Research Article

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Embedding of lattices and K3-covers of an Enriques surface

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Abstract: In this study, we establish necessary conditions for the embeddings of lattices and apply these conditions to the problem of characterizing algebraic $K3$ surfaces that cover an Enriques surface. By refining existing criteria and providing a more elementary approach, we offer a new perspective on the structure of such surfaces. Our results apply to any lattices, extending beyond specific cases and offering a comprehensive framework for understanding the embedding conditions in terms of Gram matrices.

Keywords: K3 surface, Enriques surface, lattice

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1 Introduction

In this work, we give the necessary conditions for the embeddings of lattices and present an application of the provided criterion for the problem of characterizing algebraic $K3$ surfaces covering an Enriques surface.

The main result of this article is the following theorem, whose proof is given in Section 3.

Theorem 1.1. *Let L and M be even integral lattices of* $\mathrm{rank}(L)$ *and* $\mathrm{rank}(M)$, *and let* $\mathrm{rank}_2(L)$ *and* $\mathrm{rank}_2(M)$ *denote their ranks over $\mathbb{Z}/2$. Let ϕ be an embedding of L into M. Then, one of the following conditions holds*:

(*I*) *If* $\mathrm{rank}_2(M) = 0$, *then there exists a lattice T such that $L \cong T(2)$.*

(*II*) *If* $\mathrm{rank}_2(M) > 0$ *and* $\mathrm{rank}_2(L) = 0$, *then*

$$\mathrm{rank}(L) \le \mathrm{rank}(M) - \frac{1}{2}\mathrm{rank}_2(M),$$

and there exists a lattice T such that $L \cong T(2)$.

(*III*) *If* $\mathrm{rank}_2(M) > 0$ *and* $\mathrm{rank}_2(L) > 0$, *then*

$$\mathrm{rank}(L) \le \mathrm{rank}(M) - \frac{1}{2}\mathrm{rank}_2(M) + \frac{1}{2}\mathrm{rank}_2(L),$$

and there exists an even lattice T such that $L \cong T$ and its associated Gram matrix must have the form

$$G_T = \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \dots & \dots & 2a_{\lambda\lambda} \end{pmatrix},$$

where $a_{2k-1,2k}$ is odd for each $1 \le k \le \frac{1}{2}\mathrm{rank}_2(L)$, and the remaining off-diagonal entries are even.

(*IV*) *If* rank(L) = rank(M), *$L \cong M$, provided that the embedding is primitive.*

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Recall that an *algebraic K3 surface* over $\mathbb{C}$ is a smooth projective surface X such that the canonical divisor K_X of X is trivial and $H^1(X, \mathcal{O}_X) = 0$ and an *Enriques surface* is a smooth projective surface Y such that $2K_Y$ is trivial and $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$, where K_Y is a canonical divisor of Y. The *Néron-Severi lattice* $NS(X)$ is a sublattice of the cohomology group $H^2(X, \mathbb{Z})$ of X that is a unimodular lattice of rank 22. The rank of $NS(X)$ is called the *Picard number* of X, denoted by $\rho(X)$. The orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$ is called the *transcendental lattice* T_X, which has signature $(2, 20 - \rho(X))$.

The following criterion was established by Keum, which was originally proven under an additional assumption that Ohashi subsequently showed to be unnecessary [1].

Theorem 1.2. (Keum's criterion) [2, Theorem 1] *A K3 surface X with transcendental lattice T_X covers an Enriques surface if and only if there exists a primitive embedding of T_X into $\mathbf{\Lambda}^- \coloneqq \mathbf{U} \oplus \mathbf{U}(2) \oplus \mathbf{E}_8(2)$ such that there exists no vector $v \in T_X^\perp$ with $v^2 = -2$.*

Using the criterion mentioned above, Keum proved that every algebraic Kummer surface is a K3 cover of some Enriques surface [2]. Sertöz [3] identified conditions on the entries of the Gram matrix of the transcendental lattice T_X under which X covers an Enriques surface when $\rho(X) = 20$. Subsequently, Lee [4] and Yörük [5] extended these results to cases where $\rho(X) = 18$ and $\rho(X) = 19$.

