_1+m_2)\times(m_1+m_2)\times p}$ and unitary tensors $\mathcal{V}_1 \in \mathbb{R}^{m_1\times m_1\times p}$ and $\mathcal{V}_2 \in \mathbb{R}^{m_2\times m_2\times p}$ such that*

$$\mathcal{W} = \mathcal{U} \star_c \left[\begin{array}{c|cc} \mathcal{C} & -\mathcal{S} & \mathcal{O} \\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \begin{bmatrix} \mathcal{V}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{V}_2 \end{bmatrix}^H,$$

*where $\mathcal{C} \in \mathbb{R}^{m_1\times m_1\times p}$, $\mathcal{S} \in \mathbb{R}^{m_1\times m_1\times p}$ are F-diagonal tensors and*

$$\mathcal{C}^2 - \mathcal{S}^2 = \mathcal{I}.$$

*Proof.* Since $\mathcal{W}$ is a mode-1 strong $\mathcal{J}$-orthogonal tensor, by Definition 16, we have

$$\mathcal{W}\star_c\mathcal{J}\star_c\mathcal{W}^H = \mathcal{J}_0.$$

Define $\mathcal{P} \in \mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$ as an F-diagonal tensor with

$$\mathcal{P}^{(i)} = \mathrm{diag}\left(\sqrt{(\alpha_1^{(i)})^{-1}}, \ldots, \sqrt{(\alpha_p^{(i)})^{-1}}, \sqrt{(\beta_{p+1}^{(i)})^{-1}}, \ldots, \sqrt{(\beta_n^{(i)})^{-1}}\right), \ i = 1, \ldots, p.$$

Then, we have

$$\mathcal{P}\star_c\mathcal{W}\star_c\mathcal{J}\star_c\mathcal{W}^H\star_c\mathcal{P} = \mathcal{I}.$$

Let $\mathcal{W}_0 = \mathcal{P}\star_c\mathcal{W}$. Then, $\mathcal{W}_0\star_c\mathcal{J}\star_c\mathcal{W}_0^H = \mathcal{J}$, which means that $\mathcal{W}_0$ is a $\mathcal{J}$-orthogonal tensor. By Theorem 5, we have

$$\mathcal{W}_0 = \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \left[\begin{array}{c|cc} \mathcal{C} & -\mathcal{S} & \mathcal{O} \\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \begin{bmatrix} \mathcal{V}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{V}_2 \end{bmatrix}^H,$$

where $\mathcal{C} \in \mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{S} \in \mathbb{C}^{m_1\times m_1\times p}$ are F-diagonal tensors and

$$\mathcal{C}^2 - \mathcal{S}^2 = \mathcal{I}.$$

Therefore,

$$\mathcal{W} = \mathcal{P}^{-1}\star_c \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \left[\begin{array}{c|cc} \mathcal{C} & \mathcal{S} & \mathcal{O} \\ \hline \mathcal{S} & -\mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \begin{bmatrix} \mathcal{V}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{V}_2 \end{bmatrix}^H.$$

Let $\mathcal{U} = \mathcal{P}^{-1}\star_c \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix}$. It is easy to check $\mathcal{U}\star_c\mathcal{U}^H = (\mathcal{P}^{-1})^2$, which means that $\mathcal{U}$ is a mode-1 strong unitary tensor. □

Similar as Theorem 6, we can get the hyperbolic CS decomposition of a mode-2 strong $\mathcal{J}$-orthogonal tensor as follows.

**Theorem 7.** *Let $\mathcal{W}_{11} \in \mathbb{R}^{m_1\times m_1\times p}$, $\mathcal{W}_{22} \in \mathbb{R}^{m_2\times m_2\times p}$ and*

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} \\ \mathcal{W}_{21} & \mathcal{W}_{22} \end{bmatrix} \in \mathbb{R}^{(m_1+m_2)\times(m_1+m_2)\times p}, \ m_1 \le m_2$$

*be a mode-2 strong $\mathcal{J}$-orthogonal tensor. Then, there exist unitary tensors $\mathcal{U}_1 \in \mathbb{R}^{m_1\times m_1\times p}$, $\mathcal{U}_2 \in \mathbb{R}^{m_2\times m_2\times p}$, and a mode-1 strong unitary tensor $\mathcal{V} \in \mathbb{R}^{(m_1+m_2)\times(m_1+m_2)\times p}$ such that*

$$\mathcal{W} = \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \left[\begin{array}{c|cc} \mathcal{C} & -\mathcal{S} & \mathcal{O} \\ \hline -\mathcal{S} & \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{I} \end{array}\right] \star_c \mathcal{V}^H,$$

