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Research Article

Ibtesam Bajunaid*

Subharmonic functions and associated measures in $\mathbb{R}^n$

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Abstract: For subharmonic functions s in $\mathbb{R}^n$, $n \geq 2$, there is an associated Radon measure μ that is used to represent s locally as an integral up to an additive harmonic function. We prove that the total measure $\mu(\mathbb{R}^2)$ is finite if and only if the subharmonic function s has harmonic majorant outside a compact set. However, this relationship does not hold in higher dimensions.

Keywords: subharmonic functions in $\mathbb{R}^n$, associated Radon measure, inversion

MSC 2020: 31A05, 31B05, 31D05

1 Introduction

Potential theory is primarily concerned with the investigation of subharmonic functions. Notable examples of subharmonic functions are as follows:

Logarithmic potential: The function $\log|f(z)|$, where $f(z)$ is an analytic function in the complex plane.

Absolute potential: The function $|f(z)|$, where $f(z)$ is an analytic function in the complex plane.

Newtonian potential: The function $u(x) = -G_y(x)$, where

$$G_y(x) = \frac{1}{|x - y|}$$

is the Newtonian potential with a singularity at the point y in three-dimensional space.

While the transition from the two-dimensional space $\mathbb{R}^2$ to a higher-dimensional space $\mathbb{R}^n$, $n \geq 3$, might seem logical, it introduces significant complexities. In particular, the existence of Green functions in $\mathbb{R}^n$, $n \geq 3$ but not in $\mathbb{R}^2$ fundamentally alters the nature of potential theory. This distinction leads to substantial differences in the techniques and results employed in the study of potential theory in these two settings.

This note explores the relationship between Radon measures and subharmonic functions in $\mathbb{R}^n$, $n \geq 2$. We prove the following: Two-dimensional case $\mathbb{R}^2$: For a Radon measure μ associated with a subharmonic function s,

$$\int_{\mathbb{R}^2} \mathrm{d}\mu(y)$$

is finite if and only if s has a harmonic majorant outside a compact set.

Higher-dimensional case $\mathbb{R}^n$, $n \geq 3$: if

$$\int_{\mathbb{R}^n} \mathrm{d}\mu(y)$$

is finite, then s has a harmonic majorant outside a compact set. However, we provide a counterexample to show that the converse is not always true.

*** Corresponding author: Ibtesam Bajunaid,** Department of Mathematics, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia, e-mail: Bajunaid@ksu.edu.sa

2 Preliminaries

Let $s(x)$ be a subharmonic function on an open, bounded neighborhood ω of x in $\mathbb{R}^2$. Then,

$$s(x) = \int_{\omega} \log|x-y| \mathrm{d}\mu(y) + h(x),$$

where $h(x)$ is a harmonic function on ω and μ is the associated Radon measure to $\Delta s(x)$ in the sense of distributions. For $n \geq 3$, in $\mathbb{R}^n$, the logarithmic kernel is replaced by the kernel $-|x-y|^{2-n}$. When the neighborhood ω is extended to the entire $\mathbb{R}^n$, the integral in the representation converges if and only if

$$\int_1^{\infty} \log|y| \mathrm{d}\mu(y)$$

is finite if $n = 2$; and if and only if

$$\int_1^{\infty} |y|^{2-n} \mathrm{d}\mu(y)$$

is finite when $n \geq 3$.

2.1 Harmonic measure of the point at infinity $\mathfrak{A}$

What makes potential theory on $\mathbb{R}^2$ differ so much from potential theory on $\mathbb{R}^n$, $n \geq 3$?

