Research Article

Ghulam Farid, Sanja Kovač*, Josip Pečarić, and Mihaela Ribičić Penava

Some new Fejér type inequalities for $(h, g; \alpha - m)$-convex functions

https://doi.org/10.1515/math-2025-0184
received February 26, 2025; accepted June 30, 2025

Abstract: The study of $(h, g; \alpha - m)$-convex functions extends the classical concept of convexity to more generalized forms, which provide flexible tools for analysis. The aim of this article is to yield the generalization of the Fejér type inequalities for various classes of convex functions, such as $(h, g; \alpha - m)$-convex and $(h, g; m)$-convex functions.

Keywords: Fejér inequality, convex function, $(h, g; \alpha\text{-}m)$-convex function

MSC 2020: 26D10, 26D15, 26D20, 26D99

1 Introduction

Convex functions play a pivotal role in various fields of applied mathematics, such as optimization, economics, and probability theory due to their desirable mathematical properties and wide applicability. First, let us recall the definition of a convex function.

Definition 1. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex function if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \tag{1.1}$$

holds for all points x and y in I and all $\lambda \in [0, 1]$.

It is called strictly convex if the inequality (1.1) holds strictly whenever x and y are distinct points and $\lambda \in (0, 1)$. If $-f$ is convex (respectively, strictly convex), then we say that f is concave (respectively, strictly concave). If f is both convex and concave, then f is said to be affine.

Let us recall famous Fejér inequalities for convex functions.

Theorem 1. (The Fejér inequalities) *Let $w : [a, b] \to \mathbb{R}$ be nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. If $f : [a, b] \to \mathbb{R}$ is a convex function, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x \leq \int_a^b w(x)f(x)\mathrm{d}x \leq \left[\frac{1}{2}f(a) + \frac{1}{2}f(b)\right]\int_a^b w(x)\mathrm{d}x. \tag{1.2}$$

If f is a concave function, then inequalities in (1.2) *are reversed.*

*** Corresponding author: Sanja Kovač,** Faculty of Geotechnical Engineering, University of Zagreb, Hallerova aleja 7, 42 000 Varaždin, Croatia, e-mail: sanja.kovac@gfv.unizg.hr
Ghulam Farid: Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore 54000, Pakistan, e-mail: drghulamfarid@cuilahore.edu.pk
Josip Pečarić: Department of Mathematical, Physical and Chemical Sciences, Croatian Academy of Sciences and Arts, Zagreb, Croatia, e-mail: pecaric@hazu.hr
Mihaela Ribičić Penava: School of Applied Mathematics and Informatics, Josip Juraj Strossmayer University of Osijek, Trg Ljudevita Gaja 6, 31000 Osijek, Croatia, e-mail: mihaela@mathos.hr

In mathematical analysis, Fejér-type inequalities are important, especially when studying integral inequalities and their applications. These inequalities have been extensively studied and refined, providing essential tools for evaluating integral expressions and establishing bounds in both pure and applied mathematics. Many expansions and modifications in different directions are based on the classical Fejér inequality. Some new inequalities of the Fejér type were obtained in [1–3].

A new class of convex functions was introduced in [4].

Definition 2. Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function. We say that $f : [0, b] \to \mathbb{R}$ is a $(h - m)$-convex function, if f is nonnegative and for all $x, y \in [0, b]$, $m \in [0, 1]$, and $\alpha \in (0, 1)$, we have

$$f(\alpha x + m(1 - \alpha)y) \le h(\alpha)f(x) + mh(1 - \alpha)f(y). \tag{1.3}$$

If the inequality (1.3) is reversed, then f is said to be $(h - m)$-concave function on $[0, b]$.

Obviously, if we choose $m = 1$, then we have h-convex functions. If we choose $h(x) = x$, then we obtain nonnegative m-convex functions. If we choose $m = 1$ and $h(x) = x, 1, \frac{1}{x}, x^s$, then we obtain the following classes of functions: nonnegative convex functions, P-functions, Godunova-Levin functions, and s-convex functions (in the second sense), respectively.