*where $\mathcal{C} \in \mathbb{R}^{m_1\times m_1\times p}$, $\mathcal{S} \in \mathbb{R}^{m_1\times m_1\times p}$ are F-diagonal tensors and*

$$\mathcal{C}^2 - \mathcal{S}^2 = \mathcal{I}. \tag{27}$$

In the following, we will give an algorithm to compute the hyperbolic CS decomposition of the mode-1 strong $\mathcal{J}$-orthogonal tensor based on Theorem 6. One can also analogously get an algorithm to compute the hyperbolic CS decomposition of the mode-2 strong $\mathcal{J}$-orthogonal tensor.

**Algorithm 5.2:** COMPUTE THE HYPERBOLIC CS DECOMPOSITION OF THE MODE-1 STRONG $\mathcal{J}$-ORTHOGONAL TENSOR

**Input:** $(m_1+m_2)\times(m_1+m_2)\times p$ $\mathcal{J}$-orthogonal tensor $\mathcal{W}$
**Output:** $\mathcal{U}\in\mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$, $\mathcal{V}_1\in\mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{V}_2\in\mathbb{C}^{m_2\times m_2\times p}$, $\mathcal{C}\in\mathbb{C}^{m_1\times m_1\times p}$ and $\mathcal{S}\in\mathbb{C}^{m_1\times m_1\times p}$
1. Compute $\mathcal{D}=\mathcal{W}\star_c\mathcal{J}\star_c\mathcal{W}^H$
2. Construct an F-diagonal tensor $\mathcal{P}\in\mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p}$ with $\mathcal{P}_{iij}=\sqrt{\mathcal{D}_{iij}^{-1}}$, $i=1,\dots,m_1+m_2$, $j=1,\dots,p$
3. Compute $\mathcal{W}_0=\mathcal{P}\star_c\mathcal{W}$
4. Compute $\mathcal{T}=\operatorname{exc}(\mathcal{W}_0)$, where $\operatorname{exc}(\cdot)$ was defined in (25)
5. Compute $\widehat{\mathcal{T}}=L(\mathcal{T})=\mathcal{T}\times_3 M$, where $M$ is defined in (5)
6. for $i=1,\dots,p$
   $\left[\widehat{\mathcal{U}}_1^{(i)},\widehat{\mathcal{U}}_2^{(i)},\widehat{\mathcal{V}}_1^{(i)},\widehat{\mathcal{V}}_2^{(i)},\widehat{\mathcal{C}}_0^{(i)},\widehat{\mathcal{S}}_0^{(i)}\right]=\operatorname{csd}(\widehat{\mathcal{T}}^{(i)})$
   end
7. $\mathcal{U}_1=L^{-1}(\widehat{\mathcal{U}}_1)$, $\mathcal{U}_2=L^{-1}(\widehat{\mathcal{U}}_2)$, $\mathcal{V}_1=L^{-1}(\widehat{\mathcal{V}}_1)$, $\mathcal{V}_2=L^{-1}(\widehat{\mathcal{V}}_2)$, $\mathcal{C}_0=L^{-1}(\widehat{\mathcal{C}}_0)$, $\mathcal{S}_0=L^{-1}(\widehat{\mathcal{S}}_0)$
8. $\mathcal{C}=\mathcal{C}_0+\mathcal{S}_0\star_c\mathcal{C}_0^{-1}\star_c\mathcal{S}_0$; $\mathcal{S}=\mathcal{C}_0^{-1}\star_c\mathcal{S}_0$
9. $\mathcal{U}=\mathcal{P}^{-1}\star_c\operatorname{diag}(\mathcal{U}_1,\mathcal{U}_2)$

# 6 An application to the computation of the C-eigenvalues of a tensor

Firstly, we will introduce the C-eigenvalue of the tensor $\mathcal{A}$.

