Research Article

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Maximal function and generalized fractional integral operators on the weighted Orlicz-Lorentz-Morrey spaces

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Abstract: We define the weighted Orlicz-Lorentz-Morrey and weak weighted Orlicz-Lorentz-Morrey spaces to generalize the Orlicz spaces, the weighted Lorentz spaces, the Orlicz-Lorentz spaces, and the Orlicz-Morrey spaces. Furthermore, necessary and sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator, generalized fractional integral, and maximal operators on the weighted Orlicz-Lorentz-Morrey and weak Orlicz-Lorentz-Morrey spaces are given, based on the exploration of properties of Young functions, B_p weights, and $B_{p,\infty}$ weights. Specifying the weights and the Young functions, we recover the existing results and we obtain new results in the new and old settings.

Keywords: Hardy-Littlewood maximal operator, fractional integral operator, fractional maximal operator, Orlicz-Lorentz-Morrey space, weak Orlicz-Lorentz-Morrey space

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1 Introduction

For a function $\rho : (0, \infty) \to (0, \infty)$, the generalized fractional integral operator I_ρ is defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x - y|)}{|x - y|^n} f(y)\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where we suppose that

$$\int_0^1 \frac{\rho(t)}{t}\mathrm{d}t < \infty, \tag{1.2}$$

and there exist $0 < C < \infty$, $0 < K_1 < K_2 < \infty$, such that

$$\sup_{r \le t \le 2r} \rho(t) \le C \int_{K_1 r}^{K_2 r} \frac{\rho(t)}{t}\mathrm{d}t, \quad r > 0, \tag{1.3}$$

where (1.2) is necessary for the integral in (1.1) to converge for bounded functions with compact support and condition (1.3) was taken into account in [1]. The operator I_ρ was studied in [2,3] to extend the Hardy-Littlewood-Sobolev theorem to Orlicz spaces, and the boundedness of the operator I_ρ on Orlicz-Morrey spaces was considered in [4]. If $\rho(r) = r^\alpha$, $0 < \alpha < \infty$, then I_α is the usual fractional integral operator I_α. Particularly, (1.3)

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holds if ρ satisfies the following doubling condition (2.15). I_α was studied in [5] on the Orlicz-Lorentz spaces. For instance, the operator I_ρ is bounded from $\exp L^p$ to $\exp L^q$, where $\exp L^p$ is the Orlicz space L^Φ with

$$\Phi(t) = \begin{cases} 1/\exp(1/t^p), & \text{if } 0 < t < 1, \\ \exp(t^p), & \text{if } t > 1, \end{cases} \tag{1.4}$$

and

$$\rho(r) = \begin{cases} 1/(\log(1/r))^{\alpha+1}, & \text{if } 0 < t < 1, \\ (\log r)^{\alpha-1}, & \text{if } t > 1, \end{cases} \quad \alpha > 0, \tag{1.5}$$

$0 < p, q < \infty$, $-1/p + \alpha = -1/q$ (see also [6]).

We also investigate the generalized fractional maximal operator M_ρ. For a positive function ρ on $(0, \infty)$, the operator M_ρ is defined by

$$M_\rho f(x) = \sup_{x \in B(a,r)} \frac{\rho(r)}{|B(a,r)|} \int_{B(a,r)} |f(x)| \mathrm{d}x, \tag{1.6}$$

where the supremum is taken over all balls $B(a, r)$ containing x. The operator M_ρ was studied in [7] on generalized Morrey spaces, and the boundedness of M_ρ on Orlicz-Morrey spaces was investigated in [4]. If $\rho(r) = |B(0, r)|^{\alpha/n}$, $\alpha > 0$, then M_ρ is the fractional maximal operator M_α. M_α has been investigated in [5] on the Orlicz-Lorentz spaces. Specially, M_ρ is the Hardy-Littlewood maximal operator M for $\rho = 1$, and M was studied in [8] on the weighted Lorentz spaces. Although it is well known that

$$M_\alpha(f)(x) \leq C I_\alpha(|f|)(x)$$

and the boundedness of M_α can be obtained from one of I_α, we have a better estimate of M_ρ than I_ρ.

