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Research Article

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Green's graphs of a semigroup

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Abstract: Let S be a semigroup. In this study, we first introduce the Green's digraphs and Green's graphs related to the Green's relations $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{J}$ of S. Further, the connectedness and completeness of the Green's graphs are discussed. For a finite semigroup S, we show that each of the Green's graphs of S has a transitive orientation. Moreover, we obtain that these Green's graphs are perfect. Finally, the structures of the Green's graphs are characterized using the generalized lexicographic product.

Keywords: semigroup, Green's relation, Green's graph, complete graph, connected graph

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1 Introduction and preliminaries

Graphs related to groups and semigroups have been actively investigated in the literature [1–6]. Given a semigroup, there are many different ways to associate a directed or undirected graph with the semigroup, including the zero-divisor graphs [4], divisibility graphs [2], power graphs [2,5], and Cayley graphs [1,3], etc. Let S be a semigroup, and let T be a subset of S. The *Cayley graph* $Cay(S, T)$ of S relative to T is defined as the directed graph with vertex set S and arc set consisting of those ordered pairs (x, y) where $tx = y$ and $x \neq y$ for some $t \in T$. Kelarev and Praeger [3] characterized all vertex-transitive Cayley graphs arising from periodic semigroups. Kelarev [1] described all finite inverse semigroups and all commutative inverse semigroups with bipartite Cayley graphs. The Cayley graph of a semigroup is related to finite state automata and has many applications [7–16].

The directed power graph of a semigroup S was defined by Kelarev and Quinn in [2] as the directed graph $\vec{\mathscr{P}}(S)$ with vertex set S in which there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m. Motivated by this, Chakrabarty et al. [5] focused their study on the undirected power graphs of a semigroup S, in which distinct x and y are adjacent if one is a power of the other. They characterized the class of semigroups S for which the power graph $\mathscr{P}(S)$ is connected or complete. Based on these, Cameron and Ghosh [17,18] explored the power graphs of finite groups, obtained many profound results, and promoted the research of related problems. Nowadays, power graphs of groups and semigroups are actively investigated by researchers [6,19–23]. A detailed list of results and open problems can be found in [24,25].

Dalal et al. [21], as a generalization of power graphs of semigroups, introduced a new graph: the enhanced power graph of a semigroup S, denoted by $\mathscr{P}_e(S)$, as the graph whose vertex set is S and in which two distinct vertices x, y are adjacent if $x, y \in \langle z \rangle$ for some $z \in S$. They described the structure of $\mathscr{P}_e(S)$ and discussed some of its graph-theoretic properties. Ma et al. [26], gave a recent survey on the enhanced power graphs of groups. An interesting notion of an E-extended power graph of a finite semigroup is studied recently in [27].

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It is known that Green's relations play a fundamental role in the study of semigroups. Let S be a semigroup. *Green's relations* on S are defined as follows. The relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{J}$ on S are given by

$$a\mathscr{L}b \Leftrightarrow S^1a = S^1b, \quad a\mathscr{R}b \Leftrightarrow aS^1 = bS^1, \quad a\mathscr{J}b \Leftrightarrow S^1aS^1 = S^1bS^1,$$

where S^1 is the monoid obtained from S by adjoining an identity if necessary. Let $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$ and $\mathscr{D} = \mathscr{L} \vee \mathscr{R}$, where $\mathscr{L} \vee \mathscr{R}$ denote the minimum equivalence relation containing $\mathscr{L}$ and $\mathscr{R}$. Clearly, for all $a, b \in S$,

$$a\mathscr{L}b \Leftrightarrow a = xb \quad \text{and} \quad b = ya \quad \text{for some } x, y \in S^1.$$

The relations $\mathscr{R}$ and $\mathscr{J}$ can be characterized similarly. Let L_a, R_a, and J_a denote the $\mathscr{L}$-class, $\mathscr{R}$-class, and $\mathscr{J}$-class containing a, respectively.

It is worth noting that Green's relations also play an important role in the study of the graph theory related to semigroups. In 2014, Gharibkhajeh and Doostie [28] introduced the Green graphs of a finite semigroup S by generalizing the notion of conjugacy graphs of groups. The left Green graph of S is an undirected graph whose vertices are the $\mathscr{L}$-classes of S, and two vertices L_i and L_j are adjacent if and only if $\gcd(|L_i|, |L_j|) > 1$. The other Green graphs are defined in a similar way. They gave a necessary condition for the Green graphs related to $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$, and $\mathscr{H}$ of S to coincide. Moreover, Sorouhesh et al. [29] obtained a sufficient condition on non-group semigroups that implies the coinciding of these Green graphs. In 2023, Nupo and Chaiya [15] investigated the Cayley digraphs of full transformation semigroups with respect to Green's equivalence classes, and presented structural properties and isomorphism theorems of these digraphs. Recently, Ashegh Bonabi and Khosravi [23] gave a characterization of a completely simple semigroup in terms of its power graph and Green's relations. Cheng et al. [22] explored the power graphs of certain completely 0-simple semigroups, and they showed that a G^0-normal completely 0-simple orthodox semigroup with abelian group $\mathscr{H}$-classes is characterized by its power graph.