**Definition 18.** Let $\mathcal{A}\in\mathbb{C}^{n\times n\times p}$. Suppose that $\mathcal{X}\in\mathbb{C}^{n\times1\times p}$ and $\mathcal{X}\neq\mathcal{O}$. If

$$\mathcal{A}\star_c\mathcal{X}=\lambda\cdot\mathcal{X},\ \ \lambda\in\mathbb{C},\tag{28}$$

then $\lambda$ is called a C-eigenvalue of $\mathcal{A}$ and $\mathcal{X}$ is the C-eigenvector of $\mathcal{A}$ associated to $\lambda$.

Notice that $\mathcal{A}\star_c\mathcal{X}=\lambda\cdot\mathcal{X}$ is equivalent to $\mathbf{mat}(\mathcal{A})\mathbf{mat}(\mathcal{X})=\lambda\cdot\mathbf{mat}(\mathcal{X})$. Hence, all the C-eigenvalues of $\mathcal{A}$ are the eigenvalues of the matrix $\mathbf{mat}(\mathcal{A})$ and vice versa.

The following theorem involves a full CS decomposition of a unitary tensor.

**Theorem 8.** *Let $\mathcal{W}_{11}\in\mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{W}_{22}\in\mathbb{C}^{m_2\times m_2\times p}$ and*

$$\mathcal{W}=\begin{bmatrix}\mathcal{W}_{11} & \mathcal{W}_{12}\\ \mathcal{W}_{21} & \mathcal{W}_{22}\end{bmatrix}\in\mathbb{C}^{(m_1+m_2)\times(m_1+m_2)\times p},\ \ m_1\leq m_2$$

*be a unitary tensor. Then, there exist unitary tensors $\mathcal{U}_1\in\mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{U}_2\in\mathbb{C}^{m_2\times m_2\times p}$, $\mathcal{V}_1\in\mathbb{C}^{m_1\times m_1\times p}$ and $\mathcal{V}_2\in\mathbb{C}^{m_2\times m_2\times p}$ such that*

$$\mathcal{W}=\begin{bmatrix}\mathcal{U}_1 & \mathcal{O}\\ \mathcal{O} & \mathcal{U}_2\end{bmatrix}\star_c\begin{bmatrix}\mathcal{C} & \mathcal{S}\\ -\mathcal{S} & \mathcal{C}\end{bmatrix}\star_c\begin{bmatrix}\mathcal{V}_1 & \mathcal{O}\\ \mathcal{O} & \mathcal{V}_2\end{bmatrix}^H,\tag{29}$$

*where $\mathcal{C}\in\mathbb{C}^{m_1\times m_1\times p}$, $\mathcal{S}\in\mathbb{C}^{m_1\times m_1\times p}$ are F-diagonal tensors and*

$$\mathcal{C}^H\star_c\mathcal{C}+\mathcal{S}^H\star_c\mathcal{S}=\mathcal{I}.\tag{30}$$

*Proof.* The proof is similar as Theorem 2 and follows by using [52, Theorem 2.6.3]. □

The next lemma is helpful in establish the main result.

**Lemma 8.** *Let $\mathcal{W}\in\mathbb{R}^{2n\times2n\times p}$ be an orthogonal tensor. Then, there exists an orthogonal tensor $\Omega_0\in\mathbb{R}^{2n\times2n\times p}$ such that*

$$\Omega_0^H\star_c\mathcal{W}\star_c\Omega_0=\mathcal{H}_a\star_c\mathcal{H}_b,$$

*where*

$$L(\mathcal{H}_a)^{(i)} = \operatorname{diag}\left(\begin{bmatrix} -\alpha_1^{(i)} & \beta_1^{(i)} \\ \beta_1^{(i)} & \alpha_1^{(i)} \end{bmatrix}, \begin{bmatrix} -\alpha_3^{(i)} & \beta_3^{(i)} \\ \beta_3^{(i)} & \alpha_3^{(i)} \end{bmatrix}, \dots \begin{bmatrix} -\alpha_{2n-1}^{(i)} & \beta_{2n-1}^{(i)} \\ \beta_{2n-1}^{(i)} & \alpha_{2n-1}^{(i)} \end{bmatrix}\right),$$

$$L(\mathcal{H}_b)^{(i)} = \operatorname{diag}\left(1, \begin{bmatrix} -\alpha_2^{(i)} & \beta_2^{(i)} \\ \beta_2^{(i)} & \alpha_2^{(i)} \end{bmatrix}, \begin{bmatrix} -\alpha_4^{(i)} & \beta_4^{(i)} \\ \beta_4^{(i)} & \alpha_4^{(i)} \end{bmatrix}, \dots \begin{bmatrix} -\alpha_{2n-2}^{(i)} & \beta_{2n-2}^{(i)} \\ \beta_{2n-2}^{(i)} & \alpha_{2n-2}^{(i)} \end{bmatrix}, -1\right),$$

$\beta_k^{(i)} > 0$, $(\alpha_k^{(i)})^2 + (\beta_k^{(i)})^2 = 1$, $i = 1, 2, \dots, p$, $k = 1, 2, \dots, 2n-1$.

