Research Article

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Multiple positive solutions to a *p*-Kirchhoff equation with logarithmic terms and concave terms

https://doi.org/10.1515/math-2025-0183
received October 6, 2024; accepted June 30, 2025

Abstract: In this article, we focus on a class of p-Kirchhoff-type equations that include logarithmic and concave terms. By applying the variational method, we establish the existence and multiplicity of positive solutions.

Keywords: p-Kirchhoff-type equation, logarithmic and concave terms, variational method, positive solution

MSC 2020: 35A15, 35B09, 35J60

1 Introduction

In this article, our aim is to investigate the multiplicity of positive solutions for the p-Kirchhoff-type equation with logarithmic and concave terms as follows:

$$\begin{cases} -\left(a + b\int\limits_{\Omega}|\nabla u|^p \mathrm{d}x\right)\Delta_p u = |u|^{\gamma-2}u\ln|u| + \lambda\dfrac{|u|^{q-2}u}{|x|^{\mu}}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, the constants $a > 0$, $b \geq 0$, $1 \leq q < 2 \leq p < \gamma < p^*$, $0 \leq \mu < N(p^* - q)/p^*$, and the parameter $\lambda > 0$, $p^*(p^* = Np/(N-p)$ if $N > p$ and $p^* = \infty$ if $N \leq p)$ denotes the critical Sobolev exponent for the embedding $W_0^{1,p}(\Omega)$ into $L^s(\Omega)$ for every $s \in [1, p^*]$. The Sobolev space $W_0^{1,p}(\Omega) = \{u \in L^s(\Omega) : \nabla u \in L^s(\Omega), u|\partial\Omega = 0\}$ with the norm $||u||^p = \int_\Omega|\nabla u|^p \mathrm{d}x$ and $|u|_s^s = \int_\Omega |u|^s \mathrm{d}x$ denotes the norm of $L^s(\Omega)$ with $s \in (1, +\infty)$. Indeed, in problem (1.1), if we replace $|u|^{\gamma-2}u\ln|u| + \lambda\,|u|^{q-2}u\,|x|^{-\mu}$ by $g(x,u)$ and $p = 2$, it reduces to the following Kirchhoff-type equation with the Dirichlet boundary value condition:

$$\begin{cases} -\left(a + b\int\limits_{\Omega}|\nabla u|^2 \mathrm{d}x\right)\Delta u = g(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $a, b \geq 0, a + b > 0, \Omega = \mathbb{R}^N$ or Ω is a smooth bounded domain in $\mathbb{R}^N$ and $g : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Problem (1.2) has been extensively studied [1–6] because (1.2) is related to the stationary problem of

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a model introduced by Kirchhoff [7] as follows:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, ρ_0, h, E, and L are constants, which extends the classical d'Alembert wave equation, by considering the effects of the changes in the length of the strings during the vibrations. After the pioneering work of Lions [8], Kirchhoff-type problems began to attract the attention of several researchers, where a functional analysis approach was proposed. Recently, there are many articles on the Kirchhoff-type problem involving logarithmic term, see [9–18] and the references therein. We mentioned that Bouizem [13] considered the following Kirchhoff-type problem with logarithmic terms:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = u^{\gamma-2}u\ln|u| + \lambda f(x), & x \in \Omega,\\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$, $\gamma \in (0, 2N/(N-2))$, and $\lambda > 0$ are constants, M is a continuous positive function in $\mathbb{R}^+$, and $f(x) \in C^1(\bar{\Omega})$ changes sign. Under certain assumptions about $f(x)$ and $M(\cdot)$, the authors employ the direct variational method, Galerkin approach, and subsuper solution method to address problem (1.3) and obtain the existence results. Wen et al. [14] studied the following Kirchhoff equation with logarithmic nonlinearity:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = u^{\gamma-2}u\ln|u|^2 - k(x)u, & x \in \Omega,\\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, a, b are positive constants, $4 < \gamma < 6$, $k \in C(\Omega, \mathbb{R})$ and $\inf_{x\in\Omega} k(x) > 0$. They used the constraint variational method, topological degree theory, and some new energy estimate inequalities to prove the existence of ground state solutions and ground state sign-changing solutions with precisely two nodal domains for problem (1.4). Li et al. [15] studied the following Kirchhoff-type problem with critical and logarithmic terms:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = \lambda u^{\gamma-2}u\ln|u|^2 + \mu|u|^2 u, & x \in \Omega,\\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^4$ with smooth boundary $\partial\Omega$, $a, b, \lambda, \mu > 0$ and $\gamma \in (2,4)$. By using the Mountain Pass Lemma, Brézis-Lieb's lemma, and other methods, they proved that either the norm of the sequence of approximation solutions goes to infinity or the problem admits a nontrivial weak solution for equation (1.5). Furthermore, when $\mu = u^2$, $4 < \gamma < 6$ and $\Omega \subset \mathbb{R}^3$, Li and Han [16] proved that problem (1.5) admits a ground state solution for any $\lambda > 0$ by using the mountain pass lemma and the concentration compactness principle. Li et al. [17] studied the following p-Kirchhoff-type problem with logarithmic nonlinearity:

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^p \mathrm{d}x\right)\Delta_p u = |u|^{\gamma-2}u\ln|u|^2, & x \in \Omega,\\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $a, b > 0$, $4 \leq 2p < \gamma < p^*$ and $N > p$, $\Delta_p u$ denotes the p-Laplacian operator defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. By using constraint variational method, topological degree theory, and the quantitative deformation lemma, they proved the existence of ground state sign-changing solutions with precisely two nodal domains. Cai et al. [18] studied the following p-Kirchhoff-type problem with logarithmic nonlinearity:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)\Delta_p u = |u|^{p^*-2}u - |u|^{\gamma-2}u \ln |u|^2 + \lambda\, |u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.7}$$

where Ω is a bounded domain in $\mathbb{R}^N$with a smooth boundary $\partial\Omega$, $2 < p < p^* < N$, $M(t) : [0, +\infty) \to \mathbb{R}$ is a continuously increasing function and satisfied

M_0: There exist $k_0 > 0$ and $k_1 > 0$ such that $k_0 \le M(t) \le k_1$, $\forall t \ge 0$;

M_1: There exist $p_0 \in (p, p^*)$ and $l < \frac{p_0}{p}$ such that $k_1 < lk_0 < \frac{p_0}{p}k_0$.

The authors proved that equation (1.7) has at least one nontrivial solution by using the Nehari manifold and the Mountain Pass Lemma without the Palais-Smale compactness condition.

Motivated by the researches mentioned earlier, especially by [13,17,18], we shall investigate the existence and multiplicity of positive solutions to problem (1.1). The proof method in this article is slightly different from those mentioned earlier, and the study of the equation is a generalization and supplement to them.