In his work [6, Prop. 1.15.1], Nikulin provides a criterion to enumerate all primitive embeddings of a fixed lattice T into lattices of signature (m_+, m_-) and discriminant form q for a given pair of nonnegative integers (m_+, m_-) and a finite quadratic form q. Brandhorst et al. [7], employed Nikulin's criterion to enumerate all primitive embeddings of T_X into $\mathbf{\Lambda}^-$, and applied Keum's criterion to characterize complex K3 surfaces that cover Enriques surfaces in terms of their Gram matrices.

In this article, we establish necessary conditions for the embeddings of a fixed lattice L into arbitrary lattices by analyzing their Gram matrices over $\mathbb{Z}/2$, considered up to the action of $\mathrm{GL}_n(\mathbb{Z}/2)$. By using the necessary conditions above and applying Keum's criterion, we characterize complex K3 surfaces that cover Enriques surfaces in terms of their Gram matrices.

By Keum's criterion, there are two reasons why a K3 surface X cannot cover any Enriques surface: firstly, there exists no primitive embedding of T_X into $\mathbf{\Lambda}^-$, second, for every primitive embedding of T_X into $\mathbf{\Lambda}^-$, there exists a vector v in the orthogonal complement of T_X in $\mathbf{\Lambda}^-$ with $v^2 = -2$. In the latter, the transcendental lattice T_X is called a *co-idoneal lattice* in [7].

By providing the necessary conditions for the embeddings of lattices, we obtain the following theorem, whose proof is given in Section 4.

Theorem 1.3. *Let X be a K3 surface with $10 \le \rho(X) \le 20$ whose transcendental lattice T_X is of rank λ and signature $(2, \lambda - 2)$. Then, X covers an Enriques surface if and only if one of the following conditions holds*:

(*I*) $11 \le \rho(X) \le 20$, *and there exists an even lattice T such that $T_X \cong T(2)$.*

(*II*) $11 \le \rho(X) \le 20$, *and there exists an odd lattice T such that $T_X \cong T(2)$ except when T_X is co-idoneal.*

(*III*) $11 \le \rho(X) \le 20$, *and there exists an even lattice T such that $T_X \cong T$ and its associated Gram matrix must be in the following form*:

$$G_T = \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \dots & \dots & 2a_{\lambda\lambda} \end{pmatrix},$$

such that a_{ij} is even for each $1 \le i, j \le \lambda$ except a_{11}, a_{12}.

(*IV*) $\rho(X) = 10$, *and $T_X \cong \mathbf{\Lambda}^-$.*

Furthermore, we will derive refined characterizations of the forms of exceptional lattices, referred to as co-idoneal lattices in Section 5.

2 Preliminaries

An *integral lattice* (L, β) is a finitely generated, free $\mathbb{Z}$-module equipped with an integral symmetric bilinear form β, $L \cong (\mathbb{Z}^n, \beta)$. The lattice L is *even* if $\beta(x,x) \equiv 0 \bmod 2$ for all $x \in L$, and is *odd* otherwise.

Let $B = \{e_1, \dots, e_n\}$ be a basis of $\mathbb{Z}$-module L. The symmetric matrix $G = (a_{ij})$, given by $a_{ij} = \beta(e_i, e_j)$, is the *Gram matrix* of L with respect to this basis B, written as $L \cong G_L$. The *discriminant* of a lattice L is the determinant of G_L.

Let L and M be two $\mathbb{Z}$-modules, β and β' be bilinear forms on L and M, respectively. A module homomorphism $\phi : L \to M$ is called a *lattice morphism* if it satisfies the isometry relation, i.e., $\beta(x,y) = \beta'(\phi(x), \phi(y))$ for all $x, y \in L$. Furthermore, a lattice morphism $\phi : L \hookrightarrow M$ is called a *lattice embedding* if ϕ is a monomorphism. The embedding is called *primitive* if $M/\phi(L)$ is a free module, i.e., it has a basis. $L \cong M$ if ϕ is also an epimorphism.

Two matrices T_1 and T_2 are said to be *$\mathbb{Z}$-equivalent* if there exists an element $g \in GL(n, \mathbb{Z})$ together with its transpose g^τ such that

$$T_2 = g^\tau T_1 g.$$

The negative definite exceptional lattice is denoted $\mathbf{E}_n$, and the hyperbolic lattice is denoted $\mathbf{U}$.

For any integer n, we denote by $[n]$, the lattice $\mathbb{Z}e$ of rank 1 with $\beta(e,e) = n$. For any integer n, and $L \cong G = (a_{ij})$, the lattice $L(n)$ has $G_{L(n)} = (na_{ij})$. By abuse of language, a symmetric bilinear form β will be denoted by dot product.