*Proof.* Since $\mathcal{W} \in \mathbb{R}^{2n\times 2n\times p}$ is an orthogonal tensor, one has $L(\mathcal{W})^{(i)} \in \mathbb{R}^{2n\times 2n}$, $i = 1, 2, \dots, p$ are orthogonal matrices. By [49, Algorithm 1], there exists an orthogonal matrix $L(\Omega_0)^{(i)}$ such that

$$(L(\Omega_0)^{(i)})^H L(\mathcal{W})^{(i)} L(\Omega_0)^{(i)} = L(\mathcal{H}_a)^{(i)} L(\mathcal{H}_b)^{(i)}.$$

Thus,

$$(L(\Omega_0))^H \triangle L(\mathcal{W}) \triangle L(\Omega_0) = L(\mathcal{H}_a) \triangle L(\mathcal{H}_b),$$

which implies $\Omega_0^H \star_c \mathcal{W} \star_c \Omega_0 = \mathcal{H}_a \star_c \mathcal{H}_b$. □

Define $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$ and $\alpha^2 + \beta^2 = 1$. Define $a, b \in \mathbb{R}$ with $a, b > 0$ by

$$(a, b) = \begin{cases} \left(\dfrac{\beta}{\sqrt{2(1+\alpha)}}, \dfrac{\sqrt{2(1+\alpha)}}{2}\right), & \alpha \geq 0, \\ \left(\dfrac{\sqrt{2(1-\alpha)}}{2}, \dfrac{\beta}{\sqrt{2(1-\alpha)}}\right), & \alpha < 0. \end{cases}$$

Then, we have

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}^T \begin{bmatrix} -\alpha & \beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{31}$$

and

$$\begin{bmatrix} b & a \\ -a & b \end{bmatrix}^T \begin{bmatrix} -\alpha & \beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} b & a \\ -a & b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{32}$$

Using (31), one has there exists an orthogonal tensor $\Omega_a \in \mathbb{R}^{2n\times 2n\times p}$ such that

$$\Omega_a^H \star_c \mathcal{H}_a \star_c \Omega_a = \mathcal{D},$$

where

$$L(\Omega_a)^{(i)} = \operatorname{diag}\left(\begin{bmatrix} a_1^{(i)} & b_1^{(i)} \\ b_1^{(i)} & -a_1^{(i)} \end{bmatrix}, \begin{bmatrix} a_2^{(i)} & b_2^{(i)} \\ b_2^{(i)} & -a_2^{(i)} \end{bmatrix}, \dots, \begin{bmatrix} a_n^{(i)} & b_n^{(i)} \\ b_n^{(i)} & -a_n^{(i)} \end{bmatrix}\right)$$

and

$$L(\mathcal{D})^{(i)} = \operatorname{diag}(1, -1, 1-1, \dots, 1, -1).$$

Similar, using (32), one has that there exists an orthogonal tensor $\Omega_b \in \mathbb{R}^{2n\times 2n\times p}$ such that

$$\Omega_b^H \star_c \mathcal{H}_b \star_c \Omega_b = \mathcal{D},$$

where

$$L(\Omega_b)^{(i)} = \operatorname{diag}\left(1, \begin{bmatrix} c_1^{(i)} & d_1^{(i)} \\ -d_1^{(i)} & c_1^{(i)} \end{bmatrix}, \begin{bmatrix} c_2^{(i)} & d_2^{(i)} \\ -d_2^{(i)} & c_2^{(i)} \end{bmatrix}, \dots, \begin{bmatrix} c_{n-1}^{(i)} & d_{n-1}^{(i)} \\ d_{n-1}^{(i)} & c_{n-1}^{(i)} \end{bmatrix}, 1\right)$$