In this article, we consider the generalized fractional integral operator I_ρ and the generalized fractional maximal operator M_ρ on weighted Orlicz-Lorentz-Morrey spaces $\Lambda^{\Phi,\phi}(w)$ and weak weighted Orlicz-Lorentz-Morrey spaces $\Lambda^{\Phi,\phi,\infty}(w)$. We give necessary and sufficient conditions for the boundedness of I_ρ and M_ρ on $\Lambda^{\Phi,\phi}(w)$ and $\Lambda^{\Phi,\phi,\infty}(w)$. The Orlicz-Lorentz-Morrey spaces contain L^p spaces, Orlicz spaces, weighted Lorentz spaces, generalized Morrey spaces, and Orlicz-Morrey spaces as special cases. The weak-type spaces have also similar properties.

We organize this article as follows. In Section 2, we give some necessary definitions of the related functions and function spaces. The main results, Theorems 3.1–3.3, are shown in Section 3. In Section 4, the properties of Young functions, weighted Orlicz-Lorentz-Morrey, and weak weighted Orlicz-Lorentz-Morrey spaces are given. The proof of the main results is stated in the Section 5.

Throughout this article, we agree on the convention that the expressions of the form $0 \cdot \infty$, and $\frac{0}{0}$, $\frac{\infty}{\infty}$ are equal to zero. Given $1 \leq p < \infty$, denote by p' its conjugate index that is $\frac{1}{p} + \frac{1}{p'} = 1$. The symbol $f\downarrow$ (resp. $\uparrow$) indicates that f is a non-negative non-increasing (resp. non-decreasing) function in $\mathbb{R}_+$. Note that the constant C, unless otherwise specified, may differ from one occurrence to another.

2 Preliminaries

Let (X, μ) be a σ-finite measure space and $\mathcal{M}(X, \mu)$ be the space of all μ-measurable real-valued functions on X. The decreasing rearrangement f_μ^* of $f \in \mathcal{M}(X, \mu)$ is defined by equality [9]

$$f_\mu^*(t) = \inf\{s : \lambda_f^\mu(s) \leq t\}, \quad t \geq 0,$$

where

$$\lambda_f^\mu(s) = \mu\{x \in X : |f(x)| > s\}, \quad s \geq 0$$

is a distribution function of f. The function $w:\mathbb{R}^n\to\mathbb{R}_+$ or $w:\mathbb{R}_+\to\mathbb{R}_+$ is called a weight function, or simply a weight, whenever w is Lebesgue measurable, not identically equal to zero, and integrable on sets of finite measure. If w is a weight on $\mathbb{R}_+$, then we denote $W(t)=\int_0^t w(s)\,ds$, and we always have that $W(t)<\infty$, $t>0$. If $(X,\mu)=(\mathbb{R}^n,u\mathrm{d}x)$ or $(X,\mu)=(\mathbb{R}_+,u\mathrm{d}x)$, where u is a weight on $\mathbb{R}^n$ or $\mathbb{R}_+$, then we denote $\lambda_f^\mu=\lambda_f^u, f_\mu^*=f_u^*$, and $\mu(E)=u(E)$ for every Lebesgue measurable subset E of $\mathbb{R}^n$ or $\mathbb{R}_+$. Particularly, if $X=(\mathbb{R}^n,\mathrm{d}x)$, we denote $f_\mu^*=f^*$ and $\lambda_f^\mu=\lambda_f$.