The following theorem in [3] characterizes the primitive embeddings of lattices.

Theorem 2.1. [3] *A lattice embedding is primitive if and only if the greatest common divisor of the maximal minors of the embedding matrix with respect to any choice of basis is 1.*

Under certain assumptions on the discriminant of a lattice, an odd indefinite lattice can be decomposed into one of the following forms:

Theorem 2.2. [8, Theorem 1] *Let L be an odd indefinite $\mathbb{Z}$-lattice of rank $l \geq 3$ with square free discriminant d. Then,*

$$L \cong m[1] \oplus n[-1] \oplus B, \tag{2.1}$$

where B is a lattice of rank 2, and $m, n \geq 0$. Moreover, if d is even, then B can be chosen to be definite or indefinite.

Theorem 2.3. [9, Theorem 6] *Let L be an odd indefinite $\mathbb{Z}$-lattice of rank $l \geq 3$ and cube-free discriminant $d \not\equiv 0 \pmod 4$. Then, L has an orthogonal splitting*

$$L \cong m[1] \oplus n[-1] \oplus B, \tag{2.2}$$

where B is an indefinite odd lattice of rank 3, and $m, n \geq 0$.

Lemma 2.4. *The map $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2\mathbb{Z})$ is surjective.*

Proof. Indeed, $\mathrm{GL}_n(\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{SL}_n(\mathbb{Z}/2\mathbb{Z})$ is generated by transvections (elementary), and these obviously lift to $\mathrm{GL}_n(\mathbb{Z})$. □

3 Necessary conditions of embeddings of lattices

In this section, we will provide the necessary conditions for the embedding of lattices.

Theorem 3.1. *Let F be a finite field of characteristic 2. Then, every symmetric matrix M_n over F with zero diagonal is congruent to $\oplus_{i=1}^{q} H \oplus \oplus_{i=1}^{n-2q}[0]$ or $\oplus_{i=1}^{n}[0]$, where $q \in \mathbb{N}^+$ and*

$$H \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Furthermore, two symmetric matrices with zero diagonal over the field F are congruent if and only if they have the same rank.

Proof. Suppose that $M = 0$, the result is trivial.

Suppose that $M \neq 0$, so that there exists some non-zero element a_{ij} of M. Since $\mathrm{GL}_n(\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{SL}_n(\mathbb{Z}/2\mathbb{Z})$ is generated by the elementary matrices $\mathrm{E}_{ij}(\alpha)$, where $\alpha \in \mathbb{Z}/2$, performing the elementary congruent transformations, we may write

$$M \cong \begin{bmatrix} H & B^t \\ B & A \end{bmatrix}, \quad \text{where } H \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Performing successive elementary congruent transformations of $\mathrm{E}_{ij}(\alpha)$, i varying from 3 to n, we obtain a symmetric matrix such that

$$M \cong \begin{bmatrix} H & 0 \\ 0 & A_1 \end{bmatrix}, \quad \text{where } H \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

A_1 is an $(n-2)$-rowed symmetric matrix with zero diagonal. Therefore, we may proceed recursively and obtain a symmetric matrix of the form $\oplus_{i=1}^{q} H \oplus \oplus_{i=1}^{n-2q}[0]$ congruent to the given matrix M.

Because of the fact that the congruence of matrices is an equivalence relation possessing symmetry and transitivity, and by the above reasoning, two symmetric matrices of M_n with zero diagonal over the field F are congruent if and only if they have the same rank. This completes the proof. □

As a direct consequence, we obtain the following.

Corollary 3.2. *The number $o(n)$ of orbits of the even symmetric matrix M_n of size $n \times n$ over $\mathbb{Z}/2\mathbb{Z}$ with a zero diagonal under the action of $GL_n(\mathbb{Z}/2\mathbb{Z})$ by the transposition is given by*

$$o(n) = \begin{cases} \frac{1}{2}(n+1), & \text{if } n \text{ is odd}, \\ \frac{1}{2}(n+2), & \text{if } n \text{ is even}. \end{cases}$$

The map $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2\mathbb{Z})$ is surjective by Lemma 2.4; therefore, we can consider the action of $GL_n(\mathbb{Z}/2\mathbb{Z})$ by the transposition on the set of the even symmetric matrices M_n of size $n \times n$ over $\mathbb{Z}/2\mathbb{Z}$.