and

$$L(\mathcal{D})^{(i)} = \operatorname{diag}(1, -1, 1-1, \dots, 1, -1).$$

Let $I_n$ be the $n \times n$ identity matrix and $e_j$ be the $j$th column of $I_n$. Denote the tensor $\mathcal{P} \in \mathbb{R}^{2n\times 2n\times p}$ defined by

$$L(\mathcal{P})^{(i)} = [e_1, e_3, \ldots, e_{2n-1}, e_2, e_4, \ldots, e_{2n}], \ i = 1, 2, \ldots, p$$

and $\mathcal{J} \in \mathbb{R}^{2n\times 2n\times p}$ by

$$L(\mathcal{J})^{(i)} = \operatorname{diag}(I_n, -I_n), \ i = 1, 2, \ldots, p.$$

Then, one has

$$(\Omega_a \star_c \mathcal{P})^H \star_c \mathcal{H}_a \star_c (\Omega_a \star_c \mathcal{P}) = (\Omega_b \star_c \mathcal{P})^H \star_c \mathcal{H}_b \star_c (\Omega_b \star_c \mathcal{P}) = \mathcal{J}.$$

Define

$$\mathcal{Z} = (\Omega_a \star_c \mathcal{P})^H \star_c (\Omega_b \star_c \mathcal{P}) = \left(\mathcal{P}^H \star_c \Omega_a \star_c \mathcal{P}\right)^H \star_c \left(\mathcal{P}^H \star_c \Omega_b \star_c \mathcal{P}\right).$$

By using

$$\Omega_0^H \star_c \mathcal{W} \star_c \Omega_0 = \mathcal{H}_a \star_c \mathcal{H}_b = (\Omega_a \star_c \mathcal{P}) \star_c \mathcal{J} \star_c (\Omega_a \star_c \mathcal{P})^H \star_c (\Omega_b \star_c \mathcal{P}) \star_c \mathcal{J} \star_c (\Omega_b \star_c \mathcal{P})^H,$$

one has

$$(\Omega_0 \star_c \Omega_a \star_c \mathcal{P})^H \star_c \mathcal{W} \star_c (\Omega_0 \star_c \Omega_a \star_c \mathcal{P}) = \mathcal{J} \star_c \mathcal{Z} \star_c \mathcal{J} \star_c \mathcal{Z}^H.$$

Since $\mathcal{Z}$ is an orthogonal tensor, we can get its full CS decomposition as

$$\begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix}^H \star_c \mathcal{Z} \star_c \begin{bmatrix} \mathcal{V}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{V}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{C} & \mathcal{S} \\ \mathcal{S} & -\mathcal{C} \end{bmatrix},$$

where

$$L(\mathcal{C})^{(i)} = \operatorname{diag}(c_1^{(i)}, c_2^{(i)}, \ldots, c_n^{(i)}), \quad L(\mathcal{S})^{(i)} = \operatorname{diag}(s_1^{(i)}, s_2^{(i)}, \ldots, s_n^{(i)}), \quad i = 1, 2, \ldots, p$$

with $c_k^{(i)}, s_k^{(i)} > 0$ and $(c_k^{(i)})^2 + (s_k^{(i)})^2 = 1$, $k = 1, 2, \ldots, n$. Finally, if we define

$$\Omega = \Omega_0 \star_c \Omega_a \star_c \mathcal{P} \star_c \begin{bmatrix} \mathcal{U}_1 & \mathcal{O} \\ \mathcal{O} & \mathcal{U}_2 \end{bmatrix} \star_c \mathcal{P},$$

one has

$$\Omega^H \star_c \mathcal{W} \star_c \Omega = \Psi,$$

where

$$L(\Psi)^{(i)} = \operatorname{diag}\left( \begin{bmatrix} (c_1^{(i)})^2 - (s_1^{(i)})^2 & 2c_1^{(i)} s_1^{(i)} \\ -2c_1^{(i)} s_1^{(i)} & (c_1^{(i)})^2 - (s_1^{(i)})^2 \end{bmatrix}, \ldots, \begin{bmatrix} (c_n^{(i)})^2 - (s_n^{(i)})^2 & 2c_n^{(i)} s_n^{(i)} \\ -2c_n^{(i)} s_n^{(i)} & (c_n^{(i)})^2 - (s_n^{(i)})^2 \end{bmatrix} \right),$$
$$i = 1, 2, \ldots, p.$$