Let $L^p_{\mathrm{dec}}(w)$ be the cone of all non-increasing functions in $L^p(w)=L^p(\mathbb{R}_+,w\mathrm{d}x)$, $0<p<\infty$. For the Hardy operator A defined by

$$Af(x)=\frac{1}{x}\int_0^x f(t)\mathrm{d}t,$$

Ariño and Muckenhoupt [10] gave a characterization of the boundedness of $A:L^p_{\mathrm{dec}}(w)\to L^p(w)$ in terms of the inequality on w called condition B_p. Recall that $w\in B_p$ whenever there exists $C>0$ such that

$$r^p\int_r^\infty\frac{w(x)}{x^p}\mathrm{d}x\le C\int_0^r w(x)\mathrm{d}x,\quad r>0.$$

Carro and Soria [11] obtained similar characterization of boundedness of $A:L^p_{\mathrm{dec}}(w)\to L^{p,\infty}(w)$ showing that A is bounded whenever $w\in B_{p,\infty}$, i.e., there exists $C>0$ such that if $p>1$, then

$$\left(\int_0^r\left(\frac{1}{x}\int_0^x w(t)\mathrm{d}t\right)^{-p'}w(x)\mathrm{d}x\right)^{1/p'}\left(\int_0^r w(x)\mathrm{d}x\right)^{1/p}\le Cr,\quad r>0,$$

and if $p\le 1$, then

$$\frac{1}{r^p}\int_0^r w(x)\mathrm{d}x\le\frac{C}{s^p}\int_0^s w(x)\mathrm{d}x,\quad 0<s<r.$$

It is worth indicating that $B_p=B_{p,\infty}$ if $p>1$. Soria [12] found a characterization of the boundedness of $A:L^{p,\infty}_{\mathrm{dec}}(w)\to L^{p,\infty}(w)$ on w called condition $B^\infty_{p,\infty}$ that

$$B^\infty_{p,\infty}=B_p,\quad p>0.$$

For other characterizations of B_p, $B_{p,\infty}$, we refer to [8,13,14].

Let $0<p,q<\infty$. We say that $f\in\mathcal{M}(X,\mu)$ belongs to the Lorentz space $L^{p,q}(X)$ [9,15] if

$$\|f\|_{L^{p,q}(X)}=\left(\int_0^\infty(t^{1/p}f_\mu^*(t))^q\frac{\mathrm{d}t}{t}\right)^{1/q}<\infty.$$

For $0<p\le\infty$, the space $L^{p,\infty}(X)$ is defined as a class of $\mathcal{M}(X,\mu)$ such that

$$\|f\|_{L^{p,\infty}(X)}=\sup_{t>0}t^{1/p}f_\mu^*(t)<\infty,$$

where we agree on the convention that $t^{1/p}=1$ for $p=\infty$. If $(X,\mu)=(\mathbb{R}^n,u(x)\mathrm{d}x)$ or $(X,\mu)=(\mathbb{R}_+,u(x)\mathrm{d}x)$, we use the notation $L^{p,q}(X)=L^{p,q}(u)$.

Let w be a weight on $\mathbb{R}_+$. Using the notation $\|g\|_{L^q\left(\frac{\mathrm{d}y}{y}\right)}=\left(\int_0^\infty|g(y)|^q\ \frac{\mathrm{d}y}{y}\right)^{1/q}$, following [8] or [16], define for $0<p,q<\infty$ the weighted Lorentz space $\Lambda_X^{p,q}(w)$ as a class of $f\in\mathcal{M}(X,\mu)$ such that

$$\|f\|_{\Lambda_X^{p,q}(w)}=\|f_\mu^*\|_{L^{p,q}(w)}=\left\|y\left(\int_0^{\lambda_f^\mu(y)}w(t)\mathrm{d}t\right)^{\frac{1}{p}}\right\|_{L^q\left(\frac{\mathrm{d}y}{y}\right)}<\infty,$$

and the weighted Lorentz space $\Lambda_X^{p,\infty}(w)$ consisting of $f \in \mathcal{M}(X,\mu)$ with

$$\|f\|_{\Lambda_X^{p,\infty}(w)} = \|f_\mu^*\|_{L^{p,\infty}(w)} = \sup_{y>0} y \left(\int_0^{\lambda_f^\mu(y)} w(t)\mathrm{d}t \right)^{\frac{1}{p}} < \infty.$$