The study of subharmonic functions on a bounded domain Ω in $\mathbb{R}^n$, $n \geq 2$, has no bearing on $n = 2$ or $n \geq 3$. The problems are the same and methods of solutions are same whether $n = 2$ or $n \geq 3$. However the difference comes to the fore when Ω is unbounded; that is the point at infinity $\mathfrak{A}$ belongs to $\partial\Omega$ when the topological properties are considered on $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\mathfrak{A}\}$. Consider the Dirichlet solution on the unbounded domain $|x| > 1$ with boundary values 0 on $|x| = 1$ and 1 at $\mathfrak{A}$. The solution is $1 - \frac{1}{|x|^{n-2}}$ if $n \geq 3$; however, the situation is different when $n = 2$. Here, $\log|x|$ is an unbounded positive harmonic function on $|x| > 1$. Consequently, if we seek a bounded harmonic solution that is 0 on $|x| = 1$, the only such solution is 0. This is due to the maximum principle applied to bounded harmonic functions on unbounded domains. We term this Dirichlet solution the *harmonic measure of the point at infinity* $\mathfrak{A}$. Thus, the harmonic measure of $\mathfrak{A}$ is 0 if $n = 2$ and positive if $n \geq 3$.

The difference reflects greatly when we study the properties of subharmonic functions defined outside compact sets in $\mathbb{R}^n$ depending on $n = 2$ or $n \geq 3$. For example, if Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and if $s(x)$ is an upper-bounded subharmonic function on Ω such that

$$\limsup_{x \to y \in \partial\Omega} s(x) \leq 0,$$

then $s \leq 0$ on Ω if $n = 2$ (a result that can be used to deduce the Liouville theorem for analytic functions); not so if $n \geq 3$ (example: consider the Dirichlet solution $1 - \frac{1}{|x|^{n-2}}$ in $|x| > 1$ mentioned above).

2.2 Riesz representation for superharmonic functions

Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, and ω be a relatively compact open set $\overline{\omega} \subset \Omega$. A lower semi-continuous function $s(x)$ on Ω, $-\infty < s(x) \leq \infty$, $s \not\equiv \infty$ is superharmonic on Ω if it has the mean value property for such a function $-\Delta s(x) \geq 0$ in the sense if distributions. The fundamental solution $F_n(x,y)$ of the Laplacian in Ω is $-\Delta F_n(x,y) = \delta_y(x)$ where

$$F_2(x,y) = c_2 \log \frac{1}{|x-y|}$$

and

$$F_n(x,y) = c_n \frac{1}{|x-y|^{n-2}} n \geq 3,$$

where c_n are constants.

For a superharmonic function $s(x)$ on Ω, since $-\Delta s(x) \geq 0$ in the sense of distributions, $-\Delta s(x)$ represents a Radon measure on Ω. Therefore,

$$v(x) = \int_{\omega} F_n(x,y) \mathrm{d}\mu(y).$$

Then, $\Delta v(x) = \Delta s(x)$ on ω so that $\Delta[v(x) - s(x)] = 0$ on ω, which gives

$$s(x) = \int_{\omega} F_n(x,y) \mathrm{d}\mu(y) + h(x),$$

where $h(x)$ is harmonic on ω. This representation of $s(x)$ on ω as the sum of an integral using the fundamental solution $F_n(x,y)$ and a harmonic function $h(x)$ on ω is known as **the Riesz representation of the superharmonic function on** ω.

This representation is unique and $h(x)$ is the greatest harmonic minorant of $s(x)$; that is, if $h_1(x)$ is a harmonic function on ω such that $h_1(x) \leq s(x)$, then $h_1(x) \leq h(x)$ on ω.

For more details on the Riesz representation, refer [1,2].

The Weierstrass Theorem: The theorem states that there exists an entire function with arbitrarily prescribed zeros a_n, provided that in the case of infinitely many zeros, $a_n \to \infty$ and no other zeros [3, p. 195].

Analogous theorem for subharmonic functions on $\mathbb{R}^n$, $n \geq 2$, can be proved.

Theorem 1. *If μ is a Radon measure on $\mathbb{R}^n$, $n \geq 2$, then there exists a subharmonic function $s(x)$ on $\mathbb{R}^n$ such that $\Delta s = \mu$ in the sense of distributions.*

Proof. For the proof, we use a Runge-type approximation theorem for harmonic functions: Let $u(x) : B_r(0) \to \mathbb{R}$ be a harmonic function, where $B_r(0) = \{x \in \mathbb{R}^n : |x| < r\}$. For any compact set $K \subset B_r(0)$ and any $\varepsilon > 0$, there exists an entire harmonic function $v(x) : \mathbb{R}^n \to \mathbb{R}$ such that $|u(x) - v(x)| < \varepsilon$ on K.