Let

$$\mathcal{K}_1^{(i)} = L^{-1}\left( \begin{bmatrix} (c_1^{(i)})^2 - (s_1^{(i)})^2 & 2c_1^{(i)} s_1^{(i)} \\ -2c_1^{(i)} s_1^{(i)} & (c_1^{(i)})^2 - (s_1^{(i)})^2 \end{bmatrix} \right), \quad i = 1, 2, \ldots, p,$$
$$\mathcal{K}_2^{(i)} = L^{-1}\left( \begin{bmatrix} (c_2^{(i)})^2 - (s_2^{(i)})^2 & 2c_2^{(i)} s_2^{(i)} \\ -2c_2^{(i)} s_2^{(i)} & (c_2^{(i)})^2 - (s_2^{(i)})^2 \end{bmatrix} \right), \quad i = 1, 2, \ldots, p,$$
$$\vdots$$
$$\mathcal{K}_n^{(i)} = L^{-1}\left( \begin{bmatrix} (c_n^{(i)})^2 - (s_n^{(i)})^2 & 2c_n^{(i)} s_n^{(i)} \\ -2c_n^{(i)} s_n^{(i)} & (c_n^{(i)})^2 - (s_n^{(i)})^2 \end{bmatrix} \right), \quad i = 1, 2, \ldots, p.$$

Therefore, one can compute the eigenvalues of $\mathbf{mat}(\mathcal{K}_1)$, $\mathbf{mat}(\mathcal{K}_2)$, …, $\mathbf{mat}(\mathcal{K}_n)$, which are the C-eigenvalues of $\mathcal{W}$.

**Example 7.** Let $\mathcal{W} \in \mathbb{R}^{6\times6\times3}$ be an orthogonal tensor with

$$\mathcal{W}^{(1)} = \begin{bmatrix} 0.1901 & -0.0946 & 0.2032 & 0.0951 & 0.0452 & -0.4262 \\ -0.0691 & 0.0195 & -0.2722 & 0.3694 & 0.0229 & -0.5662 \\ -0.3240 & -0.1153 & 0.5152 & 0.0422 & -0.5658 & 0.0094 \\ 0.2378 & -0.3855 & 0.2456 & 0.4556 & 0.1895 & 0.4274 \\ 0.3024 & 0.0190 & -0.2085 & -0.2563 & -0.4049 & 0.3437 \\ -0.1751 & -0.3235 & -0.2424 & -0.0563 & 0.0705 & -0.1519 \end{bmatrix},$$

$$\mathcal{W}^{(2)} = \begin{bmatrix} -0.4995 & -0.0791 & -0.5083 & 0.3609 & -0.2312 & 0.4383 \\ -0.1977 & -0.3028 & 0.2636 & -0.6572 & 0.0654 & 0.5792 \\ -0.0612 & 0.3539 & -0.6170 & -0.4071 & 0.4011 & -0.1120 \\ -0.5046 & 0.2850 & 0.1746 & -0.2857 & -0.4878 & -0.4129 \\ -0.4837 & 0.1765 & 0.2670 & 0.2843 & 0.5766 & 0.1052 \\ -0.1808 & 0.1464 & 0.3470 & 0.0962 & 0.3100 & -0.1314 \end{bmatrix},$$

$$\mathcal{W}^{(3)} = \begin{bmatrix} 0.1665 & 0.1256 & 0.1938 & -0.1036 & -0.0162 & -0.2930 \\ 0.0103 & 0.0826 & -0.1914 & 0.3241 & 0.1113 & -0.6042 \\ 0.0117 & -0.1712 & 0.4949 & 0.0969 & -0.4144 & 0.1046 \\ 0.2441 & -0.3227 & 0.0724 & 0.2303 & 0.3967 & 0.3009 \\ 0.1637 & -0.3281 & -0.3617 & -0.2799 & -0.2872 & 0.0956 \\ 0.0429 & 0.3570 & -0.1486 & 0.0118 & -0.0828 & 0.2421 \end{bmatrix}.$$