We will characterize associated Gram matrices of lattices with respect to their ranks over $\mathbb{Z}/2$ by the following.

Definition 3.3. Let L be an integral even lattice of rank λ and G_L be its associated Gram matrix. Let L' be an induced $\mathbb{Z}/2$-module of L by restriction of ring of integers $\mathbb{Z}$ to $\mathbb{Z}/2$ and their associated Gram matrices $G_L = (a_{ij})$ of L and $G_{L'} = (a_{ij}(\bmod 2))$ of L'. The rank of G_L of L over the field of characteristic 2 will be the rank of $G_{L'}$ of L', denoted by $\mathrm{rank}_2(L)$.

Lemma 3.4. *Let L be an integral even lattice of rank λ and G_L be its associated Gram matrix, with* $\operatorname{rank}_2(L) = 2q$. *Then,*

$$G_L \cong \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \dots & \dots & 2a_{\lambda\lambda} \end{pmatrix},$$

such that $a_{2k-1,2k}$ is odd for each $1 \le k \le q$, and the remaining off-diagonal entries are even.

Proof. Let L be an even lattice. The Gram matrix G_L of L is symmetric, and its diagonal entries are even.

Reduction of G_L modulo 2 yields a symmetric matrix over $\mathbb{Z}/2\mathbb{Z}$. The diagonal entries, being even, reduce to zero, so the reduced matrix has zero diagonal entries, and the off-diagonal entries are either 0 or 1.

By Lemma 2.4, the map $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2\mathbb{Z})$ is surjective. Therefore, we can use the action of $GL_n(\mathbb{Z}/2\mathbb{Z})$ to bring G_L mod 2 into a canonical form.

Over $\mathbb{Z}/2\mathbb{Z}$, by Theorem 3.1, every symmetric matrix with zero diagonal can be reduced, via congruence, to a block diagonal form consisting of 2×2 blocks of the type

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and possibly zero blocks.

The number of such blocks corresponds to the rank of the matrix modulo 2. Let $\operatorname{rank}_2(L) = 2q$, which means G_L mod 2 can be transformed into a block diagonal matrix with q blocks of the form H, while the remaining part of the matrix consists of zero blocks.

Once we have the canonical form of G_L mod 2, we lift this back to an integral matrix over $\mathbb{Z}$. For each block H in the reduced matrix, the corresponding entries in the lifted matrix G_L will be odd. That is, the off-diagonal terms $a_{2k-1,2k}$ are odd for each $k = 1, 2, \dots, q$. The remaining off-diagonal terms, which were zero modulo 2, will be even in G_L. This completes the proof. □

Proof of Theorem 1.1. Let L have a Gram matrix G_L and let M have a Gram matrix G_M. The induced $\mathbb{Z}/2$-modules are given by

$$L' \cong \bigoplus_{i=1}^{l} (\mathbb{Z}/2)x_i \quad \text{and} \quad M' \cong \bigoplus_{i=1}^{m} (\mathbb{Z}/2)u_i,$$

where $\{x_i\}_i$ and $\{u_i\}_i$ are basis for L and M of ranks l and m, respectively.

By Theorem 3.1 and Corollary 3.2, we know that

$$G_{L'} \cong \bigoplus_{i=1}^{p} H \oplus \bigoplus_{i=1}^{l-2p} [0] \quad \text{or} \quad \bigoplus_{i=1}^{l} [0],$$

where $\operatorname{rank}_2(L) = 2p$. Similarly,

$$G_{M'} \cong \bigoplus_{i=1}^{q} H \oplus \bigoplus_{i=1}^{m-2q} [0] \quad \text{or} \quad \bigoplus_{i=1}^{m} [0],$$

where $\operatorname{rank}_2(M) = 2q$. They are uniquely determined by their ranks over $\mathbb{Z}$ and $\mathbb{Z}/2$ by Theorem 3.1 and Corollary 3.2.

Consider the embedding $\phi : L \to M$ and the induced embedding $\phi' : L' \to M'$. Since ϕ and ϕ' are embeddings, the necessary conditions for the embedding of L' into M' are that

$$\operatorname{rank}(L) \le \operatorname{rank}(M), \quad \operatorname{rank}_2(L) \le \operatorname{rank}_2(M).$$

Therefore, each embedding ϕ of L into M must satisfy

$$\operatorname{rank}(L) \le \operatorname{rank}(M), \quad \operatorname{rank}_2(L) \le \operatorname{rank}_2(M).$$

Now we can analyze each case.

