Research Article

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Some new Hermite-Hadamard type inequalities for product of strongly *h*-convex functions on ellipsoids and balls

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Abstract: We establish novel Hermite-Hadamard-type inequalities for the product of two strongly h-convex functions defined on balls and ellipsoids in multidimensional Euclidean spaces. Additionally, we investigate mappings associated with these inequalities and explore their applications. Our results generalize several existing findings in the literature.

Keywords: product of functions, strongly h-convex function, Hermite-Hadamard's inequality, mapping, ellipsoid

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1 Introduction

Let $\mathcal{D}$ be a convex subset of the Euclidean space $\mathbb{R}^n$ and $h:[0,1]\to[0,\infty)$ be a nonnegative function. A function $f:\mathcal{D}\to\mathbb{R}$ is called an *h-convex function* if for any $X=(x_1,x_2,\ldots,x_n), Y=(y_1,y_2,\ldots,y_n)\in\mathcal{D}$, and $\alpha\in[0,1]$,

$$f(\alpha X+(1-\alpha)Y)\le h(\alpha)f(X)+h(1-\alpha)f(Y).$$

This concept, introduced by Varosanec [1] in 2007, generalizes several well-known classes of functions, including convex functions ($h(\alpha)=\alpha$), s-convex functions (in the second sense) ($h(\alpha)=\alpha^s(s\in(0,1)$, [2]), P-functions ($h(\alpha)\equiv 1$, [3]), and Godunova-Levin functions ($h(\alpha)=1/\alpha(0<\alpha\le 1)$, [4]).

In 1966, Polyak [5] introduced strongly convex functions, which since played a pivotal role in optimization and mathematical economics, etc. Later, Angulo et al. [6] extended this notation to strongly h-convex functions. Specifically, $f:\mathcal{D}\to\mathbb{R}$ is *strongly h-convex with modulus* $\lambda>0$, or $f\in SX(h,\lambda,\mathcal{D})$, if for all $X,Y\in\mathcal{D}$ and $\alpha\in[0,1]$,

$$f(\alpha X+(1-\alpha)Y)\le h(\alpha)f(X)+h(1-\alpha)f(Y)-\lambda\alpha(1-\alpha)|X-Y|^2, \tag{1.1}$$

where

$$|X-Y|^2=(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^2.$$

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In particular, if f satisfies (1.1) with $h(\alpha) = \alpha$, $h(\alpha) = \alpha^s (s \in (0,1))$, $h(\alpha) = 1$, and $h(\alpha) = 1/\alpha$ $(0 < \alpha \le 1)$, then f is said to be a *strongly convex function, strongly s-convex function (in the second sense), strongly P-function*, and a *strongly Godunova-Levin function*, respectively. Moreover, it is not difficult to see that $h(1/2) > 0$ if $f \ge 0$ and $f \in SX(h, \lambda, \mathcal{D})$. Many properties and applications of the aforementioned convex-type functions can be found in the literature (see, e.g., [7–20]).

A celebrated result for convex functions is the Hermite-Hadamard inequality:

Theorem A. *If* $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *is convex, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}.$$

Dragomir et al. [21] extended this inequality to Godunova-Levin functions and P-functions, while Dragomir and Fitzpatrick [22] derived analogous results for s-convex functions in the second sense. Sarikaya et al. [23] further generalized these findings to h-convex functions. For strongly h-convex functions, the following inequality holds:

Theorem B. [6] *If* $f \in SX(h, \lambda, [a,b])$ *and h is Lebesgue integrable on* $[0,1]$ *with* $h(1/2) > 0$, *then*

$$\frac{1}{2h(1/2)}\left[f\left(\frac{a+b}{2}\right) + \frac{\lambda}{12}(b-a)^2\right] \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le [f(a)+f(b)]\int_0^1 h(t)\mathrm{d}t - \frac{\lambda}{6}(b-a)^2.$$

Hermite-Hadamard-type inequalities for products of functions have also been widely studied. For example, Pachpatte [24] established the following results for convex functions:

Theorem C. *If* $f, g : [a,b] \to [0,\infty)$ *are convex and* $fg \in L^1([a,b])$, *then*

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{f(a)g(a)+f(b)g(b)}{6} - \frac{f(a)g(b)+f(b)g(a)}{3} \\ &\le \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \le \frac{f(a)g(a)+f(b)g(b)}{3} + \frac{f(a)g(b)+f(b)g(a)}{6}. \end{aligned}$$

Subsequent works extended these results to s-convex [25] *and h-convex functions* [23] *and later to strongly h-convex functions* [26].

Multidimensional analogs of these inequalities have also been explored. For instance, the authors [27–30] studied Hermite-Hadamard inequalities for convex-type functions on rectangles. Dragomir [31,32] obtained similar estimates for convex functions on disks and balls, while Matłoka [33] generalized these to h-convex functions. Recent works by [34,35] extended these results to ellipsoids and balls in $\mathbb{R}^n$.

Motivated by these developments, this article aims to derive Hermite-Hadamard-type inequalities for products of strongly h-convex functions on ellipsoids and balls in $\mathbb{R}^n$ and to explore their applications.

2 Hermite-Hadamard-type inequalities for product of functions

In the sequel, $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ and $d\sigma(X)$ is the usual surface measure. Given $X = (x_1, x_2, \ldots, x_n)$, $Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, and scalars $a, b \in \mathbb{R}$, define the linear combination of vectors by

$$aX + bY = (ax_1 + by_1, ax_2 + by_2, \ldots, ax_n + by_n),$$

the product of vectors by

$$X \otimes Y = (x_1y_1, x_2y_2, \ldots, x_ny_n),$$

and the norm of X by

$$|X| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

$B_n(C, r)$ and $\delta_n(C, r)$ denote the n-dimensional ball and its sphere centered at the point $C = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ with radius $r > 0$, respectively. $E_n(C, R)$ is the n-dimensional ellipsoid centered at the point $C = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ with semiaxes $R = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$, i.e.,

$$\frac{(x_1 - c_1)^2}{r_1^2} + \frac{(x_2 - c_2)^2}{r_2^2} + \ldots + \frac{(x_n - c_n)^2}{r_n^2} \leq 1, \quad 0 < r_1, r_2, \ldots, r_n < \infty,$$

and $S_n(C, R)$ denotes the sphere of $E_n(C, R)$. Then,

$$|B_n(C, r)| = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad |\delta_n(C, r)| = \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma\left(\frac{n}{2} + 1\right)}, \tag{2.1}$$

$$|E_n(C, R)| = \frac{\pi^{\frac{n}{2}} r_1 \ldots r_n}{\Gamma\left(\frac{n}{2} + 1\right)}, \quad |S_n(C, tR)| = t^{n-1} |S_n(C, R)|, \quad t > 0, \tag{2.2}$$

where $\Gamma(\cdot)$ is the Gamma function. For any $0 < p, q < \infty$, we denote the beta function $B(\cdot,\cdot)$ by

$$B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1} \mathrm{d}t,$$

and then,

$$B(p, q) = B(q, p) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}, \quad B(p + 1, q) = \frac{p}{p + q} B(p, q). \tag{2.3}$$

In what follows, we also assume that $h(1/2) > 0$ in the definitions of (strongly) h-convex functions and the nonnegative function $h \in L^2([0, 1])$. Now, we are in a position to state our main results.