A simple computation gives

$$L(\mathcal{W})^{(1)} = \begin{bmatrix} -0.4758 & -0.0015 & -0.4257 & 0.6098 & -0.4496 & -0.1358 \\ -0.4440 & -0.4208 & -0.1279 & -0.2969 & 0.3761 & -0.6163 \\ -0.4229 & 0.2502 & 0.2709 & -0.5781 & -0.5923 & -0.0053 \\ -0.2832 & -0.4610 & 0.7395 & 0.3449 & 0.0073 & 0.2035 \\ -0.3375 & -0.2843 & -0.3978 & -0.2476 & 0.1740 & 0.7453 \\ -0.4510 & 0.6834 & 0.1544 & 0.1598 & 0.5247 & 0.0695 \end{bmatrix},$$

$$L(\mathcal{W})^{(2)} = \begin{bmatrix} -0.4759 & -0.2993 & -0.4989 & 0.5596 & -0.1698 & 0.3051 \\ -0.2771 & -0.3659 & 0.1828 & -0.6119 & -0.0230 & 0.6172 \\ -0.3969 & 0.4098 & -0.5967 & -0.4618 & 0.2497 & -0.2071 \\ -0.5109 & 0.2222 & 0.3478 & -0.0604 & -0.6950 & -0.2864 \\ -0.3450 & 0.5236 & 0.4202 & 0.3079 & 0.4589 & 0.3533 \\ -0.3989 & -0.5340 & 0.2532 & 0.0281 & 0.4633 & -0.5255 \end{bmatrix},$$

$$L(\mathcal{W})^{(3)}=\begin{bmatrix} 0.5231 & -0.1412 & 0.5177 & -0.1622 & 0.2926 & -0.5714\\ 0.1183 & 0.2396 & -0.3444 & 0.7025 & -0.1537 & -0.5411\\ -0.2745 & -0.2981 & 0.6373 & 0.3523 & -0.5525 & 0.0168\\ 0.4983 & -0.3477 & -0.0014 & 0.5110 & 0.2806 & 0.5394\\ 0.6224 & 0.1706 & -0.1138 & -0.2607 & -0.6944 & 0.1429\\ -0.0372 & -0.8269 & -0.4408 & -0.1643 & -0.1566 & -0.2626 \end{bmatrix}.$$

Then, the full CS decompositions of $L(\mathcal{W})^{(1)}$, $L(\mathcal{W})^{(2)}$ and $L(\mathcal{W})^{(3)}$ are

$$L\left(\mathcal{U}^H\star_c\mathcal{W}\star_c\mathcal{V}\right)^{(1)}=\begin{bmatrix} 0.8305 & 0 & 0 & 0.5571 & 0 & 0\\ 0 & 0.5428 & 0 & 0 & 0.8399 & 0\\ 0 & 0 & 0.3590 & 0 & 0 & 0.9333\\ -0.5571 & 0 & 0 & 0.8305 & 0 & 0\\ 0 & -0.8399 & 0 & 0 & 0.5428 & 0\\ 0 & 0 & -0.9333 & 0 & 0 & 0.3590 \end{bmatrix},$$

$$L\left(\mathcal{U}^H\star_c\mathcal{W}\star_c\mathcal{V}\right)^{(2)}=\begin{bmatrix} 0.9961 & 0 & 0 & 0.0881 & 0 & 0\\ 0 & 0.6783 & 0 & 0 & 0.7348 & 0\\ 0 & 0 & 0.1954 & 0 & 0 & 0.9807\\ -0.0881 & 0 & 0 & 0.9961 & 0 & 0\\ 0 & -0.7348 & 0 & 0 & 0.6783 & 0\\ 0 & 0 & -0.9807 & 0 & 0 & 0.1954 \end{bmatrix},$$

$$L\left(\mathcal{U}^H\star_c\mathcal{W}\star_c\mathcal{V}\right)^{(3)}=\begin{bmatrix} 0.9737 & 0 & 0 & 0.026 & 0 & 0\\ 0 & 0.6078 & 0 & 0 & 0.3153 & 0\\ 0 & 0 & 0.0663 & 0 & 0 & 0.4978\\ -0.026 & 0 & 0 & 0.9737 & 0 & 0\\ 0 & -0.3153 & 0 & 0 & 0.6078 & 0\\ 0 & 0 & -0.4978 & 0 & 0 & 0.0663 \end{bmatrix}.$$