The following new class of $(h, g; m)$-convex function has been introduced in [5].

Definition 3. Let h be a nonnegative function on $J \subset \mathbb{R}$, $(0,\ 1) \subset J$, $h \ne 0$, and let g be a positive function on $I \subset \mathbb{R}$. Furthermore, let $m \in (0,\ 1]$. A function $f : I \to \mathbb{R}$ is said to be an $(h, g; m)$-convex function if it is nonnegative and if

$$f(\lambda x + m(1 - \lambda)y) \le h(\lambda)f(x)g(x) + mh(1 - \lambda)f(y)g(y) \tag{1.4}$$

holds for all $x, y \in I$ and all $\lambda \in (0,\ 1)$. If (1.4) holds in the reversed sense, then f is said to be an $(h, g; m)$-concave function.

The second Fejér inequality has been improved for $(h, g; m)$-convex functions [6,7]:

Theorem 2. *Let f be a nonnegative $(h, g; m)$-convex function on $[0, \infty)$, where h is a nonnegative function on $J \subset \mathbb{R}$, $h \ne 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. Let $0 \le a < b < \infty$ and $f, g, h \in L_1[a, b]$. Furthermore, let $w : [a, b] \to \mathbb{R}$ be a nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. Then the following inequality holds*

$$\int_a^b f(x)w(x)\mathrm{d}x \le \frac{1}{2}\left[f(a)g(a) + f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right] \cdot \int_a^b h\left(\frac{x - a}{b - a}\right)w(x)\mathrm{d}x. \tag{1.5}$$

In the article, we will use the following property of the minimum:

$$\min\{a, b\} = \frac{a + b - |a - b|}{2}, \tag{1.6}$$

from which we have

$$\min\{a, b\} \le \frac{a + b}{2}. \tag{1.7}$$

Power mean of two numbers a, b is

$$M_p(a, b) = \left(\frac{a^p + b^p}{2}\right)^{1/p}.$$

Specially, for $p = 1$ we have arithmetic mean

$$M_1(a, b) = \frac{a + b}{2},$$

for $p = -1$ harmonic mean

$$M_{-1}(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}},$$

for $p = 2$ quadratic mean

$$M_2(a, b) = \sqrt{\frac{a^2 + b^2}{2}},$$

for $p = 0$ geometric mean

$$M_0(a, b) = \sqrt{ab},$$

for $p = -\infty$ minimum

$$M_{-\infty}(a, b) = \min\{a, b\}$$

and for $p = \infty$ maximum

$$M_{\infty}(a, b) = \max\{a, b\}.$$

The following inequalities are valid:

$$M_{-\infty}(a, b) \le M_p(a, b) \le M_{\infty}(a, b) \tag{1.8}$$

and

$$M_p(a, b) \le M_q(a, b), \quad \text{for } p < q. \tag{1.9}$$

The two means $M_p(a, b)$ and $M_q(a, b)$ are equal if and only if $a = b$.

This article aims to present the improvement of the second Fejér inequality for $(h, g; m)$-convex and $(h, g; \alpha - m)$-convex functions. Likewise, we will show that these new results are generalizations of already known results related to various classes of convex functions, such as $(h - m)$-convex, h-convex, m-convex, and nonnegative convex functions.

2 Main result

Here is improvement of the second Fejér inequality for $(h, g; m)$-convex function obtained in [7].