Consider a random measure μ on $\mathbb{R}^n$, and for each integer $m \geq 3$, let μ_m be the restriction to the annulus $m - 1 < |x| \leq m$.

Then,

$$u_m(x) = \int F_n(x,y) \mathrm{d}\mu_m(y)$$

is harmonic outside $m - 1 < |x| \leq m$, where F_n is the fundamental solution of the Laplacian in $\mathbb{R}^n$. It is known that $u_m(x)$ is harmonic for all x outside the support of μ_m. Now, for each $m \geq 3$, we are given the existence of a harmonic function $v_m(x)$ defined on the entire space $\mathbb{R}^n$ such that

$$|u_m(x) - v_m(x)| < 2^{-m}$$

on $|x| \leq m - 2$. Therefore,

$$f(x) = \sum_{3}^{\infty} [u_m(x) - v_m(x)].$$

Each $[u_m(x) - v_m(x)]$ is harmonic on $|x| < M$ and

$$|u_m(x) - v_m(x)| \leq 2^{-m}$$

on K. The absolute and uniform convergence ensures that

$$\sum_{m>M} [u_m(x) - v_m(x)]$$

represents a harmonic function on K; and

$$\sum_{3}^{M}[u_m(x) - v_m(x)]$$

represents a subharmonic function on K with associated measure μ.

Thus, on $K, f(x)$ is a subharmonic function with associated measure μ. This means that on any bounded open set ω in $\mathbb{R}^n, f(x)$ represents a subharmonic function such that $\Delta s = \mu$ on ω. Since

$$\int_{|y|<2} F_n(x,y)\mathrm{d}\mu(y)$$

is a subharmonic function on $\mathbb{R}^n$, with associated measure inside $|x| < 2$, we obtain

$$s(x) = f(x) + \int_{|y|<2} F_n(x,y)\mathrm{d}\mu(y)$$

is a subharmonic function on $\mathbb{R}^n$ for which $\Delta s = \mu$ in the sense of distributions. □

Theorem 2. *For any non-negative Radon measure μ on an open set Ω in $\mathbb{R}^n$, $n \geq 2$, there exists a subharmonic function $s(x)$ on Ω such that $\Delta s = \mu$ in the sense of distributions.*

Proof. By giving values 0 outside Ω, the measure μ can be considered as one defined on $\mathbb{R}^n$. Then, apply the proof of the above Theorem to conclude. □

This result is often referred to as the converse of Riesz decomposition theorem. This converse has been established in various settings: Brelot [4] originally proved the converse of the Riesz decomposition theorem for mass distributions defined by continuous density functions, Arsove [5], (page 329) extended the proof to cover the case of general Radon measures in open sets of $\mathbb{R}^2$. The theorem has also been shown to hold in the more abstract setting of a Brelot harmonic space (locally compact spaces equipped with a sheaf of harmonic functions satisfying specific axioms) ([6], page 75).

3 Finite total associated measures in $\mathbb{R}^2$

When studying harmonic functions near infinity (i.e., outside a compact set), a powerful technique is the use of inversion. This transformation maps neighborhoods of infinity to neighborhoods of the origin, excluding the origin itself.

3.1 Inversion

Consider the one-point compactification of n-dimensional Euclidean space, denoted as $\overline{\mathbb{R}}^n$. Inversion is a transformation defined within $\overline{\mathbb{R}}^n$ ($n \geq 2$). For a point x, where x is neither 0 nor ∞, the inverse point x^* is given by $x^* = \frac{x}{|x|^2}$. The inversion maps 0 to ∞ and ∞ to 0, see for example, Axler et al. [7], (pages 59–61).