Theorem 1. *Let $f \in SX(h_1, \lambda_1, E_n(C, R))$, $g \in SX(h_2, \lambda_2, E_n(C, R))$ be both nonnegative functions with $0 < \lambda_1, \lambda_2 < \infty$ and g be symmetric about the center C. Then,*

$$\begin{aligned} &\frac{1}{4h_1(1/2)h_2(1/2)} \left[f(C)g(C) + (\lambda_2 f(C) + \lambda_1 g(C)) \frac{|R|^2}{n + 2} + \lambda_1 \lambda_2 \frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n + 2)(n + 4)} \right] \\ &\leq \frac{1}{|E_n(C, R)|} \int_{E_n(C,R)} f(X)g(X) \mathrm{d}X. \end{aligned} \tag{2.4}$$

Furthermore, if

$$\mathcal{K}_0(n) := 1 - 4h_1(1/2)h_2(1/2) n \int_0^1 t^{n-1} h_1(1 - t) h_2(1 - t) \mathrm{d}t > 0,$$

then we have

$$\begin{aligned} &\frac{1}{|E_n(C, R)|} \int_{E_n(C,R)} f(X)g(X) \mathrm{d}X \\ &\leq \frac{\mathcal{K}_1(n)}{\mathcal{K}_0(n)} \frac{1}{|\delta_n(0, 1)|} \int_{\delta_n(0,\,1)} f(\widetilde{X})g(\widetilde{X}) \mathrm{d}\sigma(X') - \lambda_2 \frac{\mathcal{K}_2(n)}{\mathcal{K}_0(n)} \frac{1}{|\delta_n(0, 1)|} \int_{\delta_n(0,\,1)} f(\widetilde{X}) |R \otimes X'|^2 \mathrm{d}\sigma(X') \end{aligned}$$

$$- \lambda_1 \frac{\mathcal{K}_3(n)}{\mathcal{K}_0(n)} \frac{1}{|\delta_n(0,1)|} \int\limits_{\delta_n(0,1)} g(\widetilde{X})|R \otimes X'|^2 \mathrm{d}\sigma(X') - \lambda_2 \frac{\mathcal{K}_4(n)}{\mathcal{K}_0(n)} \frac{|R|^2}{n+2} f(C) - \lambda_1 \frac{\mathcal{K}_5(n)}{\mathcal{K}_0(n)} \frac{|R|^2}{n+2} g(C)$$

$$- \lambda_1\lambda_2 \frac{\mathcal{K}_6(n)}{\mathcal{K}_0(n)} \frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}$$

$$\le \frac{\mathcal{K}_1(n)}{\mathcal{K}_0(n)} \frac{\mathfrak{C}(R)}{|S_n(C,R)|} \int\limits_{S_n(C,R)} f(X)g(X)\mathrm{d}\sigma(X) - \lambda_2 \frac{\mathcal{K}_2(n)}{\mathcal{K}_0(n)} \frac{\widetilde{\mathfrak{C}}(R)}{|S_n(C,R)|} \int\limits_{S_n(C,R)} f(X)|X-C|^2 \mathrm{d}\sigma(X) \tag{2.5}$$

$$- \lambda_1 \frac{\mathcal{K}_3(n)}{\mathcal{K}_0(n)} \frac{\widetilde{\mathfrak{C}}(R)}{|S_n(C,R)|} \int\limits_{S_n(C,R)} g(X)|X-C|^2 \mathrm{d}\sigma(X)$$

$$- \lambda_2 \frac{\mathcal{K}_4(n)}{\mathcal{K}_0(n)} \frac{|R|^2}{n+2} f(C) - \lambda_1 \frac{\mathcal{K}_5(n)}{\mathcal{K}_0(n)} \frac{|R|^2}{n+2} g(C) - \lambda_1\lambda_2 \frac{\mathcal{K}_6(n)}{\mathcal{K}_0(n)} \frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)},$$

where $\widetilde{X} = R \otimes X' + C \in S_n(C,R)$ *and*

$$\mathcal{K}_1(n) = n\int_0^1 t^{n-1}h_1(t)h_2(t)\mathrm{d}t + 2nh_1(1/2)\int_0^1 t^{n-1}h_1(1-t)h_2(t)\mathrm{d}t + 2nh_2(1/2)\int_0^1 t^{n-1}h_1(t)h_2(1-t)\mathrm{d}t,$$

$$\mathcal{K}_2(n) = n\int_0^1 t^{n-1}h_1(t)h_2(1-t)\mathrm{d}t + n\int_0^1 t^n(1-t)h_1(t)\mathrm{d}t,$$

$$\mathcal{K}_3(n) = n\int_0^1 t^{n-1}h_1(1-t)h_2(t)\mathrm{d}t + n\int_0^1 t^n(1-t)h_2(t)\mathrm{d}t,$$

$$\mathcal{K}_4(n) = (n+2)\int_0^1 t^n(1-t)h_1(1-t)\mathrm{d}t + n\int_0^1 t^{n-1}h_1(1-t)h_2(1-t)\mathrm{d}t,$$

$$\mathcal{K}_5(n) = (n+2)\int_0^1 t^n(1-t)h_2(1-t)\mathrm{d}t + n\int_0^1 t^{n-1}h_1(1-t)h_2(1-t)\mathrm{d}t,$$

$$\mathcal{K}_6(n) = n\int_0^1 t^{n-1}h_1(1-t)h_2(1-t)\mathrm{d}t - \frac{2}{(n+2)(n+3)},$$

$$\mathfrak{C}(R) = \frac{\Gamma(n/2+1)|S_n(C,R)|}{n\pi^{n/2}r^{n-1}} = \frac{|S_n(C,R)|}{|\delta_n(C,r)|},$$

$$\widetilde{\mathfrak{C}}(R) = \frac{\Gamma(n/2+1)|S_n(C,R)|}{n\pi^{n/2}\tilde{r}^{n-1}} = \frac{|S_n(C,R)|}{|\delta_n(C,\tilde{r})|},$$

with $r = \min\{r_1, r_2, \ldots, r_n\}$, $\tilde{r} = \max\{r_1, r_2, \ldots, r_n\}$.

Proof. (i) First, we prove inequality (2.4). The facts of strongly h-convexity of f and g imply that

$$f(C)g(C) = \frac{1}{|E_n(C,R)|} \int\limits_{E_n(C,R)} f\left(\frac{X}{2} + \frac{2C-X}{2}\right) g\left(\frac{X}{2} + \frac{2C-X}{2}\right) \mathrm{d}X$$

$$\le \frac{1}{|E_n(C,R)|} \int\limits_{E_n(C,R)} [h_1(1/2)f(X) + h_1(1/2)f(2C-X) - \lambda_1 |X-C|^2]$$

$$\times [h_2(1/2)g(X) + h_2(1/2)g(2C-X) - \lambda_2 |X-C|^2]\mathrm{d}X$$

$$= \frac{1}{|E_n(C,R)|} \int\limits_{E_n(C,R)} \{h_1(1/2)h_2(1/2)[f(X) + f(2C-X)][g(X) + g(2C-X)]$$

$$- \lambda_2 h_1(1/2)[f(X) + f(2C-X)]|X-C|^2$$

$$- \lambda_1 h_2(1/2)[g(X) + g(2C-X)]|X-C|^2 + \lambda_1\lambda_2 |X-C|^4\}\mathrm{d}X.$$

Since g is symmetric about the point C, we have

$$g(X) = g(2C - X),$$

for any $X \in E_n(C, R)$. Then, the basic property of Lebesgue integral means that

$$\int_{E_n(C,R)} [f(X) + f(2C - X)][g(X) + g(2C - X)]\mathrm{d}X = 4 \int_{E_n(C,R)} f(X)g(X)\mathrm{d}X$$

and

$$\int_{E_n(C,R)} [f(X) + f(2C - X)]|X - C|^2\mathrm{d}X = 2 \int_{E_n(C,R)} f(X)|X - C|^2\mathrm{d}X,$$

$$\int_{E_n(C,R)} [g(X) + g(2C - X)]|X - C|^2\mathrm{d}X = 2 \int_{E_n(C,R)} g(X)|X - C|^2\mathrm{d}X.$$

Therefore,

$$\begin{aligned} &\frac{4h_1(1/2)h_2(1/2)}{|E_n(C, R)|} \int_{E_n(C,R)} f(X)g(X)\mathrm{d}X \\ &\geq f(C)g(C) - \frac{\lambda_1\lambda_2}{|E_n(C, R)|} \int_{E_n(C,R)} |X - C|^4\mathrm{d}X + \frac{2\lambda_2 h_1(1/2)}{|E_n(C, R)|} \int_{E_n(C,R)} f(X)|X - C|^2\mathrm{d}X \\ &\quad + \frac{2\lambda_1 h_2(1/2)}{|E_n(C, R)|} \int_{E_n(C,R)} g(X)|X - C|^2\mathrm{d}X. \end{aligned} \tag{2.6}$$