Recall that the power graph $\mathscr{P}(S)$ of a semigroup S is an undirected graph whose vertex set is S such that two vertices $a, b \in S$ are adjacent if and only if $a \neq b$ and $a = b^m$ or $b = a^m$ for some positive integer m. This means that $S^1a \subseteq S^1b$ or $S^1b \subseteq S^1a$ (resp. $aS^1 \subseteq bS^1$ or $bS^1 \subseteq aS^1$, $S^1aS^1 \subseteq S^1bS^1$ or $S^1bS^1 \subseteq S^1aS^1$). That is, $a = xb$ or $b = ya$ (resp. $a = bx$ or $b = ay$, $a = xby$ or $b = xay$) for some $x, y \in S^1$. As a generalization of power graphs of semigroups, we introduce new types of graphs on semigroups called Green's digraphs and Green's graphs.

As usual, a *graph* means an undirected simple graph, and a *digraph* means a directed graph without loops. Given a graph (resp. digraph) Γ, we always use $V(\Gamma)$ and $E(\Gamma)$ to denote the *vertex set* and the *edge set* (resp. the *arc set*), respectively. The digraph O is an *orientation* for Γ if $V(O) = V(\Gamma)$ and $|\{(u, v), (v, u)\} \cap E(O)| = 1$ for all $\{u, v\} \in E(\Gamma)$. A *transitive orientation* for Γ is an orientation O such that $\{(u, v), (v, w)\} \subseteq E(O)$ implies $(u, w) \in E(O)$. A *comparability graph* is a graph that admits a transitive orientation. It has been characterized in [30,31].

The study is structured as follows. In Section 2, we first give the definitions of Green's digraphs and Green's graphs of a semigroup. We then characterize the connected components of the Green's graphs and the class of semigroups for which Green's graphs are complete. For a finite semigroup S, we construct transitive orientations for the Green's graphs of S. Moreover, we prove that the Green's graphs are perfect graphs. In Section 3, we use the generalized lexicographic product to characterize the structures of the Green's graphs of a finite semigroup.

For other notations and terminologies not given in this article, the reader is referred to the books [32] and [33].

2 Green's graphs of a semigroup

The aim of this section is to give the definitions of Green's digraphs and Green's graphs of a semigroup related to Green's relations $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{J}$. Further, we shall discuss some graph-theoretic properties of the Green's graphs.

Definition 2.1. Let S be a semigroup. The Green's $\mathscr{L}$-digraph (resp. $\mathscr{R}$-digraph, $\mathscr{J}$-digraph) of S, denoted by $D_{\mathscr{L}}(S)$ (resp. $D_{\mathscr{R}}(S)$, $D_{\mathscr{J}}(S)$), is a directed graph with vertex set S such that there is an arc from a to b if and only if $a \neq b$ and $b \in Sa$ (resp. $b \in aS$, $b \in S^1aS^1$).

It is easy to see that $D_{\mathscr{L}}(S)$ and $D_{\mathscr{R}}(S)$ are two distinct subdigraphs of $D_{\mathscr{J}}(S)$, and the directed power graph $\vec{\mathscr{P}}(S)$ of S is a subdigraph of each of the Green's digraphs of S. Obviously, the Green's $\mathscr{L}$-digraph $D_{\mathscr{L}}(S)$ and the Cayley graph $Cay(S, S)$ are the same, and the divisibility graph $Div(S)$ and Green's $\mathcal{J}$-digraph are the same.

In the following, we shall give the definitions of Green's graphs of a semigroup.

Definition 2.2. Let S be a semigroup. The Green's $\mathscr{L}$-graph of S, denoted by $\Gamma_{\mathscr{L}}(S)$, is an undirected graph whose vertex set is S such that two vertices $a, b \in S$ are adjacent if and only if $a \neq b$ and $b \in Sa$ or $a \in Sb$, i.e.,

$$V(\Gamma_{\mathscr{L}}(S)) = S, \quad E(\Gamma_{\mathscr{L}}(S)) = \{\{a, b\} \subseteq S | a \neq b, \text{ and } b \in \mathrm{Sa} \text{ or } a \in \mathrm{Sb}\}.$$

The other Green's graphs $\Gamma_{\mathscr{R}}(S)$ and $\Gamma_{\mathscr{J}}(S)$ are defined in a similar way: $V(\Gamma_{\mathcal{R}}(S)) = V(\Gamma_{\mathcal{J}}(S)) = S$ and

$$E(\Gamma_{\mathscr{R}}(S)) = \{\{a, b\} \subseteq S | a \neq b \text{ and } b \in aS \text{ or } a \in bS\},$$
$$E(\Gamma_{\mathscr{J}}(S)) = \{\{a, b\} \subseteq S | a \neq b \text{ and } b \in S^1aS^1 \text{ or } a \in S^1bS^1\}.$$