A function $u \in W_0^{1,p}(\Omega)$ is called a weak solution of problem (1.1) if and only if

$$(a + b\|u\|^p)\int_\Omega |\nabla u|^{p-1}\nabla\varphi \mathrm{d}x - \int_\Omega |u|^{\gamma-1}\varphi \ln|u| \mathrm{d}x - \lambda\int_\Omega \frac{|u|^{q-1}\varphi}{|x|^\mu}\mathrm{d}x = 0, \quad \forall\varphi \in W_0^{1,p}(\Omega).$$

Our main result is state as follows:

Theorem 1.1. *Assume $a > 0, b \ge 0, 1 \le q < 2 \le p < \gamma < p^*$, and $0 \le \mu < N(p^* - q)/p^*$, and, there exists $T_0 > 0$ such that problem* (1.1) *has at least two positive solutions for any* $0 < \lambda < T_0$.

This article is organized as follows. Some preliminaries and important lemmas are given in Section 2, and in Section 3, we prove Theorem 1.1.

2 Preliminaries

To describe our results clearly, we shall introduce some notations next. We denote by B_α (respectively, S_α) the closed ball (respectively, the sphere) of center zero and radius α, where $B_\alpha(u) = \{u \in W_0^{1,p}(\Omega) : \|u\| \le \alpha\}$ and $S_\alpha(u) = \{u \in W_0^{1,p}(\Omega) : \|u\| = \alpha\}$. C and $C_i (i = 0, 1, 2, \ldots)$ denote various positive constants, which may vary from line to line. $\to$(respectively,$\rightharpoonup$) denotes the strong (respectively, weak) convergence. Let S be the best Sobolev constant, that is,

$$S = \inf\left\{\int_\Omega |\nabla u|^p \mathrm{d}x : u \in W_0^{1,p}(\Omega)\backslash\{0\}, \int_\Omega |u|^{p^*}\mathrm{d}x = 1\right\}. \tag{2.1}$$

Let α be a constant such that $\Omega \subset B(0, \alpha) = \{x \in \mathbb{R}^N : |x| < \alpha\}$, and then, there exists a constant Π such that

$$\begin{aligned} \int_\Omega \frac{|u|^q}{|x|^\mu}\mathrm{d}x &\le \left(\int_\Omega |u|^{p^*}\mathrm{d}x\right)^{\frac{q}{p^*}}\left(\int_\Omega \frac{1}{|x|^{\frac{\mu p^*}{p^*-q}}}\mathrm{d}x\right)^{\frac{p^*-q}{p^*}} \le \left(\int_\Omega |u|^{p^*}\mathrm{d}x\right)^{\frac{q}{p^*}}\left(\int_{B(0,\alpha)} \frac{1}{|x|^{\frac{\mu p^*}{p^*-q}}}\mathrm{d}x\right)^{\frac{p^*-q}{p^*}} \\ &\le S^{-\frac{q}{p}}\|u\|^q\left(\Pi\int_0^\alpha r^{\frac{(N-1)(p^*-q)-\mu p^*}{p^*-q}}\mathrm{d}r\right)^{\frac{p^*-q}{p^*}} \le \left[\frac{\Pi(p^* - q)}{N(p^* - q) - \mu p^*}\right]^{\frac{p^*-q}{p^*}} \alpha^{\frac{N(p^*-q)-\mu p^*}{p^*}} S^{-\frac{q}{p}}\|u\|^q \end{aligned} \tag{2.2}$$

for any $u \in W_0^{1,p}(\Omega)$ by the Hölder inequality, (2.1), and $0 \le \mu < N(p^* - q)/p^*$. If $N = 3$, it is easy to verify that $\Pi = 4\pi$. This indicates that the element $x = 0$ can be included in the region Ω.

Next, we provide some definitions and lemmas that are necessary for proving the result.

The energy functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated with problem (1.1) is given by

$$I(u) = \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} + \frac{1}{\gamma^2}\int_\Omega |u|^\gamma \mathrm{d}x - \frac{1}{\gamma}\int_\Omega |u|^\gamma \ln|u| \mathrm{d}x - \frac{\lambda}{q}\int_\Omega \frac{|u|^q}{|x|^\mu}\mathrm{d}x, \quad u \in W_0^{1,p}(\Omega). \tag{2.3}$$

Clearly, I is of class $C^1(W_0^{1,p}(\Omega), \mathbb{R})$ and every weak solution of problem (1.1) is the critical point of I.

Definition 2.1. Let $(\mathcal{K}, d)$ be a complete metric space and $F : \mathcal{K} \to \mathbb{R}$ be a continuous functional in $\mathcal{K}$. For any $u \in \mathcal{K}$, we denote by $|\mathrm{d}F|(u)$ the supremum of η in $[0, \infty)$ such that there exist $\alpha > 0$ and a continuous map $\delta : B_\alpha(u) \times [0, \alpha] \to \mathcal{K}$ and satisfying

$$\begin{cases} F(\delta(v, t)) \le F(v) - \eta t, & (v, t) \in B_\alpha(u) \times [0, \alpha], \\ d(\delta(v, t), v) \le t, & (v, t) \in B_\alpha(u) \times [0, \alpha]. \end{cases} \tag{2.4}$$

The extended real number $|\mathrm{d}F|(u)$ is called the weak slope of F at u.

Definition 2.2. Recalled that a function $u \in \mathcal{K}$ is a (lower) critical point of F if $|\mathrm{d}F|(u) = 0$, and a number $c \in \mathbb{R}$ is a (lower) critical value of F if there exists a (lower) critical point $u \in \mathcal{K}$ of F with $F(u) = c$.

Definition 2.3. A sequence $\{u_n\} \subset \mathcal{K}$ is called a (PS) sequence of the functional F, if $|\mathrm{d}F|(u_n) \to 0$ as $n \to \infty$ and $F(u_n)$ is bounded.

Since $u \to |\mathrm{d}F|(u)$ is lower semicontinuous, any accumulation point of a (PS) sequence is clearly a critical point of F. In this article, since we are looking for a positive solution to problem (1.1), the functional I is considered on the closed positive cone Λ of $W_0^{1,p}(\Omega)$, namely,

$$\Lambda = \{u | u \in W_0^{1,p}(\Omega), u(x) \ge 0, \ \text{a.e. } x \in \Omega\} \subset W_0^{1,p}(\Omega),$$

where Λ is a complete metric space, and I is a continuous functional on Λ.