Theorem 3. *Let f be a nonnegative $(h, g; m)$-convex function on $[0, \infty)$, where h is a nonnegative function on $J \subset \mathbb{R}$, $h \neq 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. Let $0 \le a < b < \infty$ and $f, g, h \in L_1[a, b]$. Furthermore, let $w : [a, b] \to \mathbb{R}$ be a nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. Then the following inequality holds*

$$\int_a^b f(x)w(x)\mathrm{d}x \le \min\left\{f(a)g(a) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right), f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right\} \cdot \int_a^b h\left(\frac{x - a}{b - a}\right)w(x)\mathrm{d}x. \tag{2.1}$$

If f is nonnegative $(h, g; m)$-concave function, then the following inequality holds:

$$\int_a^b f(x)w(x)\mathrm{d}x \ge \max\left\{f(a)g(a) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right), f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right\} \cdot \int_a^b h\left(\frac{x - a}{b - a}\right)w(x)\mathrm{d}x. \tag{2.2}$$

Proof. Let f be a nonnegative $(h, g; m)$-convex function on $[0, \infty)$. First, we have

$$f(x) = f\left(\frac{b-x}{b-a}a + m\frac{x-a}{b-a}\frac{b}{m}\right)$$

and

$$f(a+b-x) = f\left(\frac{b-x}{b-a}b + m\frac{x-a}{b-a}\frac{a}{m}\right).$$

From the fact that f is $(h, g; m)$-convex function, we have

$$f\left(\frac{b-x}{b-a}a + m\frac{x-a}{b-a}\frac{b}{m}\right) \le h\left(\frac{b-x}{b-a}\right)f(a)g(a) + mh\left(\frac{x-a}{b-a}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right) \tag{2.3}$$

and

$$f\left(\frac{b-x}{b-a}b + m\frac{x-a}{b-a}\frac{a}{m}\right) \le h\left(\frac{b-x}{b-a}\right)f(b)g(b) + mh\left(\frac{x-a}{b-a}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right). \tag{2.4}$$

Integrating on $[a, b]$, we have

$$\begin{aligned}\int_a^b f(x)w(x)\mathrm{d}x &= \int_a^b f\left(\frac{b-x}{b-a}a + m\frac{x-a}{b-a}\frac{b}{m}\right)w(x)\mathrm{d}x \\ &\le \int_a^b \left[h\left(\frac{b-x}{b-a}\right)f(a)g(a) + mh\left(\frac{x-a}{b-a}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]w(x)\mathrm{d}x \\ &= f(a)g(a)\int_a^b h\left(\frac{b-x}{b-a}\right)w(x)\mathrm{d}x + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\int_a^b f(x)w(x)\mathrm{d}x &= \int_a^b f\left(\frac{b-x}{b-a}b + m\frac{x-a}{b-a}\frac{a}{m}\right)w(x)\mathrm{d}x \\ &\le \int_a^b \left[h\left(\frac{b-x}{b-a}\right)f(b)g(b) + mh\left(\frac{x-a}{b-a}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right]w(x)\mathrm{d}x \\ &= f(b)g(b)\int_a^b h\left(\frac{b-x}{b-a}\right)w(x)\mathrm{d}x + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x.\end{aligned} \tag{2.6}$$

Since w is symmetric, it is easy to check that

$$\int_a^b h\left(\frac{b-x}{b-a}\right)w(x)\mathrm{d}x = \int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x.$$

Hence, we have

$$\int_a^b f(x)w(x)\mathrm{d}x \le \left[f(a)g(a) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\cdot\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x \tag{2.7}$$

and

$$\int_a^b f(x)w(x)\mathrm{d}x \le \left[f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right]\cdot\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x, \tag{2.8}$$

so the inequality (2.1) is proved.

Now, let us prove the inequality (2.2). For $(h, g; m)$-concave function f, the following inequality holds:

$$f(\lambda x + m(1-\lambda)y) \geq h(\lambda)f(x)g(x) + mh(1-\lambda)f(y)g(y), \quad \forall \lambda \in (0,1). \tag{2.9}$$

Further, we use the substitutions:

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b \tag{2.10}$$

and

$$a + b - x = \frac{b-x}{b-a}b + \frac{x-a}{b-a}a \tag{2.11}$$

to express $f(x)$ and $f(a+b-x)$ in forms suitable for applying the inequality (2.9). Hence, we obtain:

$$f(x) \geq h\left(\frac{b-x}{b-a}\right)f(a)g(a) + mh\left(\frac{x-a}{b-a}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)$$

and

$$f(a+b-x) \geq h\left(\frac{b-x}{b-a}\right)f(b)g(b) + mh\left(\frac{x-a}{b-a}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right).$$