For a semigroup S, it is clear that both $\Gamma_{\mathscr{L}}(S)$ and $\Gamma_{\mathscr{R}}(S)$ are obtained by deleting some of the edges from $E(\Gamma_{\mathscr{J}}(S))$. Hence, $\Gamma_{\mathscr{L}}(S)$ and $\Gamma_{\mathscr{R}}(S)$ are two spanning subgraphs of $\Gamma_{\mathscr{J}}(S)$. It is easy to check that the power graph $\mathscr{P}(S)$ of S is a spanning subgraph of each of the Green's graphs. The principal left ideal graph of S is the graph S^G with $V(S^G) = S$ such that two vertices a and b $(a \neq b)$ are adjacent in S^G if and only if $S^1a \cap S^1b \neq \varnothing$. The principal right ideal graph is defined similarly. Moreover, it is clear that $\Gamma_{\mathscr{L}}(S)$ (resp. $\Gamma_{\mathscr{R}}(S)$) is a spanning subgraph of the principal left (resp. right) ideal graph of S.

Now, we shall discuss some graph-theoretic properties of the Green's graphs for a semigroup S. We only consider the Green's $\mathscr{L}$-graph $\Gamma_{\mathscr{L}}(S)$. There are analogous results for $\Gamma_{\mathscr{R}}(S)$ and $\Gamma_{\mathscr{J}}(S)$. We define two binary relations τ_1 and τ_2 on S by

$$a\tau_1 b \Leftrightarrow S^1a \cap S^1b \neq \varnothing,$$
$$a\tau_2 b \Leftrightarrow S^1a \cup S^1b \subseteq S^1c \quad \text{for some } c \in S.$$

Clearly, $\tau_1 \cup \tau_2$ is also a binary relation. Let τ denote the equivalence relation on S generated by $\tau_1 \cup \tau_2$, i.e., the minimum equivalence relation on S containing $\tau_1 \cup \tau_2$. By [33, Proposition 4.25], we have the following lemma.

Lemma 2.3. *Let S be a semigroup. Then, $\tau = (\tau_1 \cup \tau_2)^\infty$.*

Moreover, we have the following result.

Theorem 2.4. *Let S be a semigroup, and let $a, b \in S$ with $a \neq b$. Then, a and b are connected in $\Gamma_{\mathscr{L}}(S)$ if and only if $a\tau b$.*

Proof. Suppose that a and b are connected by a path, say $(a, c_1, c_2, \ldots, c_k, b)$ in $\Gamma_{\mathscr{L}}(S)$, where $c_1, c_2, \ldots, c_k \in S$. Letting $a = c_0$ and $b = c_{k+1}$, for each $i \in \{0, \ldots, k\}$, we have $c_i \in Sc_{i+1}$ or $c_{i+1} \in Sc_i$, and hence $c_i \tau c_{i+1}$. It follows by transitivity that $a\tau b$.

Conversely, let $a\tau b$. By Lemma 2.3, we have $(a, b) \in (\tau_1 \cup \tau_2)^\infty$. That is, $(a, b) \in (\tau_1 \cup \tau_2)^n$ for some positive integer n. Hence, there exist $c_1, c_2, \ldots, c_{n-1} \in S$ such that $(a, c_1) \in \tau_1 \cup \tau_2$, $(c_1, c_2) \in \tau_1 \cup \tau_2, \ldots, (c_{n-1}, b) \in \tau_1 \cup \tau_2$. Let $a = c_0$ and $b = c_n$, and consider $i \in \{0, \ldots, n-1\}$. If $(c_i, c_{i+1}) \in \tau_1$, then there exists some $d \in S^1c_i \cap S^1c_{i+1}$. If $(c_i, c_{i+1}) \in \tau_2$, then there exists some $d \in S$ such that $S^1c_i \cap S^1c_{i+1} \subseteq S^1d$. In either case, we have $\{c_i, d\}, \{d, c_{i+1}\} \in E(\Gamma_{\mathscr{L}}(S))$, and hence c_i and c_{i+1} are connected by the path (c_i, d, c_{i+1}). It follows that a and b are connected. □

The following corollary is immediate.

Corollary 2.5. *Suppose that S is a semigroup. Then, the connected components of* $\Gamma_{\mathscr{L}}(S)$ *are precisely* $\{\tau_a | a \in S\}$, *where* τ_a *is the equivalence class containing* a.

It follows from Corollary 2.5 that $\Gamma_{\mathscr{L}}(S)$ is a connected graph (resp. a null graph) if and only if τ is the universal relation (resp. the equality relation) on S.