Lemma 2.4. *If* $|\mathrm{d}I|(u) < +\infty$ *holds, for any* $v \in \Lambda$, *we have*

$$\begin{aligned}\lambda\int_\Omega \frac{u^{q-1}(v-u)}{|x|^\mu}\mathrm{d}x \le{}& a\int_\Omega |\nabla u|^{p-1}\nabla(v-u)\mathrm{d}x + b\int_\Omega |\nabla u|^p \mathrm{d}x\int_\Omega |\nabla u|^{p-1}\nabla(v-u)\mathrm{d}x - \int_\Omega u^{\gamma-1}(v-u)\ln|u|\mathrm{d}x \\ &+ |\mathrm{d}I|(u)\|v-u\|.\end{aligned} \tag{2.5}$$

Proof. From [19, Lemma 3.1], let $|\mathrm{d}I|(u) < c, \eta < \frac{1}{2}\|v - u\|$, $v \in \Lambda$, and $v \ne u$. We define the mapping $\tau : U \times [0, \eta] \to \Lambda$ by $\tau(z, t) = z + t\frac{v - z}{\|v - z\|}$, where U is a neighborhood of u. Then, we have $\|\tau(z, t) - z\| = t$, by (2.4), and there exists a pair $(z, t) \in U \times [0, \eta]$ such that $I(\tau(z, t)) > I(z) - ct$. Therefore, we shall assume that there exists a sequences $\{u_n\} \subset \Lambda$ and $\{t_n\} \subset [0, \infty)$ such that $u_n \to u$, $t_n \to 0^+$ and

$$I\left(u_n + t_n\frac{v - u_n}{\|v - u_n\|}\right) \ge I(u_n) - ct_n,$$

namely,

$$I(u_n + s_n(v - u_n)) \ge I(u_n) - cs_n\|v - u_n\|, \tag{2.6}$$

where $s_n = \frac{t_n}{\|v - u_n\|}$ and $s_n \to 0^+$ as $n \to \infty$. Set $H(u) = \frac{1}{\gamma^2}\int_\Omega |u|^\gamma \mathrm{d}x - \frac{1}{\gamma}\int_\Omega |u|^\gamma \ln|u|\mathrm{d}x$, by dividing (2.6) by s_n, we obtain the inequality

$$\begin{aligned}
&\frac{\lambda}{q}\int_\Omega \frac{[u_n+s_n(v-u_n)]^q-u_n^q}{s_n\,|x|^\mu}\mathrm{d}x\\
&\quad\le \frac{a}{p}\cdot\frac{\|(u_n+s_n(v-u_n))\|^p-\|u_n\|^p}{s_n}+\frac{b}{2p}\cdot\frac{\|u_n+s_n(v-u_n)\|^{2p}-\|u_n\|^{2p}}{s_n}\\
&\qquad+\int_\Omega \frac{H[u_n+s_n(v-u_n)]-H(u_n)}{s_n}\mathrm{d}x+c\|v-u_n\|.
\end{aligned}\tag{2.7}$$

Now, we claim that

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \frac{H[u_n+s_n(v-u_n)]-H(u_n)}{s_n}\mathrm{d}x\\
&=\lim_{n\to\infty}\int_\Omega \frac{|u_n+s_n(v-u_n)|^\gamma-|u_n|^\gamma}{\gamma^2 s_n}\mathrm{d}x-\lim_{n\to\infty}\int_\Omega \frac{|u_n+s_n(v-u_n)|^\gamma\ln|u_n+s_n(v-u_n)|-|u_n|^\gamma\ln|u_n|}{\gamma s_n}\mathrm{d}x\\
&=\frac{1}{\gamma}\int_\Omega |u|^{\gamma-1}(v-u)\mathrm{d}x-\int_\Omega |u|^{\gamma-1}(v-u)\ln|u|\mathrm{d}x-\frac{1}{\gamma}\int_\Omega |u|^{\gamma-1}(v-u)\mathrm{d}x\\
&=-\int_\Omega |u|^{\gamma-1}(v-u)\ln|u|\mathrm{d}x.
\end{aligned}\tag{2.8}$$

In addition, for all $2\le p<\gamma$ and $m\in(\gamma,p^*)$, we obtain

$$\lim_{t\to0}\frac{t^{\gamma-1}\ln|t|}{t^{p-1}}=0\quad\text{and}\quad\lim_{t\to\infty}\frac{t^{\gamma-1}\ln|t|}{t^{m-1}}=0.$$

Therefore, for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|t|^{\gamma-1}\ln|t|\le\varepsilon\,|t|^{p-1}+C_\varepsilon\,|t|^{m-1},\qquad\forall t\in\mathbb{R}\backslash\{0\}.\tag{2.9}$$

Indeed, note that $u_n(x)\to u(x)$ a.e. in Ω and $u_n\to u_n^\gamma\ln|u_n|$ is continuous, there is

$$|u_n(x)|^\gamma\ln|u_n(x)|\to|u(x)|^\gamma\ln|u(x)|,\qquad\text{a.e. } x\in\Omega.$$

By using the Lebesgue dominated convergence theorem (see [20, pp. 27]) and (2.9), we have

$$\int_\Omega |u_n|^\gamma\ln|u_n|\mathrm{d}x\to\int_\Omega |u|^\gamma\ln|u|\mathrm{d}x,$$

as $n\to\infty$. Thus, the claim (2.8) is holds.

Set

$$I_{1,n}=\int_\Omega \frac{[u_n+s_n(v-u_n)]^q-[(1-s_n)u_n]^q}{s_nq\,|x|^\mu}\mathrm{d}x\quad\text{and}\quad I_{2,n}=\frac{[(1-s_n)]^q-1}{s_nq}\int_\Omega\frac{|u_n|^q}{|x|^\mu}\mathrm{d}x.$$

Then, we have

$$\begin{aligned}
&\int_\Omega \frac{[u_n+s_n(v-u_n)]^q-u_n^q}{s_nq\,|x|^\mu}\mathrm{d}x\\
&\quad=\int_\Omega \frac{[u_n+s_n(v-u_n)]^q-[(1-s_n)u_n]^q}{s_nq\,|x|^\mu}\mathrm{d}x+\int_\Omega \frac{[(1-s_n)u_n]^q-u_n^q}{s_nq\,|x|^\mu}\mathrm{d}x\\
&\quad=\int_\Omega \frac{[u_n+s_n(v-u_n)]^q-[(1-s_n)u_n]^q}{s_nq\,|x|^\mu}\mathrm{d}x+\frac{[(1-s_n)]^q-1}{s_nq}\int_\Omega\frac{|u_n|^q}{|x|^\mu}\mathrm{d}x\\
&\quad=I_{1,n}+I_{2,n}.
\end{aligned}$$

Obviously,

$$I_{1,n} = \int_\Omega \frac{\xi_n^{q-1} s_n v}{s_n |x|^\mu} dx = \int_\Omega \frac{\xi_n^{q-1} v}{|x|^\mu} dx,$$

where $\xi_n \in (u_n - u_n s_n, u_n + s_n(v - u_n))$, which implies that $\xi_n \to u(u_n \to u)$ as $s_n \to 0^+$. By applying the Fatou's Lemma to $I_{1,n}$, we can obtain

$$\liminf_{n\to\infty} I_{1,n} \geq \int_\Omega \frac{u^{q-1} v}{|x|^\mu} dx, \quad \forall v \in \Lambda \tag{2.10}$$

since $I_{1,n} \geq 0$ for all n. By applying the dominated convergence theorem to $I_{2,n}$, we obtain

$$\lim_{n\to\infty} I_{2,n} = -\int_\Omega \frac{u^q}{|x|^\mu} dx. \tag{2.11}$$

As a result, it holds that

$$(a + b\|u\|^p)\int_\Omega |\nabla u|^{p-1}\nabla(v-u)dx - \int_\Omega u^{\gamma-1}(v-u)\ln|u|dx + c\|v-u\| \geq \liminf_{n\to\infty}(I_{1,n} + I_{2,n}) \geq \lambda\int_\Omega \frac{u^{q-1}(v-u)}{|x|^\mu}dx$$

for any $v \in \Lambda$ since $|dI|(u) < c$ is arbitrary. The proof is complete. □