By integrating both inequalities over $[a, b]$ and using the symmetry of w, we obtain:

$$\int_a^b f(x)w(x)\mathrm{d}x \geq \left[f(a)g(a) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right]\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x,$$
$$\int_a^b f(x)w(x)\mathrm{d}x \geq \left[f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right]\int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x.$$

Therefore, inequality (2.2) follows by taking the maximum of the two lower bounds. □

Remark 1. If we use (1.7), we obtain:

$$\begin{aligned}&\min\left\{f(a)g(a) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right), f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right\}\\ &\leq \frac{1}{2}\left[f(a)g(a) + f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right],\end{aligned} \tag{2.12}$$

so the Theorem 3 is the improvement of the Theorem 2.

Corollary 1. *Let f be a nonnegative $(h, g; m)$-concave function on $[0, \infty)$, where h is a nonnegative function on $J \subset \mathbb{R}$, $h \neq 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. Let $0 \leq a < b < \infty$ and $f, g, h \in L_1[a, b]$. Furthermore, let $w : [a, b] \to \mathbb{R}$ be a nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. Then the following inequality holds*

$$\begin{aligned}&\int_a^b f(x)w(x)\mathrm{d}x\\ &\geq \frac{1}{2}\left[f(a)g(a) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right), f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right] \cdot \int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x.\end{aligned} \tag{2.13}$$

Proof. We use inequality (1.8) with $p = 1$ to obtain

$$\begin{aligned}&\max\left\{f(a)g(a) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right), f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\right\}\\ &\geq \frac{1}{2}\left[f(a)g(a) + f(b)g(b) + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right].\end{aligned} \tag{2.14}$$

□

Now we shall give the special cases of the Theorem 3 by using special classes of convex functions.

Corollary 2. *Let f be a nonnegative $(h - m)$-convex function on $[0, \infty)$, where h is a nonnegative function on $J \subset \mathbb{R}$, $h \neq 0$ and $m \in (0, 1]$. Let $0 \leq a < b < \infty$ and $f, h \in L_1[a, b]$. Furthermore, let $w : [a, b] \to \mathbb{R}$ be a nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. Then the following inequality holds*

$$\int_a^b f(x)w(x)\mathrm{d}x \leq \min\left\{f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right)\right\} \cdot \int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x. \tag{2.15}$$

If f is nonnegative $(h - m)$-concave function, then the following inequality holds:

$$\int_a^b f(x)w(x)\mathrm{d}x \geq \max\left\{f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right)\right\} \cdot \int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x. \tag{2.16}$$

Proof. We apply Theorem 3 for $g(x) = 1$ to obtain the result. □

Corollary 3. *Let f be a nonnegative h-convex function on $[0, \infty)$, where h is a nonnegative function on $J \subset \mathbb{R}$, $h \neq 0$. Let $0 \leq a < b < \infty$ and $f, h \in L_1[a, b]$. Furthermore, let $w : [a, b] \to \mathbb{R}$ be a nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. Then the following inequality holds*

$$\int_a^b f(x)w(x)\mathrm{d}x \leq (f(a) + f(b)) \cdot \int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x. \tag{2.17}$$

If f is nonnegative h-concave, then the following inequality holds:

$$\int_a^b f(x)w(x)\mathrm{d}x \leq (f(a) + f(b)) \cdot \int_a^b h\left(\frac{x-a}{b-a}\right)w(x)\mathrm{d}x. \tag{2.18}$$

Proof. We apply Theorem 3 for $m = 1$ and $g = 1$ to obtain the result. □

Corollary 4. *Let f be a nonnegative m-convex function on $[0, \infty)$, for $m \in (0, 1]$. Let $0 \leq a < b < \infty$ and $f \in L_1[a, b]$. Furthermore, let $w : [a, b] \to \mathbb{R}$ be a nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. Then the following inequality holds*

$$\int_a^b f(x)w(x)\mathrm{d}x \leq \frac{1}{2}\min\left\{f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right)\right\} \cdot \int_a^b \frac{x-a}{b-a}w(x)\mathrm{d}x. \tag{2.19}$$