Recall that a *partially ordered set*, or simply *poset*, P is an ordered pair $(V(P), \leq_P)$, where $V(P)$ is called the vertex set of P, and $\leq_P$ is a partial order on $V(P)$. As usual, we write $x <_P y$ if $x \leq_P y$ and $x \neq y$. For two elements $x, y \in V(P)$, x and y are *comparable* in P if $x \leq_P y$ or $y \leq_P x$; otherwise, x and y are *incomparable*. A *chain* (resp. *antichain*) is a partially ordered set such that all elements are pairwise comparable (resp. incomparable).

Recall that a graph is *complete* if any two vertices are adjacent. The next result characterizes the class of semigroups S for which $\Gamma_{\mathscr{L}}(S)$ is complete.

Proposition 2.6. *Let S be a semigroup. Then,* $\Gamma_{\mathscr{L}}(S)$ *is complete if and only if the principal left ideals of S form a chain with respect to the usual inclusion.*

Proof. $\Gamma_{\mathscr{L}}(S)$ is complete if and only if for any $a, b \in S$ with $a \neq b$ either $a \in bS$ or $b \in aS$ if and only if the principal left ideals of S form a chain. □

Example. For a monogenic semigroup $S = \langle a \rangle$, every principal (left) ideal of S is of the form Sa^t for some positive integer t or S^1a. It is easy to check that the principal left (resp. right, two-sided) ideals of S form a chain under usual inclusion. Hence, by Proposition 2.6, $\Gamma_{\mathscr{L}}(S)$, $\Gamma_{\mathscr{R}}(S)$, and $\Gamma_{\mathscr{J}}(S)$ are complete.

Note that if S is a left simple (resp. right simple, simple) semigroup, i.e., $\mathscr{L} = S \times S$ (resp. $\mathscr{R} = S \times S$, $\mathscr{J} = S \times S$), then $\Gamma_{\mathscr{L}}(S)$ (resp. $\Gamma_{\mathscr{R}}(S)$, $\Gamma_{\mathscr{J}}(S)$) is a complete graph. In particular, if S is a left zero semigroup, then $\Gamma_{\mathscr{L}}(S)$ is complete and $\Gamma_{\mathscr{R}}(S)$ is a null graph. Dually, for a right zero semigroup S, $\Gamma_{\mathscr{R}}(S)$ is complete and $\Gamma_{\mathscr{L}}(S)$ is a null graph.

Now, let S be a finite semigroup. As usual, for each $a \in S$, $L_a = \{b \in S | (a, b) \in \mathscr{L}\}$. Write

$$L(S) := \{L_a | a \in S\} = \{L_{a_{11}}, L_{a_{21}}, \ldots, L_{a_{m1}}\}, \tag{2.1}$$

where m is the number of $\mathscr{L}$-classes of S, and let $L_{a_{i1}} = \{a_{i1}, a_{i2}, \ldots, a_{is_i}\}$ for each $i \in [m]$ $(= \{1, 2, \ldots, m\})$, where s_i denote the cardinality of the $\mathscr{L}$-class containing a_{i1}.

Definition 2.7. Let S be a finite semigroup and let $a, b \in S$ with $a \neq b$. Define $a \prec b$ if one of the following conditions holds.

(1) $L_a < L_b$, i.e., $S^1a \subsetneq S^1b$.

(2) For some $i \in [m]$, $a = a_{i\ell}$, $b = a_{ik}$, and $\ell < k$.

Define $a \preccurlyeq b$ if $a \prec b$ or $a = b$.

The proof of the following lemma is obvious.

Lemma 2.8. *Suppose that S is a finite semigroup. With reference to* (2.1), *if there exist $i, j \in [m]$ and $i \neq j$ such that $a_{i\ell_0} \prec a_{jk_0}$ for some $\ell_0 \in [s_i]$ and $k_0 \in [s_j]$, then $a_{i\ell} \prec a_{jk}$ for each $\ell \in [s_i]$ and each $k \in [s_j]$.*

Now, we define O_S as the digraph with vertex set S such that there is an arc from b to a if $a \prec b$. It is easy to see that O_S is an orientation of $\Gamma_{\mathscr{L}}(S)$. Moreover, we have the following result.

Theorem 2.9. *Let S be a finite semigroup. Then, the following statements hold.*

(i) *O_S is a transitive orientation of* $\Gamma_{\mathscr{L}}(S)$ *and a subdigraph of* $D_{\mathscr{L}}(S)$.