Lemma 2.5. *I satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset \Lambda$ be a (PS) sequence of I, there is

$$I(u_n) \to c \quad \text{and} \quad |dI|(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.12}$$

By Lemma 2.4 and for any $v \in \Lambda$, it holds

$$\lambda\int_\Omega \frac{u_n^{q-1}(v-u_n)}{|x|^\mu}dx \leq (a + b\|u\|^p)\int_\Omega |\nabla u|^{p-1}\nabla(v-u_n)dx - \int_\Omega u_n^{\gamma-1}(v-u_n)\ln|u_n|dx + o(1)\|v-u_n\|. \tag{2.13}$$

Taking $v = 2u_n \in \Lambda$ in (2.13), one has

$$a\|u_n\|^p + b\|u_n\|^{2p} - \int_\Omega u_n^\gamma \ln|u_n|dx + o(1)\|u_n\| \geq \lambda\int_\Omega \frac{u_n^q}{|x|^\mu}dx. \tag{2.14}$$

The fact $I(u_n) \to c$ means that

$$\frac{a}{p}\|u_n\|^p + \frac{b}{2p}\|u_n\|^{2p} + \frac{1}{\gamma^2}\int_\Omega |u_n|^\gamma dx - \frac{1}{\gamma}\int_\Omega u_n^\gamma \ln|u_n|dx - \frac{\lambda}{q}\int_\Omega \frac{|u_n|^q}{|x|^\mu}dx = c + o(1). \tag{2.15}$$

By combining the inequalities (2.2), (2.14), and (2.15), there exist two positive constants C_1 and C_2 such as

$$\begin{aligned} &\left(\frac{a}{p} - \frac{a}{\gamma}\right)\|u_n\|^p + \left(\frac{b}{2p} - \frac{b}{\gamma}\right)\|u_n\|^{2p} + \frac{1}{\gamma^2}\int_\Omega |u_n|^\gamma dx \\ &\leq \left(\frac{1}{q} - \frac{1}{\gamma}\right)\lambda\int_\Omega \frac{|u_n|^q}{|x|^\mu}dx + c + o(1) + o(1)\|u_n\| \\ &\leq \frac{\gamma - q}{\gamma q}\lambda\left[\frac{\Pi(p^* - q)}{N(p^* - q) - \mu p^*}\right]^{\frac{p^*-q}{p^*}} a^{\frac{N(p^*-q)-\mu p^*}{p^*}} S^{-\frac{q}{p}}\|u\|^q + c + o(1) + o(1)\|u_n\| \\ &\leq C_1\lambda\|u_n\|^q + C_2 + o(1)\|u_n\|. \end{aligned} \tag{2.16}$$

From the fact (2.16), we know that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. So, there exists a subsequence (still denoted by $\{u_n\}$) and $u_* \in W_0^{1,p}(\Omega)$, such that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{weakly in } W_0^{1,p}(\Omega),\\ u_n \to u_*, & \text{strongly in } L^s(\Omega) \text{ for } s \in [1, p^*),\\ u_n(x) \to u_*(x), & \text{a.e. } x \in \Omega. \end{cases}$$

Taking $v = u_n$ in (2.13), we have

$$\begin{aligned} \lambda \int_\Omega \frac{u_n^{q-1}(u_m - u_n)}{|x|^\mu} \mathrm{d}x \le a \int_\Omega |\nabla u_n|^{p-1} \nabla(u_m - u_n) \mathrm{d}x + b \int_\Omega |\nabla u_n|^p \mathrm{d}x \int_\Omega |\nabla u_n|^{p-1} \nabla(u_m - u_n) \mathrm{d}x \\ - \int_\Omega u_n^{\gamma-1}(u_m - u_n) \ln|u_n| \mathrm{d}x + o(1)\|u_m - u_n\|. \end{aligned} \tag{2.17}$$

Changing the role of u_m and u_n in (2.17), we obtain

$$\begin{aligned} \lambda \int_\Omega \frac{u_m^{q-1}(u_n - u_m)}{|x|^\mu} \mathrm{d}x \le a \int_\Omega |\nabla u_m|^{p-1} \nabla(u_n - u_m) \mathrm{d}x + b \int_\Omega |\nabla u_m|^p \mathrm{d}x \int_\Omega |\nabla u_m|^{p-1} \nabla(u_n - u_m) \mathrm{d}x \\ - \int_\Omega u_m^{\gamma-1}(u_n - u_m) \ln|u_m| \mathrm{d}x + o(1)\|u_n - u_m\|. \end{aligned} \tag{2.18}$$

By combining inequalities (2.17) and (2.18), we obtain

$$\begin{aligned} a\|u_n - u_m\|^p + b\|u_n - u_m\|^{2p} &\le \lambda \int_\Omega (u_n - u_m) \left(\frac{u_n^{q-1}}{|x|^\mu} - \frac{u_m^{q-1}}{|x|^\mu} \right) \mathrm{d}x \\ &\quad + \int_\Omega (u_n - u_m)(|u_n|^{\gamma-1} \ln|u_n| - |u_m|^{\gamma-1} \ln|u_m|) \mathrm{d}x + o(1)\|u_m - u_n\| \\ &\le \int_\Omega (u_n - u_m)(|u_n|^{\gamma-1} \ln|u_n| - |u_m|^{\gamma-1} \ln|u_m|) \mathrm{d}x + o(1)\|u_m - u_n\|. \end{aligned}$$

In addition, we note that

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega |u_n|^\gamma \ln|u_n| \mathrm{d}x \to \int_\Omega |u|^\gamma \ln|u| \mathrm{d}x, \\ \lim_{m\to\infty} \int_\Omega |u_m|^\gamma \ln|u_m| \mathrm{d}x \to \int_\Omega |u|^\gamma \ln|u| \mathrm{d}x, \\ \lim_{n\to\infty} \int_\Omega (u_n - u_m)(|u_n|^{\gamma-1} \ln|u_n| - |u_m|^{\gamma-1} \ln|u_m|) \mathrm{d}x \to 0, \end{aligned}$$

it holds that $\lim_{n\to\infty}\|u_n - u_m\| \to 0$. Thus, $u_n \to u$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$. The proof is complete. □

Lemma 2.6. *If* $|\mathrm{d}I|(u) = 0$, *then,* u *is a weak solution of problem* (1.1). *That is,* $\frac{u^{q-1}\varphi}{|x|^\mu} \in L^1(\Omega)$ *and*

$$(a + b\|u\|^p) \int_\Omega |\nabla u|^{p-1} \nabla \varphi \mathrm{d}x = \int_\Omega |u|^{\gamma-1} \varphi \ln|u| \mathrm{d}x + \lambda \int_\Omega \frac{|u|^{q-1}\varphi}{|x|^\mu} \mathrm{d}x, \ \forall \varphi \in W_0^{1,p}(\Omega).$$