If f is nonnegative m-concave function, then the following inequality holds:

$$\int_a^b f(x)w(x)\mathrm{d}x \geq \frac{1}{2}\max\left\{f(a) + mf\left(\frac{b}{m}\right), f(b) + mf\left(\frac{a}{m}\right)\right\} \cdot \int_a^b \frac{x-a}{b-a}w(x)\mathrm{d}x. \tag{2.20}$$

Proof. We apply Theorem 3 for $h(x) = x$ and $g = 1$ to obtain the result. □

The following corollary is the second Fejér inequality for nonnegative convex and concave functions.

Corollary 5. *Let f be a nonnegative convex function on* $[0,\infty)$. *Let* $0 \le a < b < \infty$ *and* $f \in L_1[a,b]$. *Furthermore, let* $w:[a,b]\to\mathbb{R}$ *be a nonnegative, integrable, and symmetric about* $\frac{a+b}{2}$. *Then the following inequality holds*

$$\int_a^b f(x)w(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}\cdot\int_a^b w(x)\mathrm{d}x. \tag{2.21}$$

If f is nonnegative concave function on $[0,\infty)$, *then the following inequality holds*:

$$\int_a^b f(x)w(x)\mathrm{d}x \ge \frac{f(a)+f(b)}{2}\cdot\int_a^b w(x)\mathrm{d}x. \tag{2.22}$$

Proof. We apply Theorem 3 for $m=1$, $h(x)=x$ and $g=1$ to obtain the result. □

3 Further generalization to the $(h,g;\alpha-m)$-convex functions

In this section, we shall give the further generalization of the previous results to the class of the $(h,g;\alpha-m)$-convex functions (see [8], Definition 2.1. by setting $c=0$).

Definition 4. Let h be a nonnegative function on $J\subset\mathbb{R}$, $(0,1)\subset J$, $h\ne 0$ and let g be a positive function on $I\subset\mathbb{R}$ and $\alpha, m\in(0,1]$. A function $f:I\to\mathbb{R}$ is said to be $(h,g;\alpha-m)$-convex if it is nonnegative and satisfies the following inequality:

$$f(\lambda x+m(1-\lambda)y)\le h(\lambda^\alpha)f(x)g(x)+mh(1-\lambda^\alpha)f(y)g(y) \tag{3.1}$$

for all $\lambda\in(0,1)$ and all $x,y\in I$.

If we rewrite the aforementioned definition in the following form:

$$f(x)\le h\left(\left(\frac{b-x}{b-a}\right)^\alpha\right)f(a)g(a)+m\cdot h\left(1-\left(\frac{b-x}{b-a}\right)^\alpha\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right) \tag{3.2}$$

and

$$f(x)\le h\left(\left(\frac{x-a}{b-a}\right)^\alpha\right)f(b)g(b)+m\cdot h\left(1-\left(\frac{x-a}{b-a}\right)^\alpha\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right), \tag{3.3}$$

then we have

$$\begin{aligned} f(x)\le\min\Bigg\{&h\left(\left(\frac{b-x}{b-a}\right)^\alpha\right)f(a)g(a)+m\cdot h\left(1-\left(\frac{b-x}{b-a}\right)^\alpha\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right),\\ &h\left(\left(\frac{x-a}{b-a}\right)^\alpha\right)f(b)g(b)+m\cdot h\left(1-\left(\frac{x-a}{b-a}\right)^\alpha\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\Bigg\}. \end{aligned} \tag{3.4}$$