Then,

$$\mathcal{K}_1^{(1)}=\begin{bmatrix} 0.7581 & 0.3422\\ -0.3422 & 0.7581 \end{bmatrix},\ \mathcal{K}_1^{(2)}=\begin{bmatrix} 0.0186 & 0.0624\\ -0.0624 & 0.0186 \end{bmatrix},\ \mathcal{K}_1^{(3)}=\begin{bmatrix} -0.2079 & 0.2291\\ -0.2291 & -0.2079 \end{bmatrix},$$

$$\mathcal{K}_2^{(1)}=\begin{bmatrix} 0.0431 & 0.5594\\ -0.5594 & 0.0431 \end{bmatrix},\ \mathcal{K}_2^{(2)}=\begin{bmatrix} -0.0951 & 0.3068\\ -0.3068 & -0.0951 \end{bmatrix},\ \mathcal{K}_2^{(3)}=\begin{bmatrix} -0.1318 & -0.1307\\ 0.1307 & -0.1318 \end{bmatrix},$$

$$\mathcal{K}_3^{(1)}=\begin{bmatrix} -0.4097 & 0.2674\\ -0.2674 & -0.4097 \end{bmatrix},\ \mathcal{K}_3^{(2)}=\begin{bmatrix} -0.3401 & 0.1586\\ -0.1586 & -0.3401 \end{bmatrix},\ \mathcal{K}_3^{(3)}=\begin{bmatrix} 0.1738 & 0.0427\\ -0.0427 & 0.1738 \end{bmatrix}.$$

By computing the eigenvalues of

$$\mathbf{mat}(\mathcal{K}_1)=\begin{bmatrix} \mathcal{K}_1^{(1)}+\mathcal{K}_1^{(2)} & \mathcal{K}_1^{(2)}+\mathcal{K}_1^{(3)} & \mathcal{K}_1^{(3)}\\ \mathcal{K}_1^{(2)}+\mathcal{K}_1^{(3)} & \mathcal{K}_1^{(1)} & \mathcal{K}_1^{(2)}+\mathcal{K}_1^{(3)}\\ \mathcal{K}_1^{(3)} & \mathcal{K}_1^{(2)}+\mathcal{K}_1^{(3)} & \mathcal{K}_1^{(1)}+\mathcal{K}_1^{(2)} \end{bmatrix},$$

$$\mathbf{mat}(\mathcal{K}_2) = \begin{bmatrix} \mathcal{K}_2^{(1)} + \mathcal{K}_2^{(2)} & \mathcal{K}_2^{(2)} + \mathcal{K}_2^{(3)} & \mathcal{K}_2^{(3)} \\ \mathcal{K}_2^{(2)} + \mathcal{K}_2^{(3)} & \mathcal{K}_2^{(1)} & \mathcal{K}_2^{(2)} + \mathcal{K}_2^{(3)} \\ \mathcal{K}_2^{(3)} & \mathcal{K}_2^{(2)} + \mathcal{K}_2^{(3)} & \mathcal{K}_2^{(1)} + \mathcal{K}_2^{(2)} \end{bmatrix},$$

$$\mathbf{mat}(\mathcal{K}_3) = \begin{bmatrix} \mathcal{K}_3^{(1)} + \mathcal{K}_3^{(2)} & \mathcal{K}_3^{(2)} + \mathcal{K}_3^{(3)} & \mathcal{K}_3^{(3)} \\ \mathcal{K}_3^{(2)} + \mathcal{K}_3^{(3)} & \mathcal{K}_3^{(1)} & \mathcal{K}_3^{(2)} + \mathcal{K}_3^{(3)} \\ \mathcal{K}_3^{(3)} & \mathcal{K}_3^{(2)} + \mathcal{K}_3^{(3)} & \mathcal{K}_3^{(1)} + \mathcal{K}_3^{(2)} \end{bmatrix}.$$

one can get the C-eigenvalues of the orthogonal tensor $\mathcal{W}$:

$$\lambda_{1,2} = 0.3795 \pm 0.9252i,\ \ \lambda_{3,4} = 0.9846 \pm 0.1755i,\ \ \lambda_{5,6} = 0.9474 \pm 0.0507i,$$

$$\lambda_{7,8} = 0.3336 \pm 0.9188i,\ \ \lambda_{9,10} = -0.4099 \pm 0.176i,\ \ \lambda_{11,12} = 0.129 \pm 0.2142i,$$

$$\lambda_{13,14} = -0.7423 \pm 0.67i,\ \ \lambda_{15,16} = -0.9236 \pm 0.3833i,\ \ \lambda_{17,18} = -0.2434 \pm 0.0661i.$$

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