(ii) *If O is a transitive orientation of* $\Gamma_{\mathscr{L}}(S)$ *and a subdigraph of* $D_{\mathscr{L}}(S)$, *then the graphs O and O_S are isomorphic.*

Proof. (*i*). Assume that $\{(a,b),(b,c)\} \subseteq E(O_S)$ for distinct $a,b,c \in S$. Then $b \prec a$ and $c \prec b$. This implies that $L_b \leq L_a$ and $L_c \leq L_b$. First, suppose that $(a,c) \notin \mathscr{L}$ in S. Then, $L_c < L_a$, and so $c \prec a$. It follows that $(a,c) \in E(O_S)$. Now, suppose that $(a,c) \in \mathscr{L}$, i.e., $L_a = L_c$. Then, by (2.1), there exists an index $i \in [m]$ such that $a = a_{i\ell}$, $b = a_{ir}$, and $c = a_{ik}$ for some $\ell, r, k \in [s_i]$ and $k < r < \ell$. Thus, $c \prec a$, and so $(a,c) \in E(O_S)$. This shows that O_S is a transitive orientation of $\Gamma_{\mathscr{L}}(S)$. Clearly, O_S is a subdigraph of $D_{\mathscr{L}}(S)$.

(*ii*). Suppose that a subdigraph O of $D_{\mathscr{L}}(S)$ is another transitive orientation of $\Gamma_{\mathscr{L}}(S)$. Since the induced subgraph on $L_{a_{i1}}$ of $\Gamma_{\mathscr{L}}(S)$ is a complete graph and all transitive orientations of a fixed completed graph are isomorphic, we need only to show that $(a_{i1}, a_{j1}) \in E(O_S)$ if and only if $(a_{i1}, a_{j1}) \in E(O)$ for any $i,j \in [m]$ and $i \neq j$.

If $(a_{i1}, a_{j1}) \in E(O_S)$, that is, $L_{a_{j1}} < L_{a_{i1}}$, then $a_{j1} = xa_{i1}$ for some $x \in S$, and so $\{a_{i1}, a_{j1}\} \in E(\Gamma_{\mathscr{L}}(S))$, $(a_{i1}, a_{j1}) \in E(D_{\mathscr{L}}(S))$ and $(a_{j1}, a_{i1}) \notin E(D_{\mathscr{L}}(S))$. Since O is a subdigraph of $D_{\mathscr{L}}(S)$, we have $(a_{i1}, a_{j1}) \in E(O)$.

Conversely, if $(a_{i1}, a_{j1}) \in E(O)$, then $(a_{i1}, a_{j1}) \in D_{\mathscr{L}}(S)$. That is, $a_{j1} = xa_{i1}$ for some $x \in S$, and so $L_{a_{j1}} < L_{a_{i1}}$. It follows that $a_{j1} \prec a_{i1}$, and so $(a_{i1}, a_{j1}) \in E(O_S)$. This shows that O and O_S are isomorphic. □

The following result is an immediate consequence of Theorem 2.9.

Corollary 2.10. *The Green's $\mathscr{L}$-graph $\Gamma_{\mathscr{L}}(S)$ of a finite semigroup S is a comparability graph.*

A graph Γ is *perfect* if for each induced subgraph Λ of Γ, the chromatic number and the clique number of Λ are equal. It is well known that comparability graphs are perfect [34, Chapter V, Theorem 17]. Therefore, we have the following corollary.

Corollary 2.11. *The Green's $\mathscr{L}$-graph $\Gamma_{\mathscr{L}}(S)$ of a finite semigroup S is perfect.*

3 Structure of Green's graphs for a finite semigroup

In this section, by means of the generalized lexicographic product of certain graphs, we characterize the structures of Green's graphs for a finite semigroup S. We shall only characterize the Green's $\mathscr{L}$-graph $\Gamma_{\mathscr{L}}(S)$; analogous results hold for $\Gamma_{\mathscr{R}}(S)$ and $\Gamma_{\mathscr{J}}(S)$.

Let P be a poset. For any subset $U \subseteq V(P)$, the subposet of P induced by U, denoted by $P(U)$, is a poset $(U, \leq_{P(U)})$, where for any $x,y \in U$, $x \leq_{P(U)} y$ if and only if $x \leq_P y$. It follows from Definition 2.7 that $(S, \preccurlyeq)$ is a poset. In the remainder of this study, we use L_S to denote this poset.

The comparability graph of P, denoted by $\mathscr{C}_P$, is the graph with the vertex set $V(P)$, where two distinct vertices are adjacent if and only if they are comparable.

Lemma 3.1. *Let S be a finite semigroup. Then, $\mathscr{C}_{L_S} = \Gamma_{\mathscr{L}}(S)$.*

Proof. Clearly, $V(\mathscr{C}_{L_S}) = V(\Gamma_{\mathscr{L}}(S)) = S$. For any $a,b \in S$, we have $\{a,b\} \in E(\mathscr{C}_{L_S})$ if and only if $L_a \leq L_b$ or $L_b \leq L_a$ if and only if $\{a,b\} \in E(\Gamma_{\mathscr{L}}(S))$. Thus, $E(\mathscr{C}_{L_S}) = E(\Gamma_{\mathscr{L}}(S))$. □

From [6], let P be a poset. A subset Q of P is *homogeneous* if for any $y \in P \backslash Q$, one of the following holds:
(1) For all $x \in Q$, $x \leq_P y$.
(2) For all $x \in Q$, $y \leq_P x$.
(3) For all $x \in Q$, x and y are incomparable.