Now, we will estimate the terms of right-hand side in the aforementioned inequality. Observe that

$$\int_{E_n(C,R)} |X - C|^4\mathrm{d}X = \int_{E_n(0,R)} (x_1^2 + x_2^2 + \ldots + x_n^2)^2\mathrm{d}X = \sum_{i=1}^{n} \int_{E_n(0,R)} x_i^4\mathrm{d}X + 2 \sum_{1\leq i<j\leq n} \int_{E_n(0,R)} x_i^2 x_j^2\mathrm{d}X. \tag{2.7}$$

It follows from (2.2) that

$$\begin{aligned} \int_{E_n(0,R)} x_n^4\mathrm{d}X &= 2\frac{\pi^{(n-1)/2}}{\Gamma((n - 1)/2 + 1)} r_1 \ldots r_{n-1} \int_0^{r_n} x_n^4 \left(1 - \frac{x_n^2}{r_n^2}\right)^{(n-1)/2} \mathrm{d}x_n \\ &= 2\frac{\pi^{(n-1)/2}}{\Gamma((n + 1)/2)} r_1 \ldots r_{n-1} r_n^5 \int_0^1 t^4(1 - t^2)^{(n-1)/2}\mathrm{d}t \\ &= \frac{\pi^{(n-1)/2}}{\Gamma((n + 1)/2)} r_1 \ldots r_{n-1} r_n^5 \int_0^1 t^{3/2}(1 - t)^{(n-1)/2}\mathrm{d}t \\ &= r_1 \ldots r_{n-1} r_n^5 \frac{\pi^{(n-1)/2}}{\Gamma((n + 1)/2)} B\left(\frac{5}{2}, \frac{n + 1}{2}\right). \end{aligned} \tag{2.8}$$

We infer from (2.3) and the basic properties of the gamma function that

$$\frac{\pi^{(n-1)/2}}{\Gamma((n + 1)/2)} B\left(\frac{5}{2}, \frac{n + 1}{2}\right) = \frac{\pi^{(n-1)/2}}{\Gamma((n + 1)/2)} \frac{\Gamma(5/2)\Gamma((n + 1)/2)}{\Gamma(n/2 + 3)} = \frac{3\pi^{n/2}}{(n + 2)(n + 4)\Gamma(n/2 + 1)},$$

which means that

$$\int_{E_n(0,R)} x_n^4\mathrm{d}X = \frac{3\pi^{n/2} r_1 \ldots r_{n-1} r_n^5}{(n + 2)(n + 4)\Gamma(n/2 + 1)} = \frac{3|E_n(0, R)|}{(n + 2)(n + 4)} r_n^4, \tag{2.9}$$

and

$$\sum_{i=1}^{n} \int_{E_n(0,R)} x_i^4\mathrm{d}X = \frac{3|E_n(0, R)|}{(n + 2)(n + 4)} \sum_{i=1}^{n} r_i^4. \tag{2.10}$$

On the other hand,

$$\int_{E_n(0,R)} x_{n-1}^2 x_n^2 \mathrm{d}X = 2\int_0^{r_n} x_n^2 \left(\int_{E_{n-1}(0,\tilde{R})} x_{n-1}^2 \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} \right) \mathrm{d}x_n,$$

where $\tilde{R} = \sqrt{1 - x_n^2/r_n^2}(r_1, r_2, \ldots, r_{n-1})$. Using (2.2) again,

$$\begin{aligned}\int_{E_{n-1}(0,\tilde{R})} x_{n-1}^2 \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} &= \frac{2\pi^{(n-2)/2} r_1 \ldots r_{n-2}}{\Gamma((n-2)/2+1)} (1 - x_n^2/r_n^2)^{(n-2)/2} \\ &\quad \times \int_0^{r_{n-1}\sqrt{1-x_n^2/r_n^2}} x_{n-1}^2 \left(1 - \frac{x_{n-1}^2}{(r_{n-1}\sqrt{1 - x_n^2/r_n^2})^2} \right)^{(n-2)/2} \mathrm{d}x_{n-1} \\ &= \frac{2\pi^{(n-2)/2}}{\Gamma(n/2)} r_1 \ldots r_{n-2} r_{n-1}^3 (1 - x_n^2/r_n^2)^{(n+1)/2} \int_0^1 t^2 (1 - t^2)^{(n-2)/2} \mathrm{d}t \\ &= \frac{\pi^{(n-2)/2}}{\Gamma(n/2)} r_1 \ldots r_{n-2} r_{n-1}^3 (1 - x_n^2/r_n^2)^{(n+1)/2} B\left(\frac{3}{2}, \frac{n}{2}\right),\end{aligned}$$

which implies that

$$\begin{aligned}\int_{E_n(0,R)} x_{n-1}^2 x_n^2 \mathrm{d}X &= 2\frac{\pi^{(n-2)/2}}{\Gamma(n/2)} r_1 \ldots r_{n-2} r_{n-1}^3 B\left(\frac{3}{2}, \frac{n}{2}\right) \int_0^{r_n} x_n^2 (1 - x_n^2/r_n^2)^{(n+1)/2} \mathrm{d}x_n \\ &= \frac{\pi^{(n-2)/2}}{\Gamma(n/2)} r_1 \ldots r_{n-2} r_{n-1}^3 r_n^3 B\left(\frac{3}{2}, \frac{n}{2}\right) B\left(\frac{3}{2}, \frac{n+3}{2}\right) \\ &= \frac{\pi^{n/2}}{(n+2)(n+4)\Gamma(n/2+1)} r_1 \ldots r_{n-2} r_{n-1}^3 r_n^3 = \frac{|E_n(0,R)|}{(n+2)(n+4)} r_{n-1}^2 r_n^2,\end{aligned}$$

and then,

$$\sum_{1 \le i < j \le n} \int_{E_n(0,R)} x_i^2 x_j^2 \mathrm{d}X = \frac{|E_n(0,R)|}{(n+2)(n+4)} \sum_{1 \le i < j \le n} r_i^2 r_j^2. \tag{2.11}$$

Therefore, we infer from (2.7), (2.10), and (2.11) that

$$\frac{1}{|E_n(C,R)|} \int_{E_n(C,R)} |X - C|^4 \mathrm{d}X = \frac{3\sum_{i=1}^n r_i^4 + 2\sum_{1 \le i < j \le n} r_i^2 r_j^2}{(n+2)(n+4)} = \frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}. \tag{2.12}$$

And by similar arguments as (2.7)–(2.10) or equality (29) in [34], we have

$$\frac{1}{|E_n(C,R)|} \int_{E_n(C,R)} |X - C|^2 \mathrm{d}X = \frac{|R|^2}{n+2}, \tag{2.13}$$

which yields that

$$\begin{aligned}\frac{|R|^2}{n+2} f(C) &= \frac{1}{|E_n(C,R)|} \int_{E_n(C,R)} f(C) |X - C|^2 \mathrm{d}X \\ &\le \frac{1}{|E_n(C,R)|} \int_{E_n(C,R)} [h_1(1/2)(f(X) + f(2C - X)) - \lambda_1 |X - C|^2] |X - C|^2 \mathrm{d}X \\ &= \frac{2h_1(1/2)}{|E_n(C,R)|} \int_{E_n(C,R)} f(X) |X - C|^2 \mathrm{d}X - \lambda_1 \frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}.\end{aligned}$$

This tells us that

$$\frac{2h_1(1/2)}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)|X-C|^2\mathrm{d}X \geq \frac{|R|^2}{n+2}f(C)+\lambda_1\frac{|R|^4+2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}. \tag{2.14}$$

Similarly,

$$\frac{2h_2(1/2)}{|E_n(C,R)|}\int_{E_n(C,R)} g(X)|X-C|^2\mathrm{d}X \geq \frac{|R|^2}{n+2}g(C)+\lambda_2\frac{|R|^4+2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}. \tag{2.15}$$

Therefore, we complete the proof of inequality (2.4) by (2.6), (2.12), (2.14), and (2.15).