Case I: $\operatorname{rank}_2(M) = 0$.

Since $\operatorname{rank}_2(M) = 0$, $G_{M'} \cong \oplus_{i=1}^{m}[0]$. It implies that

$$G_{L'} \cong \bigoplus_{i=1}^{l}[0],$$

where $l \leq m$.

By Lemma 3.4, there exists a lattice T such that $L \cong T(2)$. Thus, we have:

$$L \cong T(2)$$

for some lattice T. This completes the proof for Case I.

Case II: $\operatorname{rank}_2(M) > 0$ **and** $\operatorname{rank}_2(L) = 0$.

Since $\operatorname{rank}_2(L) = 0$, we conclude that $G_{L'} \cong \oplus_{i=1}^{l}[0]$. Suppose that

$$\operatorname{rank}(L) > \operatorname{rank}(M) - \frac{1}{2}\operatorname{rank}_2(M).$$

By Theorem 3.1, Corollary 3.2, we have

$$G_{L'} \cong \bigoplus_{i=1}^{t} H \oplus \bigoplus_{i=1}^{l-2t}[0] \tag{3.1}$$

for some $t \in \mathbb{N}^+$. This contradicts the condition that $\operatorname{rank}_2(L) = 0$. Therefore,

$$\operatorname{rank}(L) \leq \operatorname{rank}(M) - \frac{1}{2}\operatorname{rank}_2(M).$$

The existence of a lattice T such that $L \cong T(2)$ follows from Lemma 3.4. This completes the proof for Case II.

Case III: $\operatorname{rank}_2(M) > 0$ **and** $\operatorname{rank}_2(L) > 0$.

Suppose that

$$\operatorname{rank}(L) > \frac{1}{2}\operatorname{rank}_2(L) + \operatorname{rank}(M) - \frac{1}{2}\operatorname{rank}_2(M).$$

By Theorem 3.1 and Corollary 3.2, we have

$$G_{L'} \cong \bigoplus_{i=1}^{t} H \oplus \bigoplus_{i=1}^{l-2t}[0] \tag{3.2}$$

for some $t \in \mathbb{N}^+$ such that $t > \operatorname{rank}_2(L)$. This leads to a contradiction with the uniqueness of its canonical form. Hence, we obtain the following rank condition:

$$\operatorname{rank}(L) \leq \operatorname{rank}(M) - \frac{1}{2}\operatorname{rank}_2(M) + \frac{1}{2}\operatorname{rank}_2(L).$$

By Lemma 3.4, we can express the Gram matrix G_L of L in the specified form:

$$G_L \cong \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \dots & \dots & 2a_{\lambda\lambda} \end{pmatrix},$$

where $a_{2k-1,2k}$ is odd for each $1 \leq k \leq \frac{1}{2}\operatorname{rank}_2(L)$, and the remaining off-diagonal entries are even.

This completes the proof for Case III.

Case IV: $\operatorname{rank}(L) = \operatorname{rank}(M)$.

If rank(L) = rank(M), then $L \cong M$. The claim trivially follows by the definition of a primitive embedding.

By addressing each case, we have proved that for any embedding ϕ of L into M, one of the conditions stated in the theorem must hold. Therefore, the proof is complete. □

4 Proof of Theorem 1.3

As a consequence of the foregoing theorem, we derive the following:

Theorem 4.1. *Let T_X be transcendental lattice of signature* $(2, \lambda - 2)$. *If there exists an embedding of T_X into $\mathbf{\Lambda}^-$, then* $\mathrm{rank}_2(T_X) = 0$ *or* 2. *Particularly, the associated Gram matrix of each embedding of T_X into $\mathbf{\Lambda}^-$ must be one of the following types*:

(1) $T_X \cong T(2)$, *where T is an even lattice,*
(2) $T_X \cong T(2)$, *where T is an odd lattice,*
(3) $\mathrm{rank}_2(T_X) = 2$,

$$G_{T_X} \cong \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \dots & \dots & 2a_{\lambda\lambda} \end{pmatrix},$$

such that a_{ij} is even for each $1 \le i, j \le \lambda$ *except* a_{11}, a_{12}.

Proof. Let $\{x_i\}_i$ be a basis of the transcendental lattice T_X and let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be the standard basis of $\mathbf{U}$ and $\mathbf{U}(2)$, respectively.