Denote $\Lambda_X^p(w) = \Lambda_X^{p,p}(w)$. Note that if $0 < p, q < \infty$, then $\Lambda_X^{p,q}(w) = \Lambda_X^q(\bar{w})$, where $\bar{w} = W^{\frac{q}{p}-1}w$. We know [8, Theorem 2.2.5] that $\Lambda_X^p(w)$ is normable, i.e., there exists a norm in $\Lambda_X^p(w)$ equivalent to the expression $\|\cdot\|_{\Lambda_X^p(w)}$, if and only if $p \geq 1$ and $w \in B_{p,\infty}$, and $\Lambda_X^{p,\infty}(w)$ is normable if and only if $w \in B_p$. If $(X,\mu) = (\mathbb{R}^n, u(x)\mathrm{d}x)$, we denote $\Lambda_X^p(w) = \Lambda_u^p(w)$ and $\Lambda_1^p(w) = \Lambda^p(w)$.

For an increasing function $\Phi : [0,\infty] \to [0,\infty]$, let

$$a(\Phi) = \sup\{t \geq 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \geq 0 : \Phi(t) = \infty\}, \tag{2.1}$$

with convention $\inf\varnothing = \infty$ and $\sup\varnothing = 0$. Then,

$$0 \leq a(\Phi) \leq b(\Phi).$$

Let $\bar{\Phi}$ be the set of all increasing functions $\Phi : [0,\infty] \to [0,\infty]$ such that

$$0 \leq a(\Phi) < \infty,\ 0 < b(\Phi) \leq \infty, \tag{2.2}$$

$$\lim_{t\to 0^+} \Phi(t) = \Phi(0) = 0, \tag{2.3}$$

$$\Phi \text{ is left continuous in } [0, b(\Phi)), \tag{2.4}$$

$$\text{if } b(\Phi) = \infty, \text{ then } \lim_{t\to\infty} \Phi(t) = \Phi(\infty) = \infty, \tag{2.5}$$

$$\text{if } b(\Phi) < \infty, \text{ then } \lim_{t\to b(\Phi)^-} \Phi(t) = \Phi(b(\Phi)). \tag{2.6}$$

In the following, if an increasing and left continuous function $\Phi : [0,\infty) \to [0,\infty)$ satisfies (2.3) and $\lim_{t\to\infty}\Phi(t) = \infty$, then we always regard that $\Phi(\infty) = \infty$ and that $\Phi \in \bar{\Phi}$ (see also [4]).

Definition 2.1. For $\Phi \in \bar{\Phi}$ and $u \in [0,\infty]$, let

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \geq 0 : \Phi(t) > u\}, & u \in [0,\infty), \\ \infty, & u = \infty. \end{cases} \tag{2.7}$$

Suppose $\Phi \in \bar{\Phi}$. Then, Φ^{-1} is finite, increasing, and right continuous on $[0,\infty)$ and positive on $(0,\infty)$. If Φ is bijective from $[0,\infty]$ to itself, then Φ^{-1} is the usual inverse function of Φ. Moreover, if $\Phi \in \bar{\Phi}$, then

$$\Phi(\Phi^{-1}(u)) \leq u \leq \Phi^{-1}(\Phi(u)), \quad \forall u \in [0,\infty]. \tag{2.8}$$

For its proof, see [17, Proposition 2.2].

The notation $\Phi \sim \Psi$, for $\Phi, \Psi \in \bar{\Phi}$, indicates the existence of a universal constant $C > 0$ independent of all parameters involved, so that

$$C^{-1}\Psi(t) \leq \Phi(t) \leq C\Psi(t), \quad \forall t \in [0,\infty].$$

We write $\Phi \approx \Psi$, for $\Phi, \Psi \in \bar{\Phi}$, if there exists a positive constant C such that

$$\Phi(C^{-1}t) \leq \Phi(t) \leq \Phi(Ct), \quad \forall t \in [0,\infty].$$

Then,

$$\Phi \approx \Psi \Leftrightarrow \Psi^{-1} \sim \Psi^{-1} \tag{2.9}$$

(see [17, Lemma 2.8].)