(ii) Next, we prove the second part of Theorem 1. Changing variables yields that

$$\begin{aligned}\int_{E_n(C,R)} f(X)g(X)\mathrm{d}X &= r_1r_2\dots r_n\int_{B_n(0,\,1)} f(R\otimes X+C)g(R\otimes X+C)\mathrm{d}X\\ &= r_1r_2\dots r_n\int_0^1\int_{\delta_n(0,\,1)} f(t\widetilde{X}+(1-t)C)g(t\widetilde{X}+(1-t)C)t^{n-1}\mathrm{d}\sigma(X')\mathrm{d}t,\end{aligned}$$

here we recall the notation $\widetilde{X}=R\otimes X'+C\in S_n(C,R)$. Thus,

$$\begin{aligned}\int_{E_n(C,R)} f(X)g(X)\mathrm{d}X &\leq r_1r_2\dots r_n\int_0^1\int_{\delta_n(0,\,1)}[h_1(t)f(\widetilde{X})+h_1(1-t)f(C)-\lambda_1t(1-t)|R\otimes X'|^2]\\ &\quad\times[h_2(t)g(\widetilde{X})+h_2(1-t)g(C)-\lambda_2t(1-t)|R\otimes X'|^2]t^{n-1}\mathrm{d}\sigma(X')\mathrm{d}t\\ &= r_1r_2\dots r_n\int_0^1 t^{n-1}h_1(t)h_2(t)\mathrm{d}t\int_{\delta_n(0,\,1)} f(\widetilde{X})g(\widetilde{X})\mathrm{d}\sigma(X')\\ &\quad+r_1r_2\dots r_n\int_0^1 t^{n-1}h_1(t)h_2(1-t)\mathrm{d}t\int_{\delta_n(0,\,1)} f(\widetilde{X})g(C)\mathrm{d}\sigma(X')\\ &\quad+r_1r_2\dots r_n\int_0^1 t^{n-1}h_1(1-t)h_2(t)\mathrm{d}t\int_{\delta_n(0,\,1)} f(C)g(\widetilde{X})\mathrm{d}\sigma(X')\\ &\quad+r_1r_2\dots r_n|\delta_n(0,1)|\int_0^1 t^{n-1}h_1(1-t)h_2(1-t)\mathrm{d}t f(C)g(C)\\ &\quad-\lambda_2r_1r_2\dots r_n\int_0^1 t^n(1-t)h_1(t)\mathrm{d}t\int_{\delta_n(0,\,1)} f(\widetilde{X})|R\otimes X'|^2\mathrm{d}\sigma(X')\\ &\quad-\lambda_1r_1r_2\dots r_n\int_0^1 t^n(1-t)h_2(t)\mathrm{d}t\int_{\delta_n(0,\,1)} g(\widetilde{X})|R\otimes X'|^2\mathrm{d}\sigma(X')\\ &\quad-\lambda_2r_1r_2\dots r_n\int_0^1 t^n(1-t)h_1(1-t)\mathrm{d}t\int_{\delta_n(0,\,1)}|R\otimes X'|^2\mathrm{d}\sigma(X')f(C)\\ &\quad-\lambda_1r_1r_2\dots r_n\int_0^1 t^n(1-t)h_2(1-t)\mathrm{d}t\int_{\delta_n(0,\,1)}|R\otimes X'|^2\mathrm{d}\sigma(X')g(C)\\ &\quad+\lambda_1\lambda_2r_1r_2\dots r_n\int_0^1 t^{n+1}(1-t)^2\mathrm{d}t\int_{\delta_n(0,\,1)}|R\otimes X'|^4\mathrm{d}\sigma(X').\end{aligned} \tag{2.16}$$

Noting that

$$\int_{E_n(C,R)} |X - C|^4 \mathrm{d}X = r_1 r_2 \dots r_n \int_0^1 t^{n+3}\mathrm{d}t \int_{\delta_n(0,\,1)} |R \otimes X'|^4 \mathrm{d}\sigma(X') = \frac{r_1 r_2 \dots r_n}{n+4} \int_{\delta_n(0,\,1)} |R \otimes X'|^4 \mathrm{d}\sigma(X'),$$

then, with the aid of (2.1), (2.2), and (2.12), we have

$$\int_{\delta_n(0,\,1)} |R \otimes X'|^4 \mathrm{d}\sigma(X') = |B_n(0,1)| \frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{n+2}. \tag{2.17}$$

Similarly, by the same arguments as mentioned earlier, (2.1), (2.2), and (2.13) (or by (30) in [34]) imply that

$$\int_{\delta_n(0,\,1)} |R \otimes X'|^2 \mathrm{d}\sigma(X') = |B_n(0,1)||R|^2. \tag{2.18}$$

On the other hand,

$$\begin{aligned}\int_{\delta_n(0,\,1)} f(\widetilde{X})g(C)\mathrm{d}\sigma(X') &= \int_{\delta_n(0,\,1)} f(\widetilde{X})g\left(\frac{\widetilde{X}}{2} + \frac{2C - \widetilde{X}}{2}\right)\mathrm{d}\sigma(X') \\ &\le 2h_2(1/2)\int_{\delta_n(0,\,1)} f(\widetilde{X})g(\widetilde{X})\mathrm{d}\sigma(X') - \lambda_2 \int_{\delta_n(0,\,1)} f(\widetilde{X})|R \otimes X'|^2 \mathrm{d}\sigma(X').\end{aligned} \tag{2.19}$$

Analogously,

$$\int_{\delta_n(0,\,1)} f(C)g(\widetilde{X})\mathrm{d}\sigma(X') \le 2h_1(1/2)\int_{\delta_n(0,\,1)} f(\widetilde{X})g(\widetilde{X})\mathrm{d}\sigma(X') - \lambda_1 \int_{\delta_n(0,\,1)} g(\widetilde{X})|R \otimes X'|^2 \mathrm{d}\sigma(X'). \tag{2.20}$$

Therefore, we infer from (2.16)–(2.20) that

$$\begin{aligned}&\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)g(X)\mathrm{d}X \\ &\le \frac{\mathcal{K}_1(n)}{|\delta_n(0,1)|}\int_{\delta_n(0,\,1)} f(\widetilde{X})g(\widetilde{X})\mathrm{d}\sigma(X') - \lambda_2 \frac{\mathcal{K}_2(n)}{|\delta_n(0,1)|}\int_{\delta_n(0,\,1)} f(\widetilde{X})|R \otimes X'|^2 \mathrm{d}\sigma(X') \\ &\quad - \lambda_1 \frac{\mathcal{K}_3(n)}{|\delta_n(0,1)|}\int_{\delta_n(0,\,1)} g(\widetilde{X})|R \otimes X'|^2 \mathrm{d}\sigma(X') \\ &\quad - \lambda_2 \int_0^1 t^n(1-t)h_1(1-t)\mathrm{d}t\, |R|^2 f(C) - \lambda_1 \int_0^1 t^n(1-t)h_2(1-t)\mathrm{d}t\, |R|^2 g(C) \\ &\quad + n\int_0^1 t^{n-1}h_1(1-t)h_2(1-t)\mathrm{d}t f(C)g(C) + 2\lambda_1\lambda_2 \frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)^2(n+3)(n+4)}.\end{aligned} \tag{2.21}$$

Due to (2.4), we have

$$\begin{aligned}f(C)g(C) &\le \frac{4h_1(1/2)h_2(1/2)}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)g(X)\mathrm{d}X \\ &\quad - \frac{|R|^2}{n+2}(\lambda_2 f(C) + \lambda_1 g(C)) - \lambda_1\lambda_2 \frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}.\end{aligned} \tag{2.22}$$

Thus, we complete the proof of inequality (2.5) by (2.21) and (2.22).