A *homogeneous chain* (resp. *antichain*) in P is a chain (resp. an antichain) that is homogeneous. An equivalence relation ρ of P is *homogeneous* if all its equivalence classes are homogeneous in P, and the partition Ω corresponding to ρ is called a *homogeneous partition* of P. The quotient P/ρ is the poset

$(\Omega, \leq_{P/\rho})$ such that two subsets $\Omega_1, \Omega_2 \in \Omega$ satisfy $\Omega_1 \leq_{P/\rho} \Omega_2$ if and only if $\Omega_1 = \Omega_2$ or $x <_P y$ for each $x \in \Omega_1$ and each $y \in \Omega_2$.

L_S is defined in the second paragraph of Section 3. The following lemma is an immediate consequence of Lemma 2.8.

Lemma 3.2. *Let S be a finite semigroup. Any element in $L(S)$ is a homogeneous chain in L_S. Moreover, $L(S)$ and $\mathscr{L}$ are a homogeneous partition and a homogeneous equivalence relation of L_S, respectively.*

Recall that the lexicographical sum [35] is defined as follows. Let P be a poset and $\mathbb{P} = \{Q_x | x \in V(P)\}$ be a family of posets indexed by $V(P)$. The *lexicographical sum* of $\mathbb{P}$ over P, denoted by $P[\mathbb{P}]$, is the poset with the vertex set $V(P[\mathbb{P}]) = \{(x, y) | x \in V(P), y \in V(Q_x)\}$, where $(x_1, y_1) \leq_{P[\mathbb{P}]} (x_2, y_2)$, provided that either $x_1 = x_2$ and $y_1 \leq_{Q_{x_1}} y_2$ or $x_1 <_P x_2$.

Let ρ be a homogeneous equivalence relation of a poset P, and let $R = P/\rho$ and $\mathbb{P} = \{P(Q) | Q \in R\}$. Then, P is isomorphic to $R[\mathbb{P}]$ [6, Lemma 2.8]. Hence, the following result is an immediate consequence of Lemma 3.2.

Theorem 3.3. *Let S be a finite semigroup and let $\mathbb{P} = \{L_S(L_a) | L_a \in L(S)\}$. Then, $L_S \cong (L_S/\mathscr{L})[\mathbb{P}]$.*

Now, in order to characterize the structure of Green's $\mathscr{L}$-graph $\Gamma_{\mathscr{L}}(S)$ of a finite semigroup S, we need the definition of the generalized lexicographic product [36]. Given a graph Γ and a family of graphs $\mathbb{F} = \{\mathfrak{F}_v | v \in V(\Gamma)\}$, indexed by $V(\Gamma)$, their *generalized lexicographic product*, denoted by $\Gamma[\mathbb{F}]$, is defined as the graph with vertex set

$$V(\Gamma[\mathbb{F}]) = \{(v, w) | v \in V(\Gamma), w \in V(\mathfrak{F}_v)\}$$

and edge set

$$E(\Gamma[\mathbb{F}]) = \{\{(v_1, w_1), (v_2, w_2)\} | \{v_1, v_2\} \in E(\Gamma), \text{ or } v_1 = v_2 \text{ and } \{w_1, w_2\} \in E(\mathfrak{F}_{v_1})\}.$$

Given a poset P, let $\mathbb{P}$ be a family of posets indexed by $V(P)$. Suppose that $\mathbb{C}_{\mathbb{P}}$ consists of all comparability graphs of posets in $\mathbb{P}$. Then, $\mathscr{C}_{P[\mathbb{P}]} = \mathscr{C}_P[\mathbb{C}_{\mathbb{P}}]$ [6, Lemma 2.12].

Lemma 3.4. *Given a finite semigroup S, let $\mathbb{P} = \{L_S(L_a) | L_a \in L(S)\}$ and $\mathbb{C}_{\mathbb{P}} = \{\mathscr{C}_{L_S(L_a)} | L_S(L_a) \in \mathbb{P}\}$. Then,*

$$\mathscr{C}_{(L_S/\mathscr{L})[\mathbb{P}]} = \mathscr{C}_{(L_S/\mathscr{L})}[\mathbb{C}_{\mathbb{P}}].$$

Now, for a finite semigroup S and $a \in S$, let K_a denote the complete graph of order $|L_a|$, and write $\mathbb{K}_S = \{K_a | L_a \in L(S)\}$. Thus, we have the following result.

Theorem 3.5. *Let S be a finite semigroup. Then, the Green's $\mathscr{L}$-graph $\Gamma_{\mathscr{L}}(S)$ is isomorphic to the generalized lexicographic product $\mathscr{C}_{(L_S/\mathscr{L})}[\mathbb{K}_S]$.*

Proof. It follows from Lemma 3.1 that $\mathscr{C}_{L_S} = \Gamma_{\mathscr{L}}(S)$. Hence, we only need to show that $\mathscr{C}_{L_S} \cong \mathscr{C}_{(L_S/\mathscr{L})}[\mathbb{K}_S]$.