Now, we recall the definition of the Young function and give its generalization.

Definition 2.2. A function $\Phi \in \bar{\Phi}$ is called a Young function (or sometimes also called an Orlicz function) if Φ is convex on $[0, b(\Phi))$. Let Φ_Y be the set of all Young functions. Let $\bar{\Phi}_Y$ be the set of all $\Phi \in \bar{\Phi}$ such that $\Phi \approx \Psi$ for some $\Psi \in \Phi_Y$.

The classes Φ_Y and $\bar{\Phi}_Y \backslash \Phi_Y$ are nonempty. For instance, let

$$\Phi_1(t) = \begin{cases} t, & \text{if } 0 < t < a, \\ \infty, & \text{if } t > a, \end{cases} \qquad \Phi_2(t) = \max(0, t^3 - 8), \quad t \geq 0, \tag{2.10}$$

and

$$\Phi_3(t) = \begin{cases} e^{1-1/t^p}, & \text{if } 0 < t \leq 1, \\ e^{t^p - 1}, & \text{if } t > 1, \end{cases} \tag{2.11}$$

where Φ_3 is not convex near $t = 1$. Then, Φ_1 and Φ_2 given by (2.10) are in Φ_Y, but Φ_3 given by (2.11) is in $\bar{\Phi}_Y \backslash \Phi_Y$.

Orlicz and weak Orlicz spaces on a measure spaces (Ω, μ) are defined as follows: for $\Phi \in \overline{\Phi}_Y$, let $L^\Phi(\Omega, \mu)$ and $wL^\Phi(\Omega, \mu)$ be the set of all measurable functions f such that the following functionals are finite, respectively:

$$\|f\|_{L^\Phi(\Omega,\mu)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\},$$

$$\|f\|_{wL^\Phi(\Omega,\mu)} = \inf\left\{s > 0 : \sup_{t \in (0,\infty)} \Phi(t) \lambda^\mu_{f/s}(t) \leq 1\right\}.$$

Then, $\|\cdot\|_{L^\Phi(\Omega,\mu)}$ and $\|\cdot\|_{wL^\Phi(\Omega,\mu)}$ are quasi Banach spaces. If $\Phi \in \Phi_Y$, then $L^\Phi(\Omega, \mu)$ is a Banach space. For $\Phi, \Psi \in \overline{\Phi}_Y$, if $\Phi \approx \Psi$, then

$$L^\Phi(\Omega, \mu) = L^\Psi(\Omega, \mu)$$

and

$$wL^\Phi(\Omega, \mu) = wL^\Psi(\Omega, \mu),$$

with equivalent quasi norms, respectively.

For a Young function Φ, its complementary function is defined by

$$\widetilde{\Phi}(t) = \begin{cases} \sup\{tu - \Phi(u) : u \in [0, \infty]\}, & t \in [0, \infty), \\ \infty, & t = \infty. \end{cases}$$

Then, $\widetilde{\Phi}$ is also a Young function, and $(\Phi, \widetilde{\Phi})$ is called a complementary pair. The following inequality holds:

$$t \leq \Phi^{-1}(t) \widetilde{\Phi}^{-1}(t) \leq 2t, \quad t > 0, \tag{2.12}$$

which is [18, (1.3)]. We indicate that for $\Phi \in \bar{\Phi}_Y$, the function $\tilde{\Phi}$ is defined to be the function $\tilde{\Psi}$, where $\Psi \in \Phi_Y$ and $\Psi \approx \Phi$.

Definition 2.3.