Since $f, g \geq 0$ and $r = \min\{r_1, r_2, \dots, r_n\}$, $\tilde{r} = \max\{r_1, r_2, \dots, r_n\}$,

$$\begin{aligned}\int\limits_{S_n(C,R)} & f(X)g(X)\mathrm{d}\sigma(X) \\ &= \int\limits_{S_n(0,R)} f(X+C)g(X+C)\mathrm{d}\sigma(X) \\ &\geq \int\limits_{\mathcal{S}_n(0,r)} f\left(\frac{R}{r}\otimes X + C\right)g\left(\frac{R}{r}\otimes X + C\right)\mathrm{d}\sigma(X) \\ &= r^{n-1}\int\limits_{\mathcal{S}_n(0,\,1)} f(R\otimes X' + C)g(R\otimes X' + C)\mathrm{d}\sigma(X') \\ &= r^{n-1}\int\limits_{\mathcal{S}_n(0,\,1)} f(\widetilde{X})g(\widetilde{X})\mathrm{d}\sigma(X')\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}\int\limits_{S_n(C,R)} f(X)|X-C|^2\mathrm{d}\sigma(X) &\leq \int\limits_{\mathcal{S}_n(0,\tilde{r})} f\left(\frac{R}{\tilde{r}}\otimes X + C\right)\left|\frac{R}{\tilde{r}}\otimes X'\right|^2 \mathrm{d}\sigma(X') \\ &= \tilde{r}^{n-1}\int\limits_{\mathcal{S}_n(0,\,1)} f(\widetilde{X})|R\otimes X'|^2\mathrm{d}\sigma(X'),\end{aligned} \tag{2.24}$$

$$\int\limits_{S_n(C,R)} g(X)|X-C|^2\mathrm{d}\sigma(X) \leq \tilde{r}^{n-1}\int\limits_{\mathcal{S}_n(0,\,1)} g(\widetilde{X})|R\otimes X'|^2\mathrm{d}\sigma(X'). \tag{2.25}$$

Thus, the proof of Theorem 1 is completed by (2.5), (2.23), (2.24) and (2.25). □

Remark. By checking the proof of the preceding theorem, it is not difficult to see that Theorem 1 remains valid if we replace the symmetry of the function g by f.

It is easy to check that $\mathfrak{C}(R) = \widetilde{\mathfrak{C}}(R) \equiv 1$ and the last inequality in Theorem 1 becomes equality if ellipsoids reduce to balls. As a consequence, we immediately have

Theorem 2. *Let $f \in SX(h_1, \lambda_1, B_n(C,r))$, $g \in SX(h_2, \lambda_2, B_n(C,r))$ be both nonnegative functions with $0 < \lambda_1, \lambda_2 < \infty$ and g be symmetric about the center C. Then,*

$$\frac{1}{4h_1(1/2)h_2(1/2)}\left[f(C)g(C) + (\lambda_2 f(C) + \lambda_1 g(C))\frac{nr^2}{n+2} + \lambda_1\lambda_2\frac{nr^4}{n+4}\right] \leq \frac{1}{|B_n(C,r)|}\int\limits_{B_n(C,r)} f(X)g(X)\mathrm{d}X.$$

Furthermore, if h_1 and h_2 satisfy the same assumptions as in Theorem 1, then we have

$$\begin{aligned}\frac{1}{|B_n(C,r)|}&\int\limits_{B_n(C,r)} f(X)g(X)\mathrm{d}X \\ \leq\ & \frac{\mathcal{K}_1(n)}{\mathcal{K}_0(n)}\frac{1}{|\mathcal{S}_n(C,r)|}\int\limits_{\mathcal{S}_n(C,r)} f(X)g(X)\mathrm{d}\sigma(X) \\ &- \lambda_2\frac{\mathcal{K}_2(n)}{\mathcal{K}_0(n)}\frac{r^2}{|\mathcal{S}_n(C,r)|}\int\limits_{\mathcal{S}_n(C,r)} f(X)\mathrm{d}\sigma(X) - \lambda_1\frac{\mathcal{K}_3(n)}{\mathcal{K}_0(n)}\frac{r^2}{|\mathcal{S}_n(C,r)|}\int\limits_{\mathcal{S}_n(C,r)} g(X)\mathrm{d}\sigma(X) \\ &- \lambda_2\frac{\mathcal{K}_4(n)}{\mathcal{K}_0(n)}\frac{nr^2}{n+2}f(C) - \lambda_1\frac{\mathcal{K}_5(n)}{\mathcal{K}_0(n)}\frac{nr^2}{n+2}g(C) - \lambda_1\lambda_2\frac{\mathcal{K}_6(n)}{\mathcal{K}_0(n)}\frac{nr^4}{n+4},\end{aligned}$$

where $\mathcal{K}_0(n)$–$\mathcal{K}_6(n)$ and $\widetilde{X}$ are defined in Theorem 1.

Particularly, if letting $\lambda_2 \to 0$, i.e., the function g reduces to the h-convex function in Theorems 1 and 2, then we have the following concise results.

Theorem 3. *Let $f, g : E_n(C, R) \to [0, +\infty)$ and $f \in SX(h_1, \lambda_1, E_n(C, R))$, g be an h_2-convex function and be symmetric about the center C. Then,*

$$\frac{1}{4h_1(1/2)h_2(1/2)}\left[f(C)g(C) + \lambda_1 g(C)\frac{|R|^2}{n+2}\right] \leq \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)g(X)\mathrm{d}X.$$

Furthermore, if h_1 and h_2 satisfy the same assumptions as in Theorem 1, then we have

$$\begin{aligned}
&\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)g(X)\mathrm{d}X \\
&\leq \frac{\mathcal{K}_1(n)}{\mathcal{K}_0(n)}\frac{1}{|\mathcal{S}_n(0,1)|}\int_{\mathcal{S}_n(0,1)} f(\widetilde{X})g(\widetilde{X})\mathrm{d}\sigma(X') \\
&\quad - \lambda_1\frac{\mathcal{K}_3(n)}{\mathcal{K}_0(n)}\frac{1}{|\mathcal{S}_n(0,1)|}\int_{\mathcal{S}_n(0,1)} g(\widetilde{X})|R \otimes X'|^2\mathrm{d}\sigma(X') - \lambda_1\frac{\mathcal{K}_5(n)}{\mathcal{K}_0(n)}\frac{|R|^2}{n+2}g(C) \\
&\leq \frac{\mathcal{K}_1(n)}{\mathcal{K}_0(n)}\frac{\mathfrak{C}(R)}{|S_n(C,R)|}\int_{S_n(C,R)} f(X)g(X)\mathrm{d}\sigma(X) \\
&\quad - \lambda_1\frac{\mathcal{K}_3(n)}{\mathcal{K}_0(n)}\frac{\widetilde{\mathfrak{C}}(R)}{|S_n(C,R)|}\int_{S_n(C,R)} g(X)|X - C|^2\mathrm{d}\sigma(X) - \lambda_1\frac{\mathcal{K}_5(n)}{\mathcal{K}_0(n)}\frac{|R|^2}{n+2}g(C),
\end{aligned}$$

where $\mathcal{K}_0(n), \mathcal{K}_1(n), \mathcal{K}_3(n), \mathcal{K}_5(n), \widetilde{X}, \mathfrak{C}(R)$, and $\widetilde{\mathfrak{C}}(R)$ are defined in Theorem 1.

Theorem 4. *Let $f, g : B_n(C, r) \to [0, +\infty)$ and $f \in SX(h_1, \lambda_1, B_n(C, r))$, g be an h_2-convex function and be symmetric about the center C. Then,*

$$\frac{1}{4h_1(1/2)h_2(1/2)}\left[f(C)g(C) + \lambda_1 g(C)\frac{nr^2}{n+2}\right] \leq \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)g(X)\mathrm{d}X.$$

Furthermore, if h_1 and h_2 satisfy the same assumptions as in Theorem 1, then we have

$$\begin{aligned}
&\frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)g(X)\mathrm{d}X \\
&\leq \frac{\mathcal{K}_1(n)}{\mathcal{K}_0(n)}\frac{1}{|\mathcal{S}_n(C,r)|}\int_{\mathcal{S}_n(C,r)} f(X)g(X)\mathrm{d}\sigma(X) \\
&\quad - \lambda_1\frac{\mathcal{K}_3(n)}{\mathcal{K}_0(n)}\frac{r^2}{|\mathcal{S}_n(C,r)|}\int_{\mathcal{S}_n(C,r)} g(X)\mathrm{d}\sigma(X) - \lambda_1\frac{\mathcal{K}_5(n)}{\mathcal{K}_0(n)}\frac{nr^2}{n+2}g(C),
\end{aligned}$$

where $\mathcal{K}_0(n), \mathcal{K}_1(n), \mathcal{K}_3(n)$, and $\mathcal{K}_5(n)$ are defined in Theorem 1.

Subsequently, taking $h_1(t) = h_2(t) = h(t)$ and $\lambda_1 = \lambda_2 = \lambda$ in Theorems 1 and 2, we immediately yield the following two conclusions.