By (2.1) and Definition 2.7, for each $a \in S$, the subposet $L_S(L_a)$ is a chain, i.e., every pair of distinct elements in L_a are comparable. Hence, the comparability graph $\mathscr{C}_{L_S(L_a)}$ is the complete graph of order $|L_a|$. That is, $\mathscr{C}_{L_S(L_a)} \cong K_a$ for each $a \in S$. Moreover,

$$\mathscr{C}_{(L_S/\mathscr{L})}[\{\mathscr{C}_{L_S(L_a)} | L_a \in L(S)\}] \cong \mathscr{C}_{(L_S/\mathscr{L})}[\mathbb{K}_S].$$

It follows from Theorem 3.3 that

$$L_S \cong (L_S/\mathscr{L})[\mathbb{P}] = (L_S/\mathscr{L})[\{L_S(L_a) | L_a \in L(S)\}].$$

Therefore, by Lemma 3.4,

$$\mathscr{C}_{L_S} \cong \mathscr{C}_{(L_S/\mathscr{L})[\{L_S(L_a) | L_a \in L(S)\}]} \cong \mathscr{C}_{(L_S/\mathscr{L})}[\{\mathscr{C}_{L_S(L_a)} | L_a \in L(S)\}] \cong \mathscr{C}_{(L_S/\mathscr{L})}[\mathbb{K}_S].$$

Thus, $\Gamma_{\mathscr{L}}(S) \cong \mathscr{C}_{(L_S/\mathscr{L})}[\mathbb{K}_S]$. This completes the proof. □

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References

[1] A. V. Kelarev, *On Cayley graphs of inverse semigroups*, Semigroup Forum **72** (2006), no. 3, 411–418, DOI: https://doi.org/10.1007/s00233-005-0526-9.
[2] A. V. Kelarev and S. J. Quinn, *Directed graph and combinatorial properties of semigroups*, J. Algebra **251** (2002), 16–26, DOI: https://doi.org/10.1006/jabr.2001.9128.
[3] A. V. Kelarev and C. E. Praeger, *On transitive Cayley graphs of groups and semigroups*, European J. Combin. **24** (2003), no. 1, 59–72, DOI: https://doi.org/10.1016/S0195-6698(02)00120-8.
[4] F. DeMeyer and L. DeMeyer, *Zero divisor graphs of semigroups*, J. Algebra **283** (2005), no. 1, 190–198, DOI: https://doi.org/10.1016/j.jalgebra.2004.08.028.
[5] I. Chakrabarty, S. Ghosh, and M. K. Sen, *Undirected power graphs of semigroups*, Semigroup Forum **78** (2009), no. 3, 410–426, DOI: https://doi.org/10.1007/s00233-008-9132-y.
[6] M. Feng, X. Ma, and K. Wang, *The structure and metric dimension of the power graph of a finite group*, European J. Combin. **43** (2015), 82–97, DOI: https://doi.org/10.1016/j.ejc.2014.08.019.
[7] A. V. Kelarev, *Graph Algebras and Automata*, Marcel Dekker, New York, 2003.
[8] B. Khosravi, *Cayley graphs of groupoids and generalized fat-trees*, Indian J. Pure Appl. Math. **54** (2023), no. 4, 1125–1131, DOI: https://doi.org/10.1007/s13226-022-00326-6.
[9] B. Khosravi, B. Khosravi, and B. Khosravi, *On the automorphism groups of vertex-transitive Cayley digraphs of monoids*, J. Algebraic Combin. **53** (2021), no. 1, 227–251, DOI: https://doi.org/10.1007/s10801-019-00927-1.
[10] E. Ilić-Georgijević, *A description of the Cayley graphs of homogeneous semigroups*, Comm. Algebra **48** (2020), no. 12, 5203–5214, DOI: https://doi.org/10.1080/00927872.2020.1783279.
[11] E. Ilić-Georgijević, *On the connected power graphs of semigroups of homogeneous elements of graded rings*, Mediterr. J. Math. **19** (2022), no. 3, 119, DOI: https://doi.org/10.1007/s00009-022-02041-2.
[12] E. Ilić-Georgijević, *On the power graphs of semigroups of homogeneous elements of graded semisimple Artinian rings*, Comm. Algebra **52** (2024), no. 11, 4961–4972, DOI: https://doi.org/10.1080/00927872.2024.2362338.
[13] F. Hassani Hajivand and B. Khosravi, *On Two-Sided Cayley graphs of semigroups and groups*, Ann. Comb. **27** (2023), no. 2, 413–432, DOI: https://doi.org/10.1007/s00026-022-00618-y.
[14] K. Limkul and S. Panma, *On the independence number of Cayley digraphs of Clifford semigroups*, Mathematics **11** (2023), no. 16, 3445, DOI: https://doi.org/10.3390/math11163445.
[15] N. Nupo and Y. Chaiya, *Structural properties and isomorphism theorems for Cayley digraphs of full transformation semigroups with respect to Greenas equivalence classes*, Heliyon **9** (2023), e12976, DOI: https://doi.org/10.1016/j.heliyon.2023.e12976.
[16] S. Panma, U. Knauer, and S. Arworn, *On transitive Cayley graphs of strong semilattices of right (left) groups*, Discrete Math. **309** (2009), no. 17, 5393–5403, DOI: https://doi.org/10.1016/j.disc.2008.11.038.
[17] P. J. Cameron and S. Ghosh, *The power graph of a finite group*, Discrete Math. **311** (2011), no. 6, 1220–1222, DOI: https://doi.org/10.1016/j.disc.2010.02.011.
[18] P. J. Cameron, *The power graph of a finite group II*, J. Group Theory **13** (2010), 779–783, DOI: https://doi.org/10.1515/JGT.2010.023.