Proof. Since $|\mathrm{d}I|(u) = 0$, by Lemma 2.4, it holds

$$\lambda \int_\Omega \frac{u^{q-1}(v - u)}{|x|^\mu} \mathrm{d}x \le a \int_\Omega |\nabla u|^{p-1} \nabla(v - u) \mathrm{d}x + b \int_\Omega |\nabla u|^p \mathrm{d}x \int_\Omega |\nabla u|^{p-1} \nabla(v - u) \mathrm{d}x - \int_\Omega u^{\gamma-1}(v - u) \ln|u| \mathrm{d}x \tag{2.19}$$

for every $v \in \Lambda$. Let $t \in \mathbb{R}$, $\varphi \in W_0^{1,p}(\Omega)$ and take $v = (u + t\varphi)^+ \in \Lambda$ as a test function in (2.19), then

$$\begin{aligned}
0 \le\ & a\int_\Omega |\nabla u|^{p-1}\nabla((u+t\varphi)^+ - u)\mathrm{d}x + b\int_\Omega |\nabla u|^p\mathrm{d}x\int_\Omega |\nabla u|^{p-1}\nabla((u+t\varphi)^+ - u)\mathrm{d}x \\
& - \int_\Omega u^{\gamma-1}((u+t\varphi)^+ - u)\ln|u|\mathrm{d}x - \lambda\int_\Omega \frac{u^{q-1}((u+t\varphi)^+ - u)}{|x|^\mu}\mathrm{d}x \\
=\ & t\left(a\int_\Omega |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x + b\int_\Omega |\nabla u|^p\mathrm{d}x\int_\Omega |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x - \int_\Omega u^{\gamma-1}\varphi\ln|u|\mathrm{d}x \right.\\
& \left. - \lambda\int_\Omega \frac{u^{q-1}\varphi}{|x|^\mu}\mathrm{d}x\right) - a\int_{u+t\varphi<0} |\nabla u|^{p-1}\nabla(u+t\varphi)\mathrm{d}x + \lambda\int_{u+t\varphi<0} \frac{u^{q-1}(u+t\varphi)}{|x|^\mu}\mathrm{d}x \\
& - b\int_\Omega |\nabla u|^p\mathrm{d}x\int_{u+t\varphi<0} |\nabla u|^{p-1}\nabla(u+t\varphi)\mathrm{d}x + \int_{u+t\varphi<0} u^{\gamma-1}(u+t\varphi)\ln|u|\mathrm{d}x \\
\le\ & t\left(a\int_\Omega |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x + b\int_\Omega |\nabla u|^p\mathrm{d}x\int_\Omega |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x - \int_\Omega u^{\gamma-1}\varphi\ln|u|\mathrm{d}x \right.\\
& \left. - \lambda\int_\Omega \frac{u^{q-1}\varphi}{|x|^\mu}\mathrm{d}x\right) - at\int_{u+t\varphi<0} |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x \\
& - tb\int_{u+t\varphi<0} |\nabla u|^p\mathrm{d}x\int_\Omega |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x + t\int_{u+t\varphi<0} u^{\gamma-1}\varphi\ln|u|\mathrm{d}x.
\end{aligned}$$

Since $\nabla u(x) = 0$ a.e. in Ω for $u(x) = 0$ and

$$\operatorname{meas}\{x \in \Omega | u(x) + t\varphi < 0, u(x) > 0\} \to 0, \quad \text{as } t \to 0,$$

we can obtain that

$$\begin{aligned}
\int_{u+t\varphi<0} |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x &= \int_{u+t\varphi<0,u>0} |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x \to 0, \\
\int_{u+t\varphi<0} |u|^{\gamma-1}(u+t\varphi)\ln|u|\mathrm{d}x &= \int_{u+t\varphi<0,u>0} |u|^{\gamma-1}(u+t\varphi)\ln|u|\mathrm{d}x \to 0.
\end{aligned}$$

Thus,

$$0 \le t\left[(a + b\|u\|^p)\int_\Omega |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x - \int_\Omega u^{\gamma-1}\varphi\ln|u|\mathrm{d}x - \lambda\int_\Omega \frac{u^{q-1}\varphi}{|x|^\mu}\mathrm{d}x\right] + o(t)$$

as $t \to 0$, we obtain that

$$(a + b\|u\|^p)\int_\Omega |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x - \int_\Omega u^{\gamma-1}\varphi\ln|u|\mathrm{d}x - \lambda\int_\Omega \frac{u^{q-1}\varphi}{|x|^\mu}\mathrm{d}x \ge 0. \tag{2.20}$$

By the arbitrariness of φ, there will be same result in (2.20) if φ is taken as $-\varphi$. Therefore, it holds that

$$\left(a + b\int_\Omega |\nabla u|^p\mathrm{d}x\right)\int_\Omega |\nabla u|^{p-1}\nabla\varphi\mathrm{d}x - \int_\Omega u^{\gamma-1}\varphi\ln|u|\mathrm{d}x - \lambda\int_\Omega \frac{u^{q-1}\varphi}{|x|^\mu}\mathrm{d}x = 0$$

for any $\varphi \in W_0^{1,p}(\Omega)$. The proof is complete. □

3 Proof of Theorem 1.1

Now, we will prove that problem (1.1) has a positive solution u_* with $I(u_*) < 0$.

Lemma 3.1. *There exist $\alpha, \beta > 0$ and $T_0 > 0$ such that, for any $\lambda \in (0, T_0)$, we have*

$$I(u)|_{u\in S_\alpha} \geq \beta > 0, \quad \text{and} \quad \inf_{u\in \overline{B}_\alpha} I(u) < 0, \tag{3.1}$$

where $\alpha = \left[\frac{(a\gamma-1)(p-q)}{C_3 p(m-q)}\right]^{\frac{1}{m-p}}$, $\beta = \frac{(Np+p-N)(a\gamma-1)}{Np^2\gamma}\left[\frac{(a\gamma-1)(p-q)}{C_3 p(m-q)}\right]^{\frac{p}{m-p}}$ *and*

$$T_0 = \frac{qS^{\frac{q}{p}}}{p^* p\gamma}\alpha^{-\frac{N(p^*-q)-\mu p^*}{p^*}}\left[\frac{N(p^*-q)-\mu p^*}{\Pi(m-q)}\right]^{\frac{p^*-q}{p^*}}\left[\frac{(p-q)(a\gamma-1)}{C_3 p(m-q)}\right]^{\frac{p-q}{m-p}}.$$

Proof. By using the results (2.1) and (2.9), we obtain

$$\int_\Omega |u|^\gamma \ln|u| dx \leq \frac{1}{p}\|u\|^p + C_3\|u\|^m, \quad \forall u \in W_0^{1,p}(\Omega).$$