Corollary 1. *Let $f, g : E_n(C, R) \to [0, +\infty)$ be both strongly h-convex functions with modulus $\lambda > 0$ and g be symmetric about the center C. Then, we have the same results as Theorem 1 with $\lambda_1 = \lambda_2 = \lambda$ and*

$$\mathcal{K}_0(n) = 1 - 4h^2(1/2)n\int_0^1 t^{n-1}h^2(1-t)\mathrm{d}t > 0, \tag{2.26}$$

$$\mathcal{K}_1(n) = n\int_0^1 t^{n-1}h^2(t)\mathrm{d}t + 4nh(1/2)\int_0^1 t^{n-1}h(t)h(1-t)\mathrm{d}t, \tag{2.27}$$

$$\mathcal{K}_2(n) = \mathcal{K}_3(n) = n\int_0^1 t^{n-1}h(t)h(1-t)\mathrm{d}t + n\int_0^1 t^n(1-t)h(t)\mathrm{d}t, \tag{2.28}$$

$$\mathcal{K}_4(n) = \mathcal{K}_5(n) = (n+2)\int_0^1 t^n(1-t)h(1-t)\mathrm{d}t + n\int_0^1 t^{n-1}h^2(1-t)\mathrm{d}t, \tag{2.29}$$

$$\mathcal{K}_6(n) = n\int_0^1 t^{n-1}h^2(1-t)\mathrm{d}t - \frac{2}{(n+2)(n+3)}. \tag{2.30}$$

Corollary 2. *Let $f, g : B_n(C, r) \to [0, +\infty)$ be both strongly h-convex functions with modulus $\lambda > 0$ on the ball $B_n(C, r)$ and g be symmetric about the center C. Then, we have the same estimates as in Theorem 2 with $\mathcal{K}_0(n)$–$\mathcal{K}_6(n)$ being stated in Corollary 1.*

Particulary, letting $\lambda \to 0$ in the aforementioned two corollaries, we have the following results for product of h-convex functions.

Corollary 3. *Let $f, g : E_n(C, R) \to [0, +\infty)$ be both h-convex functions on $E_n(C, R)$ and g be symmetric about the center C. Then,*

$$\frac{f(C)g(C)}{4h^2(1/2)} \le \frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)g(X)\mathrm{d}X.$$

If h satisfies (2.26), *then*

$$\frac{1}{|E_n(C,R)|}\int_{E_n(C,R)} f(X)g(X)\mathrm{d}X \le \frac{\mathcal{K}_1(n)}{\mathcal{K}_0(n)}\frac{1}{|\mathcal{S}_n(0,1)|}\int_{\mathcal{S}_n(0,1)} f(\widetilde{X})g(\widetilde{X})\mathrm{d}\sigma(X')$$
$$\le \frac{\mathcal{K}_1(n)}{\mathcal{K}_0(n)}\frac{\mathfrak{C}(R)}{|S_n(C,R)|}\int_{S_n(C,R)} f(X)g(X)\mathrm{d}\sigma(X),$$

where $\mathcal{K}_0(n)$, $\mathcal{K}_1(n)$, $\widetilde{X}$, and $\mathfrak{C}(R)$ are defined in Corollary 1.

Corollary 4. *Let $f, g : B_n(C, r) \to [0, +\infty)$ be both h-convex functions on $B_n(C, r)$ and g be symmetric about the center C. Then,*

$$\frac{f(C)g(C)}{4h^2(1/2)} \le \frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)g(X)\mathrm{d}X.$$

And if h satisfies (2.26), *we have*

$$\frac{1}{|B_n(C,r)|}\int_{B_n(C,r)} f(X)g(X)\mathrm{d}X \le \frac{\mathcal{K}_1(n)}{\mathcal{K}_0(n)}\frac{1}{|\mathcal{S}_n(C,r)|}\int_{\mathcal{S}_n(C,r)} f(X)g(X)\mathrm{d}\sigma(X),$$

where $\mathcal{K}_0(n)$ and $\mathcal{K}_1(n)$ are defined in Corollary 1.

Furthermore, taking $h(t) = t^s (0 < s < \infty)$ in (2.26)–(2.30), by iteration properties of beta functions, a direct calculation shows that

$$\mathcal{K}_0(n) = 1 - 4h^2(1/2)n\int_0^1 t^{n-1}h^2(1-t)\mathrm{d}t = 1 - \frac{4n}{2^{2s}}\int_0^1 t^{n-1}(1-t)^{2s}\mathrm{d}t = 1 - \frac{4^{1-s}n!}{\prod_{i=1}^n(2s+i)}, \tag{2.31}$$

which means that $\mathcal{K}_0(n) > 0$ holds for $h(t) = t^s (0 < s < \infty)$ is equivalent to

$$4^s \prod_{i=1}^{n} (2s + i) > 4n!, \tag{2.32}$$

and, it is not difficult to check that

$$\mathcal{K}_1(n) = \frac{2n\prod_{i=1}^{n}(s + i - 1)}{2^s\prod_{i=1}^{n}(2s + i)} B(s, s) + \frac{n}{2s + n}, \tag{2.33}$$

$$\mathcal{K}_2(n) = \mathcal{K}_3(n) = \frac{n\prod_{i=1}^{n}(s + i - 1)}{2\prod_{i=1}^{n}(2s + i)} B(s, s) + \frac{n}{(n + s + 1)(n + s + 2)}, \tag{2.34}$$

$$\mathcal{K}_4(n) = \mathcal{K}_5(n) = \frac{(n + 2)n!}{\prod_{i=0}^{n}(s + 2 + i)} + \frac{n!}{\prod_{i=1}^{n}(2s + i)}, \tag{2.35}$$

$$\mathcal{K}_6(n) = \frac{n!}{\prod_{i=1}^{n}(2s + i)} - \frac{2}{(n + 2)(n + 3)}. \tag{2.36}$$

Then, according to Corollaries 1–4, respectively, these facts yield the following two results.

Corollary 5. *Let $f, g : E_n(C, R) \to [0, +\infty)$ be both strongly s-convex functions (in the second sense, $0 < s < 1$) with modulus $\lambda > 0$ and g be symmetric about the center C. If inequality (2.32) holds, then we have the same results as Corollary 1 with $h(t) = t^s$ and $\mathcal{K}_0(n)$–$\mathcal{K}_6(n)$ being defined by (2.31) and (2.33)–(2.36), respectively.*

Particularly, if the ellipsoid $E_n(C, R)$ reduces to the ball $B_n(C, r)$, then we have the same results as Corollary 2, where $h(t) = t^s$ and $\mathcal{K}_0(n)$–$\mathcal{K}_6(n)$ are also defined by (2.31) and (2.33)–(2.36).

Corollary 6. *Let $f, g : E_n(C, R) \to [0, +\infty)$ be both s-convex functions (in the second sense, $0 < s < 1$) and g be symmetric about the center C. If inequality (2.32) holds, then we have the same results as Corollary 3 with $h(t) = t^s$ and $\mathcal{K}_0(n)$, $\mathcal{K}_1(n)$ being defined by (2.31) and (2.33), respectively.*

Particularly, if the ellipsoid $E_n(C, R)$ reduces to the ball $B_n(C, r)$, then we have the same results as Corollary 4, where $h(t) = t^s$ and $\mathcal{K}_0(n)$, $\mathcal{K}_1(n)$ are also defined by (2.31) and (2.33).

Additionally, choosing $s = 1$ in (2.31) and (2.33)–(2.36), inequality (2.32) holds obviously and

$$\mathcal{K}_0(n) = \mathcal{K}_1(n) = \frac{n(n + 3)}{(n + 1)(n + 2)}, \quad \mathcal{K}_2(n) = \mathcal{K}_3(n) = \frac{2n}{(n + 1)(n + 3)}, \tag{2.37}$$

$$\mathcal{K}_4(n) = \mathcal{K}_5(n) = \frac{2(2n + 5)}{(n + 1)(n + 2)(n + 3)}, \quad \mathcal{K}_6(n) = \frac{4}{(n + 1)(n + 2)(n + 3)}, \tag{2.38}$$

and

$$\frac{\mathcal{K}_1(n)}{\mathcal{K}_0(n)} = 1, \quad \frac{\mathcal{K}_2(n)}{\mathcal{K}_0(n)} = \frac{\mathcal{K}_3(n)}{\mathcal{K}_0(n)} = \frac{2(n + 2)}{(n + 3)^2}, \tag{2.39}$$

$$\frac{\mathcal{K}_4(n)}{\mathcal{K}_0(n)} = \frac{\mathcal{K}_5(n)}{\mathcal{K}_0(n)} = \frac{2(2n + 5)}{n(n + 3)^2}, \quad \frac{\mathcal{K}_6(n)}{\mathcal{K}_0(n)} = \frac{4}{n(n + 3)^2}. \tag{2.40}$$

Using (2.37)–(2.40), we obtain more explicit results for strongly convex functions and convex functions as follows.