A key property of inversion is that

- It preserves subharmonicity and harmonicity. If $u(x)$ is a subharmonic (or harmonic) function defined outside a compact set in $\mathbb{R}^n$, then the function $v(x^*) = u\left(\frac{x}{|x|^2}\right)$ is also subharmonic (or harmonic) in a neighborhood of the origin, excluding the origin itself.
- It preserves monotonicity, that is, if $f(x)$ and $g(x)$ are functions on $N\backslash\{0\}$, such that $f(x) \geq g(x)$, then $f^*(x^*) = f(x^{**}) \geq g(x^{**}) = g^*(x^*)$. Hence, the monotonicity: $f \geq g$ implies that $f^* \geq g^*$.

For a set E in $\overline{\mathbb{R}}^n$, the set E^* denotes the set of all inverses of points in E.

3.2 Total associated measure

Consider a subharmonic function $s(x)$ defined on $\mathbb{R}^2$ with associated measure μ. If the logarithmic potential

$$\int_{\mathbb{R}^2} \log|x - y| d\mu(y)$$

is finite at even one point, then $s(x)$ can be decomposed into the sum of a logarithmic potential and a harmonic function on $\mathbb{R}^2$. Additionally, the total associated measure $\mu(\mathbb{R}^2)$ is finite. In particular, this occurs when the integral

$$\int_1^\infty \log|y| d\mu(y)$$

is finite.

In this section, we will investigate the following: For a subharmonic function $s(x)$ in $\mathbb{R}^2$, under what additional conditions, beyond the known sufficient condition above, is the associated Radon measure $\mu(\mathbb{R}^2)$ finite? The following lemma provides an answer to this question.

Lemma 1. *For a subharmonic function $u(x)$ defined outside a compact set K in $\mathbb{R}^2$, the total associated measure $\rho(\mathbb{R}^2/K)$ is finite if and only if $u(x)$ has a harmonic function that dominates it outside a compact set.*

Proof. Let u be a subharmonic function defined outside the compact set K in $\mathbb{R}^2$. Let ρ be the associated measure of u such that $\rho(\mathbb{R}^2/K) < \infty$. Then, the inversion u^* of u is a subharmonic function on $N\backslash\{0\}$, where N is a bounded neighborhood of 0. Moreover, the associated measure ρ^* of u^* is given by $\rho^*(A) = \rho(A^*)$ for any compact set A in $N\backslash\{0\}$.

Define the Radon measure σ on $\mathbb{R}^2$ by $\sigma(A) = \rho^*[(N\backslash\{0\}) \cap A]$ for any compact set A. Then, the measure σ has compact support and $\sigma(\{0\}) = 0$. Moreover, the function

$$s(x) = \int_{\mathbb{R}^2} \log|x - y| d\sigma(y)$$

is subharmonic on $\mathbb{R}^2$. Consequently, there exists a harmonic function $h(x) \geq s(x)$ on N. Note that $s(x) = u^*(x) + v(x)$ on $N\backslash\{0\}$ for some harmonic function $v(x)$ (locally the associated measures of the subharmonic functions $s(x)$ and u^* are the same, which implies that v is harmonic). Therefore, $h(x) \geq u^*(x) + v(x)$ on $N\backslash\{0\}$. By inversion, we have $u(x) = u^{**}(x) \leq h^*(x) - v^*(x)$ in a neighborhood of infinity, proving that $u(x)$ has a harmonic majorant near infinity.

Conversely, given a positive real number r, define $u^*(x)$ as the inversion of $u(x)$, where $u(x)$ is subharmonic on the punctured disk $\omega = \{x : 0 < |x| < \frac{1}{r}\}$. Since $u^*(x) - h^*(x) \leq 0$ on ω, there exists a subharmonic extension $s(x)$ to $\{x : |x| < \frac{1}{r}\}$ with associated measure ρ^*. Note that $\rho^*(E) = \rho(E^*)$ for any compact set E in ω, and $\rho^*\left(|x| < \frac{1}{r}\right)$ is finite.