Theorem 4. *Let f be a nonnegative* $(h,g;\alpha-m)$*-convex function on* $[0,\infty)$, *where h is a nonnegative function on* $J\subset\mathbb{R}$, $h\ne 0$, *g is a positive function on* $[0,\infty)$ *and* $\alpha, m\in(0,1]$. *Let* $0\le a<b<\infty$ *and* $f,g,h\in L_1[a,b]$. *Furthermore, let* $w:[a,b]\to\mathbb{R}$ *be a nonnegative, integrable, and symmetric about* $\frac{a+b}{2}$. *Then the following inequality holds*

$$\int_a^b f(x)w(x)\mathrm{d}x\le\min\left\{f(a)g(a)I_1+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)I_2, f(b)g(b)I_1+mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)I_2\right\}, \tag{3.5}$$

where

$$I_1=\int_a^b h\left(\left(\frac{b-x}{b-a}\right)^{\alpha}\right)w(x)\mathrm{d}x$$

and

$$I_2=\int_a^b h\left(1-\left(\frac{b-x}{b-a}\right)^{\alpha}\right)w(x)\mathrm{d}x.$$

For a nonnegative $(h,g;\alpha-m)$-concave function f, the inequality in (3.5) *is reversed by using the maximum instead of the minimum.*

Proof. Let f be a nonnegative $(h,g;\alpha-m)$-convex function on $[0,\infty)$. Then we have:

$$\begin{aligned}f(x)\le \min\Bigg\{&h\left(\left(\frac{b-x}{b-a}\right)^{\alpha}\right)f(a)g(a)+m\cdot h\left(1-\left(\frac{b-x}{b-a}\right)^{\alpha}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right),\\ &h\left(\left(\frac{x-a}{b-a}\right)^{\alpha}\right)f(b)g(b)+m\cdot h\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\Bigg\}.\end{aligned}\tag{3.6}$$

Now, since w is symmetric with respect to $\frac{a+b}{2}$, we have

$$I_1=\int_a^b h\left(\left(\frac{b-x}{b-a}\right)^{\alpha}\right)w(x)\mathrm{d}x=\int_a^b h\left(\left(\frac{x-a}{b-a}\right)^{\alpha}\right)w(x)\mathrm{d}x\tag{3.7}$$

and

$$I_2=\int_a^b h\left(1-\left(\frac{b-x}{b-a}\right)^{\alpha}\right)w(x)\mathrm{d}x=\int_a^b h\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)w(x)\mathrm{d}x,\tag{3.8}$$

so by using (3.6), we obtain

$$\begin{aligned}\int_a^b f(x)w(x)\mathrm{d}x &\le \int_a^b \min\Bigg\{h\left(\left(\frac{b-x}{b-a}\right)^{\alpha}\right)f(a)g(a)+m\cdot h\left(1-\left(\frac{b-x}{b-a}\right)^{\alpha}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right),\\ &\qquad h\left(\left(\frac{x-a}{b-a}\right)^{\alpha}\right)f(b)g(b)+m\cdot h\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\Bigg\}w(x)\mathrm{d}x\\ &\le \min\left\{f(a)g(a)I_1+mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)I_2, f(b)g(b)I_1+mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)I_2\right\},\end{aligned}\tag{3.9}$$

which completes the proof of inequality (3.5).

Finally, if f is a nonnegative $(h,g;\alpha-m)$-concave function, the inequality presented in (3.6) is reversed with the maximum substituting the minimum. Consequently, the inequality in (3.5) is also reversed with the maximum replacing the minimum. □

Remark 2. If we put $\alpha=1$ in Theorem 4, then $(h,g;\alpha-m)$-convexity reduces to $(h,g;m)$-convexity, so the inequality from Theorem 3 is obtained.