Corollary 7. *Let $f, g : E_n(C,R) \to [0,+\infty)$ be both strongly convex functions with modulus λ and g be symmetric about the center C. Then,*

$$\begin{aligned}
& f(C)g(C) + \lambda\frac{|R|^2}{n+2}(f(C)+g(C)) + \lambda^2\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)} \\
\le\ & \frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(X)g(X)\mathrm{d}X \\
\le\ & \frac{1}{|\delta_n(0,1)|}\int\limits_{\delta_n(0,1)} f(\widetilde{X})g(\widetilde{X})\mathrm{d}\sigma(X') \\
& - \lambda\frac{2(n+2)}{(n+3)^2}\frac{1}{|\delta_n(0,1)|}\int\limits_{\delta_n(0,1)|} [f(\widetilde{X})+g(\widetilde{X})]|R\otimes X'|^2\mathrm{d}\sigma(X') \\
& - \lambda\frac{2(2n+5)|R|^2}{n(n+2)(n+3)^2}(f(C)+g(C)) - \lambda^2\frac{4(|R|^4 + 2\sum_{i=1}^n r_i^4)}{n(n+2)(n+3)^2(n+4)} \\
\le\ & \frac{\mathfrak{C}(R)}{|S_n(C,R)|}\int\limits_{S_n(C,R)} f(X)g(X)\mathrm{d}\sigma(X) \\
& - \lambda\frac{2(n+2)}{(n+3)^2}\frac{\widetilde{\mathfrak{C}}(R)}{|S_n(C,R)|}\int\limits_{S_n(C,R)} [f(X)+g(X)]|X-C|^2\mathrm{d}\sigma(X) \\
& - \lambda\frac{2(2n+5)|R|^2}{n(n+2)(n+3)^2}(f(C)+g(C)) - \lambda^2\frac{4(|R|^4 + 2\sum_{i=1}^n r_i^4)}{n(n+2)(n+3)^2(n+4)},
\end{aligned}$$

where $\widetilde{X}$, $\mathfrak{C}(R)$, and $\widetilde{\mathfrak{C}}(R)$ are defined in Theorem 1.

Corollary 8. *Let $f, g : B_n(C,r) \to [0,+\infty)$ be both strongly convex functions with modulus λ and g be symmetric about the center C. Then,*

$$\begin{aligned}
& f(C)g(C) + \lambda\frac{nr^2}{n+2}(f(C)+g(C)) + \lambda^2\frac{nr^4}{n+4} \\
\le\ & \frac{1}{|B_n(C,r)|}\int\limits_{B_n(C,r)} f(X)g(X)\mathrm{d}X \\
\le\ & \frac{1}{|\delta_n(C,r)|}\int\limits_{\delta_n(C,r)} f(X)g(X)\mathrm{d}\sigma(X) - \lambda\frac{2(n+2)}{(n+3)^2}\frac{r^2}{|\delta_n(C,r)|}\int\limits_{\delta_n(C,r)} [f(X)+g(X)]\mathrm{d}\sigma(X) \\
& - \lambda\frac{2(2n+5)r^2}{(n+2)(n+3)^2}(f(C)+g(C)) - \lambda^2\frac{4r^4}{(n+3)^2(n+4)}.
\end{aligned}$$

Corollary 9. *Let $f, g : E_n(C,R) \to [0,+\infty)$ be both convex functions and g be symmetric about the center C. Then,*

$$f(C)g(C) \le \frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(X)g(X)\mathrm{d}X$$

$$\le \frac{1}{|\delta_n(0,1)|}\int\limits_{\delta_n(0,1)} f(\widetilde{X})g(\widetilde{X})\mathrm{d}\sigma(X') \le \frac{\mathfrak{C}(R)}{|S_n(C,R)|}\int\limits_{S_n(C,R)} f(X)g(X)\mathrm{d}\sigma(X),$$

where $\widetilde{X}$, $\mathfrak{C}(R)$ are defined in Theorem 1.

Furthermore, if $E_n(C,R)$ reduces to $B_n(C,r)$, we have

$$f(C)g(C) \leq \frac{1}{|B_n(C,r)|}\int\limits_{B_n(C,r)} f(X)g(X)\mathrm{d}X \leq \frac{1}{|\delta_n(C,r)|}\int\limits_{\delta_n(C,r)} f(X)g(X)\mathrm{d}\sigma(X).$$

3 Some mappings related to Hermite-Hadamard-type inequalities

The second purpose in this article is to give some applications of the Hermite-Hadamard inequalities for product of strongly h-convex functions.

Theorem D. [34] *Define the mapping* $\mathfrak{H} : [0,1] \to \mathbb{R}$ *by*

$$\mathfrak{H}(t) = \frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(tX + (1-t)C)\mathrm{d}X.$$

If $f \in SX(h,\lambda,E_n(C,R))$, *then*

(*i*) *the function* $\mathfrak{H}$ *is a strongly h-convex function with modulus* $\frac{\lambda}{n+2}|R|^2$ on $[0,1]$,

(*ii*) *for any* $t \in (0,1]$,

$$\frac{1}{2h(1/2)}\left(f(C) + \lambda\frac{|R|^2t^2}{n+2}\right) \leq \mathfrak{H}(t) \leq \mathfrak{H}(1)[h(t) + 2h(1/2)h(1-t)] - \lambda\frac{|R|^2}{n+2}[h(t) + t(1-t)].$$

Next, we will extend the aforementioned theorem to product of functions as follows.

Theorem 5. *Let* $\tilde{\mathfrak{H}} : [0,1] \to [0,+\infty)$ *be defined as*

$$\tilde{\mathfrak{H}}(t) = \frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(tX + (1-t)C)g(tX + (1-t)C)\mathrm{d}X. \tag{3.1}$$

Let $f \in SX(h_1,\lambda_1,E_n(C,R))$, $g \in SX(h_2,\lambda_2,E_n(C,R))$ *be both nonnegative functions with* $0 < \lambda_1,\lambda_2 < \infty$ *and g be symmetric about the center C. Then, for any* $t \in (0,1]$,

$$\begin{aligned}
&\frac{1}{4h_1(1/2)h_2(1/2)}\left[f(C)g(C) + t^2\frac{|R|^2}{n+2}(\lambda_2 f(C) + \lambda_1 g(C)) + \lambda_1\lambda_2 t^4\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}\right]\\
&\leq \tilde{\mathfrak{H}}(t) \leq \tilde{\mathfrak{H}}(1)\mathcal{F}_1(t) - [\lambda_2 f(C)\mathcal{F}_2(t) + \lambda_1 g(C)\mathcal{F}_3(t)]\frac{|R|^2}{n+2} - \lambda_1\lambda_2\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}\mathcal{F}_4(t),
\end{aligned}$$

where

$$\mathcal{F}_1(t) = [h_1(t) + 2h_1(1/2)h_1(1-t)][h_2(t) + 2h_2(1/2)h_2(1-t)],$$

$$\mathcal{F}_2(t) = \frac{1}{2h_1(1/2)}[h_1(t) + 2h_1(1/2)h_1(1-t)][t(1-t) + h_2(1-t)],$$

$$\mathcal{F}_3(t) = \frac{1}{2h_2(1/2)}[h_2(t) + 2h_2(1/2)h_2(1-t)][t(1-t) + h_1(1-t)],$$

$$\mathcal{F}_4(t) = \frac{1}{2h_1(1/2)}h_1(t)[h_2(1-t) + t(1-t)] + \frac{1}{2h_2(1/2)}h_2(t)[h_1(1-t) + t(1-t)] + h_1(1-t)h_2(1-t) - t^2(1-t)^2.$$