Combining (2.3) and $a > \frac{1}{\gamma}$, one can further obtain

$$\begin{aligned}
I(u) &= \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} + \frac{1}{\gamma^2}\int_\Omega |u|^\gamma dx - \frac{1}{\gamma}\int_\Omega |u|^\gamma \ln|u| dx - \frac{\lambda}{q}\int_\Omega \frac{|u|^q}{|x|^\mu}dx \\
&\geq \frac{a}{p}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{1}{p\gamma}\|u\|^p - \frac{C_3}{\gamma}\|u\|^m - \frac{\lambda}{q}\int_\Omega \frac{|u|^q}{|x|^\mu}dx \\
&\geq \frac{a\gamma-1}{p\gamma}\|u\|^p + \frac{b}{2p}\|u\|^{2p} - \frac{C_3}{\gamma}\|u\|^m - \frac{\lambda\alpha^{\frac{N(p^*-q)-\mu p^*}{p^*}}}{qS^{\frac{q}{p}}}\left[\frac{\Pi(p^*-q)}{N(p^*-q)-\mu p^*}\right]^{\frac{p^*-q}{p^*}}\|u\|^q \\
&\geq \frac{a\gamma-1}{p\gamma}\|u\|^p - \frac{C_3}{\gamma}\|u\|^m - \frac{\lambda\alpha^{\frac{N(p^*-q)-\mu p^*}{p^*}}}{qS^{\frac{q}{p}}}\left[\frac{\Pi(p^*-q)}{N(p^*-q)-\mu p^*}\right]^{\frac{p^*-q}{p^*}}\|u\|^q \\
&= \|u\|^q\left(\frac{a\gamma-1}{p\gamma}\|u\|^{p-q} - \frac{C_3}{\gamma}\|u\|^{m-q} - \frac{\lambda\alpha^{\frac{N(p^*-q)-\mu p^*}{p^*}}}{qS^{\frac{q}{p}}}\left[\frac{\Pi(p^*-q)}{N(p^*-q)-\mu p^*}\right]^{\frac{p^*-q}{p^*}}\right).
\end{aligned}$$

Set

$$\Phi(t) = \frac{a\gamma-1}{p\gamma}t^{p-q} - \frac{C_3}{\gamma}t^{m-q} - \frac{\lambda\alpha^{\frac{N(p^*-q)-\mu p^*}{p^*}}}{qS^{\frac{q}{p}}}\left[\frac{\Pi(p^*-q)}{N(p^*-q)-\mu p^*}\right]^{\frac{p^*-q}{p^*}}$$

with $t \geq 0$, and then, there exist constants

$$t_{\max} = \left[\frac{(a\gamma-1)(p-q)}{C_3 p(m-q)}\right]^{\frac{1}{m-p}} > 0$$

and

$$T_0 = \frac{qS^{\frac{q}{p}}}{p^* p\gamma}\alpha^{-\frac{N(p^*-q)-\mu p^*}{p^*}}\left[\frac{N(p^*-q)-\mu p^*}{\Pi(m-q)}\right]^{\frac{p^*-q}{p^*}}\left[\frac{(p-q)(a\gamma-1)}{C_3 p(m-q)}\right]^{\frac{p-q}{m-p}} > 0$$

such that

$$\max_{t>0}\Phi(t) = \Phi(t_{\max}) = \frac{a\gamma-1}{p\gamma}t_{\max}^{p-q} - \frac{C_3}{\gamma}t_{\max}^{m-q} - \frac{\lambda\alpha^{\frac{N(p^*-q)-\mu p^*}{p^*}}}{qS^{\frac{q}{p}}}\left[\frac{\Pi(p^*-q)}{N(p^*-q)-\mu p^*}\right]^{\frac{p^*-q}{p^*}}$$

$$= \frac{a\gamma - 1}{p\gamma}\left[\frac{(a\gamma-1)(p-q)}{C_3 p(m-q)}\right]^{\frac{p-q}{m-p}} - \frac{\lambda \alpha^{\frac{N(p^*-q)-\mu p^*}{p^*}}}{qS^{\frac{q}{p}}}\left[\frac{\Pi(p^*-q)}{N(p^*-q)-\mu p^*}\right]^{\frac{p^*-q}{p^*}}$$

$$\geq \frac{a\gamma - 1}{p\gamma}\left[\frac{(a\gamma-1)(p-q)}{C_3 p(m-q)}\right]^{\frac{p-q}{m-p}} - \frac{T_0 \alpha^{\frac{N(p^*-q)-\mu p^*}{p^*}}}{qS^{\frac{q}{p}}}\left[\frac{\Pi(p^*-q)}{N(p^*-q)-\mu p^*}\right]^{\frac{p^*-q}{p^*}}$$

$$= \frac{(Np+p-N)(a\gamma-1)}{Np^2\gamma}\left[\frac{(a\gamma-1)(p-q)}{C_3 p(m-q)}\right]^{\frac{p-q}{m-p}} > 0$$

for any $\lambda \in (0, T_0)$. Therefore, we have

$$I(u) \geq t_{\max}^q \Phi(t_{\max}) = \left[\frac{(a\gamma-1)(p-q)}{C_3 p(m-q)}\right]^{\frac{q}{m-p}} \frac{(Np+p-N)(a\gamma-1)}{Np^2\gamma}\left[\frac{(a\gamma-1)(p-q)}{C_3 p(m-q)}\right]^{\frac{p-q}{m-p}}$$

$$= \frac{(Np+p-N)(a\gamma-1)}{Np^2\gamma}\left[\frac{(a\gamma-1)(p-q)}{C_3 p(m-q)}\right]^{\frac{p}{m-p}} \coloneqq \beta > 0, \quad \forall \lambda \in (0, T_0).$$

Choosing $\alpha = t_{\max}$, then, there exists a constant $\beta > 0$ such that $I(u)|_{u\in S_\alpha} \geq \beta > 0$ for all $\lambda \in (0, T_0)$. Moreover, choosing $u \in \bar{B}_\alpha$ with $u \neq 0$, it holds

$$\lim_{t\to 0^+} \frac{I(tu)}{t^q} = -\frac{\lambda}{q}\int_\Omega \frac{|u|^q}{|x|^\mu}\mathrm{d}x < 0,$$

then, we obtain that $I(tu) < 0$ for t small enough. Therefore, one has

$$m_0 \coloneqq \inf_{u\in \bar{B}_\alpha} I(u) < 0. \tag{3.2}$$

The proof is complete. □

Theorem 3.2. *For any* $0 < \lambda < T_0$, *then problem* (1.1) *has a positive solution* u_* *with* $I(u_*) < 0$.

Proof. By the definition of m_0 in (3.2), it can be inferred that there exists a minimization sequence $\{u_n\} \subset B_\alpha \subset \Lambda$ such that $\lim_{n\to\infty} I(u_n) = m_0 < 0$. Since $\{u_n\}$ is bounded in B_α, up to a subsequence, there exists $u_* \in W_0^{1,p}$ such that $u_n \rightharpoonup u_* \in W_0^{1,p}$, $u_n \to u_*$ in $L^s(\Omega)(1 \leq s < p^*)$, $u_n(x) \to u_*$, $a.\ e.$ in Ω as $n \to \infty$. Next, we will prove that $u_n \to W_0^{1,p}$ as $n \to \infty$. Let $w_n = u_n - u_*$ and by the Brezis-Lieb lemma (see [21, Theorem 1]), there holds

$$||u_n||^p = ||w_n||^p + ||u_*||^p + o(1).$$

By Lemma 2.5, it can be inferred that

$$m_0 = \lim_{n\to\infty} I(u_n) = I(u_*) + \lim_{n\to\infty}\left(\frac{a}{p}||w_n||^p + \frac{b}{2p}||w_n||^{2p} + \frac{b}{p}||w_n||^p||u_*||^p\right) \geq I(u_*) \geq m_0,$$

which implies that $||w_n|| \to 0$ as $n \to \infty$. Because B_α is a closed convex, it holds $u_* \in B_\alpha$. Hence, we can deduce that $I(u_*) = m_0 < 0$ and $u_* \not\equiv 0$ in Ω, which implies that u_* is a local minimizer of I.