Corollary 6. *Let f be a nonnegative $(h,g;\alpha-m)$-convex function on $[0,\infty)$, where h is a nonnegative function on $J\subset\mathbb{R}$, $h\neq 0$, g is a positive function on $[0,\infty)$ and $\alpha,m\in(0,1]$. Let $0\le a<b<\infty$ and $f,g,h\in L_1[a,b]$. Furthermore, let $w:[a,b]\to\mathbb{R}$ be a nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. Then the following inequality holds*

$$\int_a^b f(x)w(x)\mathrm{d}x\le\frac{1}{2}\left[(f(a)g(a)+f(b)g(b))I_1+m\left(f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right)I_2\right].\tag{3.10}$$

where

$$I_1 = \int_a^b h\left(\left(\frac{b-x}{b-a}\right)^{\alpha}\right) w(x)\mathrm{d}x$$

and

$$I_2 = \int_a^b h\left(1 - \left(\frac{b-x}{b-a}\right)^{\alpha}\right) w(x)\mathrm{d}x.$$

For a nonnegative $(h, g; \alpha - m)$-concave function f, the inequality in (3.10) is reversed.

Proof. If f is nonnegative $(h, g; \alpha - m)$-convex, we apply inequality (1.7) to the right-hand side of (3.5) to obtain

$$\begin{aligned} &\min\left\{f(a)g(a)I_1 + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)I_2, f(b)g(b)I_1 + mf\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)I_2\right\} \\ &\leq \frac{1}{2}\left[(f(a)g(a) + f(b)g(b))I_1 + m\left(f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right)I_2\right], \end{aligned} \tag{3.11}$$

which concludes the proof of inequality (3.10). On the other hand, if f is nonnegative $(h, g; \alpha - m)$-concave function, we use inequality (1.8) for $p = 1$ to obtain the assertion. □

Remark 3. If we put $\alpha = 1$ in Corollary 6, then $(h, g; \alpha - m)$-convexity reduces to $(h, g; m)$-convexity, so the inequality from Theorem 2 is obtained.

In the following result, we consider additional assumptions for the function h.

Theorem 5. *Let f be a nonnegative $(h, g; \alpha - m)$-convex function on $[0, \infty)$, where h is a nonnegative concave function on $J \subset \mathbb{R}$, $h \neq 0$, g is a positive function on $[0, \infty)$ and $\alpha, m \in (0, 1]$. Let $0 \leq a < b < \infty$ and $f, g, h \in L_1[a, b]$. Furthermore, let $w : [a, b] \to \mathbb{R}$ be a nonnegative, integrable, and symmetric about $\frac{a+b}{2}$. Let us denote $W \coloneqq \int_a^b w(x)\mathrm{d}x$. Then the following inequality holds*

$$\begin{aligned} \int_a^b f(x)w(x)\mathrm{d}x \leq \frac{W}{2}\Bigg[&(f(a)g(a) + f(b)g(b))h\left(\frac{1}{W}\int_a^b \left(\frac{b-x}{b-a}\right)^{\alpha} w(x)\mathrm{d}x\right) \\ &+ m\left(f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right)\cdot h\left(\frac{1}{W}\int_a^b \left(1 - \left(\frac{b-x}{b-a}\right)^{\alpha}\right) w(x)\mathrm{d}x\right)\Bigg]. \end{aligned} \tag{3.12}$$

If f is a nonnegative $(h, g; \alpha - m)$-concave function and h is a nonnegative convex function, the inequality in (3.12) is reversed.

Proof. First, let us assume that f is a nonnegative $(h, g; \alpha - m)$-convex function and h is a nonnegative concave function. After applying the integral Jensen inequality, we have the following inequalities:

$$I_1 = \int_a^b h\left(\left(\frac{b-x}{b-a}\right)^{\alpha}\right) w(x)\mathrm{d}x \leq W \cdot h\left(\frac{1}{W} \cdot \int_a^b \left(\frac{b-x}{b-a}\right)^{\alpha} w(x)\mathrm{d}x\right) \tag{3.13}$$

and

$$I_2 = \int_a^b h\left(1 - \left(\frac{b-x}{b-a}\right)^{\alpha}\right) w(x)\mathrm{d}x \leq W \cdot h\left(\frac{1}{W} \cdot \int_a^b \left(1 - \left(\frac{b-x}{b-a}\right)^{\alpha}\right) w(x)\mathrm{d}x\right). \tag{3.14}$$