If $\phi : T_X \hookrightarrow \mathbf{\Lambda}^-$ is an embedding defined generically by

$$\phi(x_i) = a'_{i1}u_1 + a'_{i2}u_2 + a'_{i3}v_1 + a'_{i4}v_2 + w_i, \tag{4.1}$$

where a'_{ij} are integers and $w_i \in \mathbf{E}_8(2)$ for $1 \le i \le \lambda$ and $1 \le j \le 4$ then, we have that

$$\phi(x_i) \cdot \phi(x_i) = 2a'_{i1}a'_{i2} + 4a'_{i3}a'_{i4} + w_1^2 = 2a_{ii}, \tag{4.2}$$

for $1 \le i \le \lambda$ and

$$\phi(x_i) \cdot \phi(x_k) = a'_{i1}a'_{k2} + a'_{i2}a'_{k1} + 2a'_{i3}a'_{k4} + 2a'_{i4}a'_{k4} + w_i w_k = a_{ik} \tag{4.3}$$

for $1 \le i < k \le \lambda$.

By Theorem 1.1, equations (4.2) and (4.3) are solvable over $\mathbb{Z}/2$ if and only if $\mathrm{rank}_2(T_X) \le \mathrm{rank}_2(\mathbf{\Lambda}^-)$. Since $\mathrm{rank}_2(\mathbf{\Lambda}^-) = 2$, $\mathrm{rank}_2(T_X)$ is either 0 or 2.

Case I: $\mathrm{rank}_2(T_X) = 0$.

Since $\mathrm{rank}_2(T_X) = 0$, $T'_X \cong \oplus_{i=1}^{\lambda}[0]$, $\lambda \le 11$ using Theorem 1.1. By Lemma 3.4, there are two types of associated Gram matrices of T_X rising after lifting up to $\mathbb{Z}$:

- $T_X \cong T(2)$, where T is an even lattice,
- $T_X \cong T(2)$, where T is an odd lattice.

Case II: $\mathrm{rank}_2(T_X) = 2$

If $\mathrm{rank}_2(T_X) = 2$, by Theorem 1.1 and Lemma 3.4,

$$G_{T_X} \cong \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \dots & \dots & 2a_{\lambda\lambda} \end{pmatrix},$$

such that a_{12} is odd and the remaining off-diagonal entries are even.

To determine the parities of diagonal entries of T_X, we need to consider the equations (4.2) and (4.3). If both a_{11} and a_{22} are odd, it contradicts with equations (4.2) and (4.3). Suppose both a_{11} and a_{22} are even. Then, under the action of element $g \in \mathrm{GL}(\lambda, \mathbb{Z})$ where all diagonal entries are 1, the (2,1)-entry is 1, and all other off-diagonal entries are 0, both a'_{11} and a'_{12} will be odd, the parities of remaining entries of T_X will be invariant. Hence, without loss of generality, assume that a_{11} is odd and a_{22} is even. Since a_{11} is odd, it enforces that a'_{11} and a'_{12} are odd by equation (4.2). a_{12} is odd, it also enforces that a'_{21} and a'_{22} have a different parity by equation (4.3). Thus, by equations (4.2) and (4.3), both a'_{i1} and a'_{i2} are even for $3 \le i \le \lambda$, it implies that $a_{ii} \in 2\mathbb{Z}$ for $3 \le i \le \lambda$. This completes the proof. □

We will now present and prove the subsequent theorems that broaden the criteria for K3-covers an Enriques surface, showing that for each embedding of T_X into $\mathbf{\Lambda}^-$ or an induced embedding T_X into $\mathbf{\Lambda}^-$, there exists no $v \in T_X^\perp$ with $v^2 = -2$, as opposed to relying solely on primitive embeddings. By Lemma 3.4 and Theorem 1.1, the following result can be generalized to any embedding of a lattice L into another lattice M, provided that $\operatorname{rank}_2(L) = \operatorname{rank}_2(M)$. However, we will limit our focus to the specific case of embedding T_X into $\mathbf{\Lambda}^-$.