Now, we claim that u_* is a critical point of I. Indeed, notice that $I(\pm u) = I(|u|)$, we can take $u_* \geq 0$ and $u_* \not\equiv 0$. Let $t > 0$ such that $u_* + t\psi \in W_0^{1,p}(\Omega)$ for any $\psi \in \Lambda \subset W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}0 \leq I(u_* + t\psi) - I(u_*) = \frac{a}{p}||u_* + t\psi||^p + \frac{b}{2p}||u_* + t\psi||^{2p} + \frac{1}{\gamma^2}\int_\Omega |u_* + t\psi|^\gamma \mathrm{d}x \\ - \frac{1}{\gamma}\int_\Omega |u_* + t\psi|^\gamma \ln|u_* + t\psi|\mathrm{d}x - \frac{\lambda}{q}\int_\Omega \frac{|u_* + t\psi|^q}{|x|^\mu}\mathrm{d}x - \frac{a}{p}||u_*||^p \\ - \frac{b}{2p}||u_*||^{2p} - \frac{1}{\gamma^2}\int_\Omega |u_*|^\gamma \mathrm{d}x + \frac{1}{\gamma}\int_\Omega |u_*|^\gamma \ln|u_*|\mathrm{d}x + \frac{\lambda}{q}\int_\Omega \frac{|u_*|^q}{|x|^\mu}\mathrm{d}x.\end{aligned} \tag{3.3}$$

From (3.3), we obtain

$$\begin{aligned}\frac{\lambda}{q}\int_\Omega\left[\frac{|u_*+t\psi|^q}{|x|^\mu}-\frac{|u_*|^q}{|x|^\mu}\right]dx &\le \frac{a}{p}(\|u_*+t\psi\|^p-\|u_*\|^p)+\frac{b}{2p}(\|u_*+t\psi\|^{2p}-\|u_*\|^{2p})\\ &+\frac{1}{\gamma^2}\int_\Omega[|u_*+t\psi|^\gamma-|u_*|^\gamma]dx\\ &-\frac{1}{\gamma}\int_\Omega[|u_*+t\psi|^\gamma\ln|u_*+t\psi|-|u_*|^\gamma\ln|u_*|]dx.\end{aligned}\tag{3.4}$$

According to the inequality (3.4) and passing to the limit as $t\to 0^+$, it holds that

$$\frac{\lambda}{q}\liminf_{t\to0^+}\int_\Omega\frac{|u_*+t\psi|^q-|u_*|^q}{t\,|x|^\mu}dx\le(a+b\|u_*\|^p)\int_\Omega\nabla u_*\nabla\psi dx-\int_\Omega|u_*|^{\gamma-1}\psi\ln|u_*|dx.\tag{3.5}$$

Notice that

$$\frac{\lambda}{q}\int_\Omega\frac{|u_*+t\psi|^q-|u_*|^q}{t\,|x|^\mu}dx=\lambda\int_\Omega\frac{|u_*+t\psi|^{q-1}\psi}{|x|^\mu}dx+o(t)$$

as $t\to0^+$, we can obtain that $|x|^{-\mu}(u_*+\varepsilon t\psi)^{q-1}\psi\to|x|^{-\mu}u_*^{q-1}\psi$ as $\varepsilon t\to0^+$ for any $\psi\in W_0^{1,p}(\Omega)$. Owing to $|x|^{-\mu}(u_*+\varepsilon t\psi)^{q-1}\psi\ge0$, we obtain

$$\frac{\lambda}{q}\liminf_{t\to0^+}\int_\Omega\frac{|u_*+t\psi|^q-|u_*|^q}{t\,|x|^\mu}dx\ge\lambda\int_\Omega\frac{u_*^{q-1}\psi}{|x|^\mu}dx.\tag{3.6}$$

Combining (3.5) and (3.6), we have

$$(a+b\|u_*\|^p)\int_\Omega|\nabla u_*|^{p-1}\nabla\psi dx-\int_\Omega u_*^{\gamma-1}\psi\ln|u_*|dx-\lambda\int_\Omega\frac{u_*^{q-1}\psi}{|x|^\mu}dx\ge0\tag{3.7}$$

for any $\psi\in W_0^{1,p}(\Omega)$ with $\psi\ge0$. From Lemma 3.1, the inequality $I(u_*)<0$ implies that $u_*\notin S_\alpha$, then, $\|u_*\|<\alpha$. So, there exists $\varrho_1\in(0,1)$ such that $(1+t)u_*\in B_\alpha$ with $|t|<\varrho_1$. We define $J:[-\varrho_1,\varrho_1]$ by $J(t)=I((1+t)u_*)$. Obviously, $J(t)$ achieves its minimum at $t=0$, we have

$$J'(0)=a\|u_*\|^p+b\|u_*\|^{2p}-\int_\Omega u_*^\gamma\ln|u_*|dx-\lambda\int_\Omega\frac{u_*^q}{|x|^\mu}dx=0.\tag{3.8}$$

For any $v\in W_0^{1,p}(\Omega)$ and $\varepsilon>0$, we define $\phi\in\Lambda$ by

$$\phi=(u_*+\varepsilon v)^+,$$

then, the inequalities (3.5) and (3.6) imply that

$$\begin{aligned}0&\le(a+b\|u_*\|^p)\int_\Omega|\nabla u_*|^{p-1}\nabla\phi dx-\int_\Omega u_*^{\gamma-1}\phi\ln|u_*|dx-\lambda\int_\Omega\frac{u_*^{q-1}\phi}{|x|^\mu}dx\\ &=\int_{u_*+\varepsilon v>0}\left[(a+b\|u_*\|^p)|\nabla u_*|^{p-1}\nabla(u_*+\varepsilon v)-u_*^{\gamma-1}(u_*+\varepsilon v)\ln|u_*|-\frac{\lambda u_*^{q-1}(u_*+\varepsilon v)}{|x|^\mu}\right]dx\\ &=\left(\int_\Omega-\int_{u_*+\varepsilon v\le0}\right)\left[(a+b\|u_*\|^p)|\nabla u_*|^{p-1}\nabla(u_*+\varepsilon v)-u_*^{\gamma-1}(u_*+\varepsilon v)\ln|u_*|-\frac{\lambda u_*^{q-1}(u_*+\varepsilon v)}{|x|^\mu}\right]dx\\ &\le a\|u_*\|^p+b\|u_*\|^{2p}-\int_\Omega u_*^\gamma\ln|u_*|dx-\lambda\int_\Omega\frac{|u_*|^q}{|x|^\mu}dx+\varepsilon\int_\Omega[(a+b\|u_*\|^p)|\nabla u_*|^{p-1}\nabla v\\ &\quad-u_*^{\gamma-1}v\ln|u_*|-\frac{\lambda\,|u_*|^{q-1}v}{|x|^\mu}\Big]dx-\int_{u_*+\varepsilon v\le0}(a+b\|u_*\|^p)|\nabla u_*|^{p-1}\nabla v dx\end{aligned}\tag{3.9}$$