(i) A function $G : [0, \infty] \to [0, \infty]$ is said to satisfy the Δ_2-condition, denoted by $G \in \Delta_2$, if there exists a constant C such that

$$G(2t) \leq C G(t), \quad \forall t > 0. \tag{2.13}$$

(ii) A function $G : [0, \infty) \to [0, \infty)$ is said to satisfy the ∇_2-condition, if there exists a constant $k > 1$ such that

$$\Phi(t) \leq \frac{1}{2k} \Phi(kt), \quad \forall t > 0. \tag{2.14}$$

(iii) A function $G : [0, \infty) \to [0, \infty)$ is said to satisfy the doubling condition if there exists a positive constant C such that for all $r, s > 0$,

$$\frac{1}{C} \le \frac{\theta(r)}{\theta(s)} \le C, \quad \text{for } \frac{1}{2} \le \frac{r}{s} \le 2. \tag{2.15}$$

(iv) A function $G : [0, \infty) \to [0, \infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a positive constant C such that for all $r, s > 0$,

$$\theta(r) \le C\theta(s) \quad (\text{resp. } \theta(s) \le C\theta(r)), \quad \text{for } r < s. \tag{2.16}$$

In this article, we take into account the following class.

Definition 2.4. [4] Let $\mathcal{G}^{\text{dec}}$ be the set of all functions $\phi : (0, \infty) \to (0, \infty)$ such that ϕ is almost decreasing and that $r \mapsto \phi(r)r^n$ is almost increasing, i.e., there exists a positive constant C such that, for all $r, s \in (0, \infty)$,

$$C\phi(r) \ge \phi(s), \quad \phi(r)r^n \le C\phi(s)s^n, \quad \text{if } r < s.$$

Remark 2.1. [6, Proposition 3.4] Let $\phi \in \mathcal{G}^{\text{dec}}$. Thus, there exists $\tilde{\phi} \in \mathcal{G}^{\text{dec}}$ such that $\phi \sim \phi$ and $\tilde{\phi}$ is continuous and strictly decreasing. Furthermore, if

$$\lim_{r\to 0} \phi(r) = \infty, \quad \lim_{r\to\infty} \phi(r) = 0, \tag{2.17}$$

then $\tilde{\phi}$ is bijective from $(0, \infty)$ to itself.

Given $\Phi \in \bar{\Phi}_Y$ and a weight w on $\mathbb{R}_+$, the Orlicz-Lorentz space $\Lambda_X^G(w)$ (resp. $\Lambda_X^{G,\infty}(w)$) [19–25] is the set of $f \in \mathcal{M}(X, \mu)$ such that for some $\lambda > 0$, we have $I_{X,w}^G(\lambda f) < \infty$ (resp. $I_{X,w}^{G,\infty}(\lambda f) < \infty$), where

$$I_{X,w}^G(f) = \int_0^\infty G(f_\mu^*(t))w(t)\mathrm{d}t, \quad (\text{resp. } I_{X,w}^{G,\infty}(f) = \sup_{t>0} G(f_\mu^*(t))W(t)),$$

and we let

$$\|f\|_{\Lambda_X^G(w)} = \inf\left\{\varepsilon > 0 : I_{X,w}^G\left(\frac{f}{\varepsilon}\right) \le 1\right\} \quad \left(\text{resp. } \|f\|_{\Lambda_X^{G,\infty}(w)} = \inf\left\{\varepsilon > 0 : I_{X,w}^{G,\infty}\left(\frac{f}{\varepsilon}\right) \le 1\right\}\right).$$

We will assume further, without loss of generality, that the weight w vanishes on the interval $[\mu(X), \infty)$ if $\mu(X) < \infty$. When $w \downarrow$ and $\Phi \in \Phi_Y$, $\|\cdot\|_{\Lambda_X^G(w)}$ is a norm [26] and when $W \in \Delta_2$, $\|\cdot\|_{\Lambda_X^G(w)}$ and $\|\cdot\|_{\Lambda_X^{G,\infty}(w)}$ are quasi-norms. If $(X, \mu) = (\mathbb{R}^n, \mathrm{d}x)$, we denote $\Lambda_X^G(w) = \Lambda^G(w)$, $\Lambda_X^{G,\infty}(w) = \Lambda^{G,\infty}(w)$, and if $w = 1$ and $(X, \mu) = (\mathbb{R}^n, u(x)\mathrm{d}x)$, then $\Lambda_X^G(w) = L^G(u)$, which are Orlicz spaces.