Lemma 4.2. *Let T_X be an even lattice of signature $(2, \lambda - 2)$, $\operatorname{rank}_2(T_X) = 2$. Then, for each embedding of T_X into $\mathbf{\Lambda}^-$, there exists no $v \in T_X^\perp$ with $v^2 = -2$.*

Proof. Let $\{x_i\}_i$ be a basis of the transcendental lattice T_X, and let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be the standard basis of $\mathbf{U}$ and $\mathbf{U}(2)$, respectively. Consider the embedding $\phi : T_X \hookrightarrow \mathbf{\Lambda}^-$ defined generically by

$$\phi(x_i) = a'_{i1}u_1 + a'_{i2}u_2 + a'_{i3}v_1 + a'_{i4}v_2 + w_i, \tag{4.4}$$

where a'_{ij} are integers and $w_i \in \mathbf{E}_8(2)$ for $1 \le i \le \lambda$ and $1 \le j \le 4$.

By Theorem 4.1, a_{11} is odd, it enforces that a'_{11} and a'_{12} are odd. Similarly, a_{12} is odd, it also enforces that a'_{21} and a'_{22} have a different parity.

To prove the orthogonal complement of the image of ϕ in $\mathbf{\Lambda}^-$ contains no self-intersection -2 vector, let $f = Xu_1 + x'u_2 + Yv_1 + y'v_2 + e \in \mathbf{\Lambda}^-$, where $e \in E_8(2)$ with $e \cdot e = -4k,\ k \ge 0$. From the equation,

$$f \cdot \phi(x_2) = (Xu_1 + x'u_2 + Yv_1 + y'v_2 + e)\cdot(a'_{21}u_1 + a'_{22}u_2 + a'_{23}v_1 + a'_{24}v_2 + w_i) = 0,$$

we obtain that

$$x'a'_{21} + Xa'_{22}x \equiv 0 \bmod 2. \tag{4.5}$$

Since a'_{21} and a'_{22} have a different parity, x' or X must be even. We obtain

$$f \cdot f = 2Xx' + 4Yy' + e \cdot e \equiv 0 \bmod 4. \tag{4.6}$$

Therefore, the orthogonal complement of the image of ϕ in $\mathbf{\Lambda}^-$ contains no self-intersection -2 vector. □

Lemma 4.3. *Let T_X be an even lattice of signature $(2, \lambda - 2)$ and $T_X \cong T(2)$, where T is an even lattice. Then, for each embedding of T_X into $\mathbf{\Lambda}^-$, there exists an induced embedding such that there exists no $v \in T_X^\perp$ with $v^2 = -2$.*

Proof. Let $\{x_i\}_i$ be a basis of the transcendental lattice T_X, and let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be the standard basis of $\mathbf{U}$ and $\mathbf{U}(2)$, respectively. Consider the embedding $\phi : T_X \hookrightarrow \mathbf{\Lambda}^-$ defined generically by

$$\phi(x_i) = a'_{i1}u_1 + a'_{i2}u_2 + a'_{i3}v_1 + a'_{i4}v_2 + w_i, \tag{4.7}$$

where a'_{ij} are integers and $w_i \in \mathbf{E}_8(2)$ for $1 \le i \le \lambda$ and $1 \le j \le 4$.

Suppose $T_X \cong T(2)$, where T is an even lattice, a_{ij} is even for $1 \le i, j \le \lambda \le 11$ by Theorem 4.1.

Since T_X is an even lattice of signature $(2, \lambda - 2)$, there exists a'_{ij} such that $a'_{ij} \ne 0$ for $1 \le i \le \lambda$ and $1 \le j \le 2$.

Suppose that a'_{i1} and a'_{i2} have different parity for $1 \le i \le \lambda$, then by the same reasoning as in Lemma 4.2, the orthogonal complement of the image of ϕ in $\mathbf{\Lambda}^-$ contains no self-intersection -2 vector.

Suppose that a'_{i1} and a'_{i2} are of the form $a'_{ij} = 2^{k_{ij}}.\ m_{ij}$, where $k_{ij} \in \mathbb{Z}^+$, $m_{ij} \notin 2\mathbb{Z}$ for all $a'_{ij} \ne 0$, $1 \le i \le \lambda$, $1 \le j \le 2$. Let k be a minimum among the all $\upsilon_2(a'_{i1})$ the exponent of the largest power of 2 of $a'_{i1} \ne 0$. Without loss of generality, let $k = k_{11}$. If we insert $a''_{i1} = 2^{k_{i1}-k}.\ m_{i1}$ and $a''_{i2} = 2^{k_{i2}+k}.\ m_{i2}$ in the place of $a'_{i1} \ne 0$, $a'_{i2} \ne 0$, respectively; equations (4.2) and (4.3) are satisfied for the new embedding ϕ' induced by the embedding ϕ for i, j, $1 \le i \le \lambda$, $1 \le j \le 2$. Particularly, if ϕ is primitive, then the induced embedding ϕ' is also primitive by Theorem 2.1.