[19] A. V. Kelarev and S. J. Quinn, *A combinatorial property and power graphs of groups*, Comment. Math. Univ. Carolin. **45** (2004), no. 1, 1–7.

[20] A. V. Kelarev, S. J. Quinn, and R. Smoliková, *Power graphs and semigroups of matrices*, Bull. Aust. Math. Soc. **63** (2001), no. 2, 341–344, DOI: https://doi.org/10.1017/S0004972700019390.

[21] S. Dalal, J. Kumar, and S. Singh, *On the enhanced power graph of a semigroup*, Algebra Colloq. **31** (2024), no. 1, 83–96, DOI: https://doi.org/10.1142/S1005386724000099.

[22] Y. L. Cheng, Y. Shao, and L. Zeng, *Power graphs of a class of completely 0-simple semigroups*, J. Algebraic Combin. **59** (2024), 697–710, DOI: https://doi.org/10.1007/s10801-024-01306-1.

[23] Y. Ashegh Bonabi and B. Khosravi, *On characterization of a completely simple semigroup by its power graph and green relations*, J. Algebraic Combin. **55** (2022), no. 4, 1123–1137, DOI: https://doi.org/10.1007/s10801-021-01087-x.

[24] A. Kumar, L. Selvaganesh, P. J. Cameron, and T. Tamizh Chelvam, *Recent developments on the power graph of finite groups-a survey*, AKCE Int. J. Graphs Comb. **18** (2021), no. 2, 65–94, DOI: https://doi.org/10.1080/09728600.2021.1953359.

[25] J. Abawajy, A. Kelarev, and M. Chowdhury, *Power graphs: A survey*, Electron. J. Graph Theory Appl. **1** (2013), no. 2, 125–147, DOI: https://doi.org/10.5614/ejgta.2013.1.2.6.

[26] X. Ma, A. Kelarev, Y. Lin, and K. Wang, *A survey on enhanced power graphs of finite groups*, *Electron. J. Graph Theory Appl.* **10** (2022), 89–111, DOI: https://doi.org/10.5614/ejgta.2022.10.1.6.

[27] Y. Wang, P. Gao, and X. Guo, *The number of spanning trees of E-extended power graphs associated with finite semigroups*, Discrete Math. **347** (2024), no. 10, 114–137, DOI: https://doi.org/10.1016/j.disc.2024.114137.

[28] A. Gharibkhajeh and H. Doostie, *On the graphs related to Green relations of finite semigroups*, Iran. Inform. **9** (2014), no. 1, 43–51, DOI: https://doi.org/10.7508/IJMSI.2014.01.004.

[29] M. R. Sorouhesh, H. Doostie, and C. M. Campbell, *A sufficient condition for coinciding the Green graphs of semigroups*, J. Math. Computer Sci. **17** (2017), 216–219, DOI: https://doi.org/10.22436/jmcs.017.02.03.

[30] P. C. Gilmore and A. J. Hoffman, *A characterization of comparability graphs and of interval graphs*, Canad. J. Math. **16** (1964), 539–548, DOI: https://doi.org/10.4153/CJM-1964-055-5.

[31] T. Gallai, *Transitive orientierbare graphen*, Acta Math. Acad. Sci. Hung. **18** (1967), 25–66, DOI: https://doi.org/10.1007/BF02020961.

[32] C. Godsil and G. F. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.

[33] J. M. Howie, *An Introduction to Semigroup Theory*, Academic Press, London, 1976.

[34] B. Bollobas, *Mordern Graph Theory*, Springer, New York, 1998.

[35] P. Ille and J. Rampon, *Reconstruction of posets with the same comparability graph*, J. Combin. Theory Ser. B **74** (1998), no. 2, 368–377, DOI: https://doi.org/10.1006/jctb.1998.1859.

[36] G. Sabidussi, *Graph derivates*, Math. Z. **76** (1961), 385–401, DOI: https://doi.org/10.1007/BF01210984.