For a measurable set $G \in \mathbb{R}^n$, we denote by $|G|$ its Lebesgue measure. $B(a, r)$ is the open ball centered at $a \in \mathbb{R}^n$ and of radius r. In the following, we give the definitions of the Orlicz-Lorentz-Morrey spaces and the weak Orlicz-Lorentz-Morrey spaces.

Definition 2.5. For $\Phi \in \bar{\Phi}_Y$, a function $\phi : (0, \infty) \to (0, \infty)$, a weight function w on $\mathbb{R}_+$, a function $f : \mathbb{R}^n \to (0, \infty)$, and a ball $B = B(a, r)$, let

$$\|f\|_{\Phi,\phi,w,B} = \inf\left\{\lambda > 0 : \left\|\Phi\left(\frac{f\chi_B}{\lambda}\right)\right\|_{\Lambda^1_{\phi_B}(w)} \le 1\right\}$$

and

$$\|f\|_{\Phi,\phi,w,B,\text{weak}} = \inf\left\{\lambda > 0 : \left\|\Phi\left(\frac{f\chi_B}{\lambda}\right)\right\|_{\Lambda^{1,\infty}_{\phi_B}(w)} \le 1\right\},$$

where the weight ϕ_B on $\mathbb{R}^n$ is equal to $\frac{1}{|B|\,\phi(r)}$. Let the weighted Orlicz-Lorentz-Morrey spaces $\Lambda^{\Phi,\phi}(w)$ and the weak weighted Orlicz-Lorentz-Morrey spaces $\Lambda^{\Phi,\phi,\infty}(w)$ be the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{\Lambda^{\Phi,\phi}(w)} = \sup_B \|f\|_{\Phi,\phi,w,B}$$

and

$$\|f\|_{\Lambda^{\Phi,\phi,\infty}(w)} = \sup_B \|f\|_{\Phi,\phi,w,B,\,\mathrm{weak}}$$

is finite, respectively, where the supremum is taken over all balls B in $\mathbb{R}^n$.

Clearly, we have

$$\|f\|_{\Phi,\phi,w,B} = \inf\left\{\lambda > 0 : \int_0^\infty \Phi\left(\frac{(f\chi_B)^*(\phi(r)|B|t)}{\lambda}\right) w(t)\mathrm{d}t \le 1\right\} \tag{2.18}$$

and

$$\|f\|_{\Phi,\phi,w,B,\,\mathrm{weak}} = \inf\left\{s > 0 : \sup_{t>0} \Phi(t) W\left(\frac{\lambda_{\frac{f\chi_B}{s}}(t)}{\phi(r)|B|}\right) \le 1\right\}, \tag{2.19}$$

where r is the radius of the ball B and χ_B is the characteristic function of B.

Obviously, $\Lambda^{\Phi,\phi}(w)$ and $\Lambda^{\Phi,\phi,\infty}(w)$ are quasi-Banach spaces if $W \in \Delta_2$. Furthermore, we indicated that, for $\Phi, \Psi \in \bar{\Psi}_Y$, if $\Phi \approx \Psi$ and $\phi \approx \psi$, then

$$\Lambda^{\Phi,\phi}(w) = \Lambda^{\Psi,\psi}(w),\ \Lambda^{\Phi,\phi,\infty}(w) = \Lambda^{\Psi,\psi,\infty}(w),$$

with equivalent quasi-norms.