Since a''_{11} and a''_{12} have different parity, again by the same reasoning as in Theorem 4.2, the orthogonal complement of the image of ϕ in $\mathbf{\Lambda}^-$ contains no self-intersection -2 vector. □

Proof of Theorem 1.3. Let X be a K3 surface of $11 \le \rho(X) \le 20$ with its transcendental lattice T_X of rank λ and signature $(2, \lambda - 2)$. Each primitive embedding of T_X into Λ^- is also an embedding, by Theorem 4.1, the associated Gram matrix of T_X of each embedding of T_X into Λ^- must be one of the following types:

- $T_X \cong T(2)$, where T is an even lattice,
- $T_X \cong T(2)$, where T is an odd lattice,
- $\mathrm{rank}_2(T_X) = 2$,

$$G_{T_X} \cong \begin{pmatrix} 2a_{11} & a_{12} & \dots & a_{1\lambda} \\ a_{12} & 2a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1\lambda} & \dots & \dots & 2a_{\lambda\lambda} \end{pmatrix},$$

such that a_{ij} is even for each $1 \le i, j \le \lambda$ except a_{11}, a_{12}.

If $T_X \cong T(2)$, where T is an even lattice, by Lemma 4.3, for each embedding of T_X into Λ^-, there exists an induced embedding such that there exists no $v \in T_X^\perp$ with $v^2 = -2$. By Theorem 1.2, the claim follows.

If $T_X \cong T(2)$, where T is an odd lattice, and T_X is not a co-idoneal lattice, by the definition of the co-idoneal lattice and Theorem 1.2, the claim follows.

If $\mathrm{rank}_2(T_X) = 2$, by Lemma 4.2, for each embedding of T_X into Λ^-, there exists no $v \in T_X^\perp$ with $v^2 = -2$. Hence, by Theorem 1.2, the claim follows.

Finally, if $\rho(X) = 10$, then $\mathrm{rank}(T_X) = \mathrm{rank}(\Lambda^-)$, $T_X \cong \Lambda^-$, so the claim is trivial. □

5 Exceptional lattices

Recall that a transcendental lattice T_X is called a co-idoneal lattice if, for every primitive embedding of T_X into Λ^-, there exists a vector v in the orthogonal complement of T_X in Λ^- with $v^2 = -2$.

The following theorem, obtained as Corollary 3.13 in [7], will be proven using elementary techniques.

Theorem 5.1. *If T_X is a co-idoneal lattice, $T_X \cong T(2)$, where T is an odd lattice.*

Proof. The proof of this theorem is a direct consequence of Theorem 4.1 and Lemmas 4.2 and 4.3. □

For the transcendental lattices T_X of rank $\lambda \ge 3$, we prove the following theorems:

Theorem 5.2. *If T_X is a co-idoneal lattice such that $T_X \cong T(2)$, where T has a square free discriminant d, then T_X must be the following forms: $m[2] \oplus n[-2] \oplus B(2)$ for some $m, n \in \mathbb{N}$, where B is a lattice of rank 2, $0 \le m \le 2$. Moreover, if d is even, T_X must be the one of the following forms: $2[2] \oplus n[-2] \oplus B(2)$ for some n, where B is a negative definite lattice of rank 2, or $n[-2] \oplus B(2)$, where B is a positive definite lattice of rank 2.*

Proof. By Theorem 5.1, since T_X is a co-idoneal lattice, $T_X \cong T(2)$, where T is an odd lattice. By Theorem 2.2, all odd lattices with a square-free discriminant d can be diagonalized in that form. Hence, the result follows. □

Theorem 5.3. *If T_X is a co-idoneal lattice such that $T_X \cong T(2)$, where T has a cube free discriminant d, then T_X must be the following forms: $m[2] \oplus n[-2] \oplus B(2)$ for some $m, n \in \mathbb{N}$, where B is a lattice of rank 3, and $0 \le m \le 2$.*

Proof. By Theorem 5.1, since T_X is a co-idoneal lattice, $T_X \cong T(2)$, where T is an odd lattice. By Theorem 2.3, all odd lattices with a cube-free discriminant d can be diagonalized in that form. Hence, the result follows. □

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