Applying the inversion process once more, we find that

$$\rho(x : |x| > r) \leq \rho^*\left(x : |x| < \frac{1}{r}\right).$$

Since $\rho^*\left(|x| < \frac{1}{r}\right)$ is finite, it follows that $\rho(x : |x| > r)$ is also finite. This completes the proof. □

Theorem 3. *For the subharmonic function s on $\mathbb{R}^2$ with associated measure μ, the total measure $\mu(\mathbb{R}^2)$ is finite if and only if s has a harmonic majorant outside a compact set.*

Proof. Since μ is a Radon measure, the theorem follows from the previous lemma. □

Corollary 1. *Let s be subharmonic on* $\mathbb{R}^2$ *with a finite total associated measure. If u is any subharmonic function on* $\mathbb{R}^2$ *such that* $u(x) \leq s(x)$ *outside a compact set, then the total associated measure of u is finite.*

Remark 1.
(1) A related study can be found in Arsove's work [8], where he conducts a comprehensive investigation of subharmonic functions of potential-type on $\mathbb{R}^2$ by utilizing the characteristic function and associated order of such functions. Additionally, Brelot [9] has achieved similar results using series expansions.
(2) In the context of a Brelot harmonic space Ω without positive potentials, Bajunaid et al. [10] explored the properties of superharmonic functions that possess a harmonic minorant outside a compact set within Ω, without relying on the concept of associated measures. Their research culminates in a Riesz-type decomposition theorem for these superharmonic functions.

4 Study in $\mathbb{R}^n$, $n \geq 3$

In $\mathbb{R}^n$, $n \geq 3$, the condition

$$\int_1^\infty |y|^{2-n} \mathrm{d}\mu(y)$$

being finite is sufficient but not necessary for $\mu(\mathbb{R}^n)$ to be finite. To illustrate this, we consider a specific example in three dimensions, using the terminology of superharmonic functions instead of subharmonic functions for convenience.

A non-negative superharmonic function $s(x)$ in $\mathbb{R}^3$ can be expressed as

$$\int_{\mathbb{R}^3} \frac{1}{|x-y|} \mathrm{d}\mu(y) + h(x),$$

where $h(x)$ is a non-negative harmonic function and μ is a Radon measure proportional to $-\Delta s(x)$ in the sense of distributions. The term

$$p(x) = \int_{\mathbb{R}^3} \frac{1}{|x-y|} \mathrm{d}\mu(y)$$

is referred to as the potential part of the superharmonic function s. The integral is well-defined (not identically equal to infinity) for any Radon measure ρ if and only if

$$\int_{|y|>1} \frac{\mathrm{d}\rho(y)}{|y|}$$

is finite. To show that the mentioned theorem in $\mathbb{R}^2$ does not hold in $\mathbb{R}^3$, it suffices to construct a Radon measure μ on $\mathbb{R}^3$ such that

$$\int_{|y|>1} \frac{\mathrm{d}\rho(y)}{|y|} \text{ is finite and } \int_{\mathbb{R}^3} \mathrm{d}\mu(y) = \infty.$$

Construction: Let μ be the Radon measure on $\mathbb{R}^n$, $n \geq 3$, defined as follows:
(1) $\mu(\{x : |x| \leq 1\}) = 0$,
(2) for x with $|x| = r > 1$, let

$$\mu(\{x : |x| = r\}) = \frac{1}{r^{2+\alpha}},$$

where $0 < \alpha < 1$ and the total measure $\frac{1}{r^{2+\alpha}}$ is uniformly distributed on $|x| = r$.

(3) Let S_n be the surface area of the unit ball in $\mathbb{R}^n$.

Then,

$$\int\limits_{|y|>1} \frac{\mathrm{d}\mu(y)}{|y|^{n-2}} = S_n \int\limits_1^\infty \frac{1}{r^{n-2}} \frac{1}{r^{2+\alpha}} r^{n-1} \mathrm{d}r < \infty.$$

Therefore, $\int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \mathrm{d}\mu(y)$ is a positive superharmonic function on $\mathbb{R}^n$. However,

$$\mu(\mathbb{R}^n) = S_n \int\limits_1^\infty \frac{1}{r^{2+\alpha}} r^{n-1} \mathrm{d}r = \infty.$$

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