Now, after applying (3.13) and (3.14) to the inequality (3.10), we obtain

$$\begin{aligned}\int_a^b f(x)w(x)\mathrm{d}x &\le \frac{1}{2}\left[(f(a)g(a)+f(b)g(b))I_1 + m\left(f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right)I_2\right]\\ &\le \frac{W}{2}\left[(f(a)g(a)+f(b)g(b))\cdot h\left(\frac{1}{W}\cdot\int_a^b\left(\frac{b-x}{b-a}\right)^{\alpha}w(x)\mathrm{d}x\right)\right.\\ &\quad\left.+\,m\cdot\left(f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\right)\cdot h\left(\frac{1}{W}\int_a^b\left(1-\left(\frac{b-x}{b-a}\right)^{\alpha}\right)w(x)\mathrm{d}x\right)\right].\end{aligned}\tag{3.15}$$

Hence, the proof in this case is finished.

In the second case, we apply the Jensen's integral inequality to the convex function h and derive the reversed forms of inequalities in (3.13) and (3.14). Analogously, we use those bounds to the reversed version of the inequality (3.10) to complete the proof. □

In this section, we have established new Fejér-type integral inequalities for the class of $(h, g; \alpha - m)$-convex functions, thereby providing a comprehensive generalization of existing results within the framework of generalized convexity.

4 Conclusion

In this study, the several new Fejér-type inequalities for $(h, g; m)$-convex functions have been established. In addition, the application of the new inequalities on various types of convex types of functions are shown, such as $(h - m)$-convex, h-convex, m-convex, and nonnegative convex functions. Further, some new Fejér type inequalities for $(h, g;\ \alpha - m)$-convex functions have been obtained. Our results generalize and extend classical inequalities, offering broader applicability in the field of integral inequalities.

Acknowledgments: The authors sincerely thank the reviewers for their valuable comments and suggestions that significantly enhanced this work.

Funding information: The authors state no external funding involved.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during this study.

References

[1] J. Barić, Lj. Kvesić, J. Pečarić, and M. Ribičić Penava, *Fejér type inequalities for higher order convex functions and quadrature formulae*, Aequationes Math. **96** (2022), no. 2, 417–430, DOI: https://doi.org/10.1007/s00010-021-00825-7.

[2] M. Jleli, *Extension of Fejéras inequality to the class of sub-biharmonic functions*, Open Math. **22** (2024), no. 1, 369–390, DOI: https://doi.org/10.1515/math-2024-0035.

[3] S. S. Dragomir, M. Jleli, and B. Samet, *On Fejér-type inequalities for generalized trigonometrically and hyperbolic k-convex functions*, Demonstr. Math. **57** (2024), no 1, 171–215, DOI: https://doi.org/10.1515/dema-2024-0001.
[4] M. E. Ozdemir, A. O. Akdemir, and E. Set, *On $(h-m)$-convexity and Hadamard type inequalities*, Transylv. J. Math. Mech. **8** (2016), no. 1, 51–58.
[5] M. Andrić and J. Pečarić, *On $(h,g;m)$-convexity and the Hermite-Hadamard inequality*, J. Convex Anal. **29** (2022), no. 1, 257–268.
[6] M. Andrić, *Féjer type inequalities for $(h,g;m)$-convex functions*, TWMS J. Pure Appl. Math. **14** (2023), no. 2, 185–194, DOI: https://doi.org/10.30546/2219-1259.14.2.2023.185.
[7] M. Andrić, V. Culjak, D. Pečarić, J. Pečarić, and J. Perić, *New Developments for Jensen and Lah-Ribarič Inequalities*, Element, Zagreb, 2023.
[8] Y. Liu, G. Farid, J. Pečarić, J. Rod, M. Elamine, and S. Abdel-Khalek, *Strongly $(g,h;\alpha-m)$-convex functions and the consequent Hermite-Hadamard-type inequalities*, Appl. Math. Sci. Eng. **33** (2025), no. 1, 2471386, DOI: https://doi.org/10.1080/27690911.2025.2471386.