$$+ \int\limits_{u_*+\varepsilon v\le 0}\left[u_*^{\gamma-1}(u_*+\varepsilon v)\ln|u_*| + \frac{\lambda u_*^{q-1}(u_*+\varepsilon v)}{|x|^\mu}\right]\mathrm{d}x$$
$$\le \varepsilon\int\limits_\Omega\left[(a+b\|u_*\|^p)|\nabla u_*|^{p-1}\nabla v - u_*^{\gamma-1}v\ln|u_*| - \frac{\lambda\,|u_*|^{q-1}v}{|x|^\mu}\right]\mathrm{d}x$$
$$- \int\limits_{u_*+\varepsilon v\le 0}(a+b\|u_*\|^p)|\nabla u_*|^{p-1}\nabla v\mathrm{d}x.$$

Since $\nabla u_*(x) = 0$ for a.e. in Ω with $u_*(x) = 0$ and

$$\mathrm{meas}\{x\in\Omega | u_*(x)+\varepsilon v<0, u_*(x)>0\}\to 0, \quad \text{as } \varepsilon\to 0,$$

we have

$$\int\limits_{u_*+\varepsilon v<0}|\nabla u_*|^{p-1}\nabla v\mathrm{d}x = \int\limits_{u+\varepsilon v<0,u>0}|\nabla u_*|^{p-1}\nabla v\mathrm{d}x\to 0,$$
$$\int\limits_{u_*+\varepsilon v<0}|u_*|^{\gamma-1}(u_*+\varepsilon v)\ln|u_*|\mathrm{d}x = \int\limits_{u_*+\varepsilon v<0,u>0}|u_*|^{\gamma-1}(u_*+\varepsilon v)\ln|u_*|\mathrm{d}x\to 0.$$

Thus, dividing by ε and $\varepsilon\to 0$ in (3.9), we obtain

$$(a+b\|u_*\|^p)\int\limits_\Omega|\nabla u_*|^{p-1}\nabla v\mathrm{d}x - \int\limits_\Omega u_*^{\gamma-1}v\ln|u_*|\mathrm{d}x - \lambda\int\limits_\Omega\frac{u_*^{q-1}v}{|x|^\mu}\mathrm{d}x\ge 0. \tag{3.10}$$

So, we have

$$(a+b\|u_*\|^p)\int\limits_\Omega|\nabla u_*|^{p-1}\nabla v\mathrm{d}x - \int\limits_\Omega u_*^{\gamma-1}v\ln|u_*|\mathrm{d}x - \lambda\int\limits_\Omega\frac{u_*^{q-1}v}{|x|^\mu}\mathrm{d}x = 0$$

because v is arbitrarily in (3.10). Therefore, u_* is a critical point of I. Thus, u_* is a nonzero negative solution of problem (1.1) with $I(u_*) = m_0 < 0$. Let

$$\vartheta(t) = \ln t + \lambda t^{q-\gamma}\,|x|^{-\mu},$$

it is easy to see that

$$\lim_{t\to+0}\vartheta(t) = +\infty, \lim_{t\to+\infty}\vartheta(t) = +\infty$$

and

$$t_* = \left[\frac{(\gamma-q)\lambda}{|x|^\mu}\right]^{\frac{1}{\gamma+1-q}}$$

is the unique minimum point of function ϑ, which implies that

$$\min_{t>0}\vartheta(t) = \vartheta(t_*) = \frac{1}{\gamma+1-q}\ln\frac{\lambda(\gamma-q)}{|x|^\mu} + \frac{1}{\gamma-q}\left[\frac{(\gamma-q)\lambda}{|x|^\mu}\right]^{\frac{1}{\gamma+1-q}} =: C_4 > 0.$$

Consequently, we have

$$-\Delta_p u_* = \frac{u_*^{\gamma-1}\ln u_* + \lambda u_*^{q-1}}{a+b\|u_*\|^p}\ge\frac{C_4 u_*^{\gamma-1}}{a+b\|u_*\|^p}\ge 0.$$

By using the strong maximum principle in [22], we obtain that u_* is a positive solution of problem (1.1) and satisfies $I(u_*) = m_0 < 0$. This proof is complete. □

Theorem 3.3. *For any* $0 < \lambda < T_0$, *problem* (1.1) *has a positive solution* u_{**} *such that* $I(u_{**}) > 0$.

Proof. According to Lemma 3.1, I satisfies the geometric structure of mountain pass. Set

$$c = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I(\eta(t)),$$

where $\Gamma = \{\eta(t) \in C([0, 1], W_0^{1,p}(\Omega)) : \eta(0) = 0, \eta(1) = e\}$. By applying the Mountain Pass Lemma [23] and Lemma 3.1, there exists a sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that

$$I(u_n) \to c > 0 \quad \text{and} \quad |\mathrm{d}I|(u_n) \to 0 \quad \text{as } n \to \infty.$$

By Lemma 2.5, we know that $\{u_n\} \subset W_0^{1,p}(\Omega)$ has a convergent subsequence (still denoted by itself) and there is a function $u_{**} \in W_0^{1,p}(\Omega)$, such that $u_n \to u_{**}$ in $W_0^{1,p}(\Omega)$, $I(u_{**}) = \lim_{n\to\infty} I(u_n) = c$ and $|\mathrm{d}I|(u_n) \to 0$, which implies that $u_{**} \not\equiv 0$. Similar to Theorem 3.2, we obtain $u_{**} > 0$ satisfies problem (1.1) with $I(u_{**}) = c > 0$. Thus, u_{**} is the second positive solution of problem (1.1). This completes the proof of Theorem 1.1. □

Acknowledgments: The authors are grateful to the referees and editors for their careful reading of the manuscript and valuable comments.

Funding information: This work was supported by the Youth Science and Technology Talent Growth Project of the Department of Education of the Guizhou Province (No. Qian Jiao Ji[2024]64), the Science and Technology Plan Project of Guizhou Province (No. Qian Ke He Jichu-[2024]Qingnian 209), and the Scientific Research Staring Foundation of Guizhou Minzu University (No. GZMUZK[2024]QD06).

Author contributions: YW provided the funding support, supervised and led the planning and execution of this research, and reviewed and evaluated the manuscript. JPL and RRW proposed the research idea, formed the overall research objective, and wrote the first draft. JPL and YW revised the final draft. All authors have read and agreed to the published version of the manuscript.

Conflict of interest: The authors declare there is no conflict of interest.

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