Proof. For any fixed $t \in (0,1]$, taking the substitution $Y = (y_1, y_2, \ldots, y_n)$, where $y_i = tx_i + (1-t)c_i$, we have

$$\begin{aligned}\tilde{\mathfrak{H}}(t) &= \frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(tX+(1-t)C)g(tX+(1-t)C)\mathrm{d}X\\ &= \frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,tR)} f(Y)g(Y)\left|\frac{\partial(x_1,x_2,\ldots,x_n)}{\partial(y_1,y_2,\ldots,y_n)}\right|\mathrm{d}Y\\ &= \frac{1}{t^n|E_n(C,R)|}\int\limits_{E_n(C,tR)} f(Y)g(Y)\mathrm{d}Y\\ &= \frac{1}{|E_n(C,tR)|}\int\limits_{E_n(C,tR)} f(X)g(X)\mathrm{d}X \text{ On the other hand.}\end{aligned}$$

Then, (2.4) yields that

$$\tilde{\mathfrak{H}}(t) \geq \frac{1}{4h_1(1/2)h_2(1/2)}\left[f(C)g(C) + t^2\frac{|R|^2}{n+2}(\lambda_2 f(C) + \lambda_1 g(C)) + \lambda_1\lambda_2 t^4\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}\right]. \tag{3.2}$$

Thus, we obtain the first part of the inequality.

It follows from the definition of strongly h-convexity that

$$\begin{aligned}\tilde{\mathfrak{H}}(t) &\leq \frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,R)}[h_1(t)f(X) + h_1(1-t)f(C) - \lambda_1 t(1-t)|X-C|^2]\\ &\quad\times[h_2(t)g(X) + h_2(1-t)g(C) - \lambda_2 t(1-t)|X-C|^2]\mathrm{d}X\\ &= \frac{h_1(t)h_2(t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(X)g(X)\mathrm{d}X\\ &\quad+\frac{h_1(t)h_2(1-t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(X)g(C)\mathrm{d}X + \frac{h_1(1-t)h_2(t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(C)g(X)\mathrm{d}X\\ &\quad-\lambda_2\frac{t(1-t)h_1(t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(X)|X-C|^2\mathrm{d}X - \lambda_1\frac{t(1-t)h_2(t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)} g(X)|X-C|^2\mathrm{d}X\\ &\quad-\lambda_2\frac{t(1-t)h_1(1-t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)}|X-C|^2\mathrm{d}Xf(C) - \lambda_1\frac{t(1-t)h_2(1-t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)}|X-C|^2\mathrm{d}Xg(C)\\ &\quad+h_1(1-t)h_2(1-t)f(C)g(C) + \lambda_1\lambda_2\frac{t^2(1-t)^2}{|E_n(C,R)|}\int\limits_{E_n(C,R)}|X-C|^4\mathrm{d}X.\end{aligned} \tag{3.3}$$

Then, with the aid of (2.12), (2.13), (2.14), (2.15), (3.2), and (3.3), we have

$$\begin{aligned}\tilde{\mathfrak{H}}(t) &\leq h_1(t)h_2(t)\tilde{\mathfrak{H}}(1) + \frac{h_1(t)h_2(1-t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(X)g(C)\mathrm{d}X\\ &\quad+\frac{h_1(1-t)h_2(t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(C)g(X)\mathrm{d}X - \lambda_2\frac{t(1-t)h_1(t)}{2h_1(1/2)}\left[\lambda_1\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)} + \frac{|R|^2}{n+2}f(C)\right]\\ &\quad-\lambda_1\frac{t(1-t)h_2(t)}{2h_2(1/2)}\left[\lambda_2\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)} + \frac{|R|^2}{n+2}g(C)\right]\\ &\quad-\lambda_2 t(1-t)h_1(1-t)\frac{|R|^2}{n+2}f(C) - \lambda_1 t(1-t)h_2(1-t)\frac{|R|^2}{n+2}g(C)\\ &\quad+h_1(1-t)h_2(1-t)\left\{4h_1(1/2)h_2(1/2)\tilde{\mathfrak{H}}(1) - \frac{|R|^2}{n+2}[\lambda_2 f(C) + \lambda_1 g(C)]\right.\\ &\quad\left.-\lambda_1\lambda_2\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}\right\} + \lambda_1\lambda_2 t^2(1-t)^2\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&= [h_1(t)h_2(t) + 4h_1(1/2)h_2(1/2)h_1(1-t)h_2(1-t)]\tilde{\mathfrak{H}}(1) - \lambda_2\left[\frac{t(1-t)h_1(t)}{2h_1(1/2)} + t(1-t)h_1(1-t)\right.\\
&\left.+ h_1(1-t)h_2(1-t)\right]\frac{|R|^2}{n+2}f(C) - \lambda_1\left[\frac{t(1-t)h_2(t)}{2h_2(1/2)} + t(1-t)h_2(1-t)\right.\\
&\left.+ h_1(1-t)h_2(1-t)\right]\frac{|R|^2}{n+2}g(C) - \lambda_1\lambda_2\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}\\
&\times\left[\frac{t(1-t)h_1(t)}{2h_1(1/2)} + \frac{t(1-t)h_2(t)}{2h_2(1/2)} + h_1(1-t)h_2(1-t) - t^2(1-t)^2\right]\\
&+ \frac{h_1(t)h_2(1-t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(X)g(C)\mathrm{d}X + \frac{h_1(1-t)h_2(t)}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(C)g(X)\mathrm{d}X.
\end{aligned}
$$

On the other hand, inequality (2.14) shows that

$$
\begin{aligned}
&\frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(X)g(C)\mathrm{d}X\\
&\le \frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(X)[h_2(1/2)(g(X) + g(2C-X)) - \lambda_2\,|X-C|^2]\mathrm{d}X\\
&= 2h_2(1/2)\tilde{\mathfrak{H}}(1) - \lambda_2\frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(X)|X-C|^2\mathrm{d}X\\
&\le 2h_2(1/2)\tilde{\mathfrak{H}}(1) - \lambda_2\frac{1}{2h_1(1/2)}\left[\lambda_1\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)} + \frac{|R|^2}{n+2}f(C)\right],
\end{aligned}
\tag{3.5}
$$

and (2.15) means that

$$
\frac{1}{|E_n(C,R)|}\int\limits_{E_n(C,R)} f(C)g(X)\mathrm{d}X \le 2h_1(1/2)\tilde{\mathfrak{H}}(1) - \lambda_1\frac{1}{2h_2(1/2)}\left[\lambda_2\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)} + \frac{|R|^2}{n+2}g(C)\right], \tag{3.6}
$$

Thus, we complete the proof of theorem by (3.4)–(3.6). □

Remark. Letting $\lambda_2 \to 0$ and $g \equiv 1$ in Theorem 5, then the result reduces to Theorem D (ii).

As a consequence of Theorem 5,

Corollary 10. *Let $\tilde{\mathfrak{H}}(t)$ be defined by* (3.1). *If f and g are both strongly convex functions with modulus $\lambda > 0$ on the ellipsoid $E_n(C,R)$ and g is symmetric about the center C, then, for any $t \in (0,1]$,*

$$
\begin{aligned}
&f(C)g(C) + \lambda t^2\frac{|R|^2}{n+2}(f(C) + g(C)) + \lambda^2 t^4\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}\\
&\le \tilde{\mathfrak{H}}(t) \le \tilde{\mathfrak{H}}(1) - \lambda(1-t^2)\frac{|R|^2}{n+2}(f(C) + g(C)) - \lambda^2(1-t^4)\frac{|R|^4 + 2\sum_{i=1}^n r_i^4}{(n+2)(n+4)}.
\end{aligned}
$$

Corollary 11. *Let $\tilde{\mathfrak{H}}(t)$ be defined by* (3.1). *If f and g are both nonnegative h-convex functions on the ellipsoid $E_n(C,R)$ and g is symmetric about the center C, then, for any $t \in (0,1]$,*

$$
\frac{f(C)g(C)}{4h^2(1/2)} \le \tilde{\mathfrak{H}}(t) \le \tilde{\mathfrak{H}}(1)\mathcal{F}_1(t),
$$

where $\mathcal{F}_1(t)$ is defined in Theorem 5 with $h_1(t) = h_2(t) = h(t)$.

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