If $w = 1$, then $\Lambda^{\Phi,\phi}(w) = L^{\Phi,\phi}$ and $\Lambda^{\Phi,\phi,\infty}(w) = \mathrm{w}L^{\Phi,\phi}$, which are Orlicz-Morrey and weak Orlicz-Morrey spaces in [4]. If $\phi(r) = 1/r^n$, then $\Lambda^{\Phi,\phi}(w) = \Lambda^{\Phi}(w)$ and $\Lambda^{\Phi,\phi,\infty}(w) = \Lambda^{\Phi,\infty}(w)$, which represent the Orlicz-Lorentz and weak Orlicz-Lorentz spaces. If $\phi(r) = r^{\lambda/q-n/p}$, $\Phi(t) = t^q$ and $w(t) = t^{q/p-1}$, then $\Lambda^{\Phi,\phi}(w) = \mathcal{M}_{p,q,\lambda}$, which are Morrey-Lorentz spaces in [27] and [28]. If $\phi(r) = 1/r^n$, $\Phi(t) = t^p$, then $\Lambda^{\Phi,\phi}(w) = \Lambda^p(w)$ and $\Lambda^{\Phi,\phi,\infty}(w) = \Lambda^{p,\infty}(w)$, which take the weighted Lorentz and weak weighted Lorentz spaces. If $w = 1$ and $\phi(r) = 1/r^n$, then $\Lambda^{\Phi,\phi}(w) = L^{\Phi}$ and $\Lambda^{\Phi,\phi,\infty}(w) = \mathrm{w}L^{\Phi}$, which stand for usual Orlicz and weak Orlicz spaces. If $\Phi(t) = t^p$, $w = 1$, $1 \le p < \infty$, then the spaces $\Lambda^{\Phi,\phi}(w)$ and $\Lambda^{\Phi,\phi,\infty}(w)$ are reduced to the spaces $L^{p,\phi}$ and $wL^{p,\phi}$, which are generalized Morrey and weak Morrey spaces, respectively. Orlicz spaces were investigated in [29,30]. Weak Orlicz spaces were studied by, for example, [31–33]. Morrey spaces were introduced by [34] and were generalized in, i.e., [35–38]. Morrrey-Lorentz spaces were studied in [27] and [28]. Orlicz-Morrey and weak Orlicz-Morrey spaces were explored in [6,39–42]. For other kinds of Orlicz-Morrey spaces, see, e.g., [7,43–46]. Recently, a kind of generalized Orlicz-Morrey space $\mathcal{M}^u_{\Phi,v}$ was defined in [47] and when $w = 1$, $v(x, r) = 1$ and $u(x, r) = \phi(r)r^n$, then $\Lambda^{\Phi,\phi}(w) = \mathcal{M}^u_{\Phi,v}$. Weighted Lorentz spaces and weak weighted Lorentz spaces were studied in, e.g., [10,11,14,16,48].

3 Main results

We first consider boundedness of the Hardy-Littlewood maximal operator on weighted Orlicz-Lorentz-Morrey spaces and weak weighted Orlicz-Lorentz-Morrey spaces.

Theorem 3.1. *Let $\Phi \in \bar{\Phi}_Y$, $\phi \in \mathcal{G}^{\mathrm{dec}}$, and $w \in B_{1,\infty}$, $w(0) < \infty$. Then, the Hardy-Littlewood maximal operator M is bounded from $\Lambda^{\Phi,\phi}(w)$ to $\Lambda^{\Phi,\phi,\infty}(w)$. Moreover, if $\Phi \in \nabla_2$ and*

$$\int_0^s \widetilde{\Phi}\left(\frac{t}{W(t)}\right) w(t)\mathrm{d}t + \int_s^\infty \widetilde{\Phi}\left(\frac{s}{W(t)}\right) w(t)\mathrm{d}t \le CW(s), \quad s > 0, \tag{3.1}$$

then the Hardy-Littlewood maximal operator M is bounded from $\Lambda^{\Phi,\phi}(w)$ to itself and if $\Phi \in \nabla_2$,

$$\int_r^\infty W^{-1}\left(\frac{1}{\Phi(t)}\right)dt \le C\frac{r}{W(\Phi(r))}, \quad 0 < r < \infty \tag{3.2}$$

and

$$J = \sup_{r>0} W^{-1}(r)W\left(\frac{1}{r}\right) < \infty, \tag{3.3}$$

then M is bounded from $\Lambda^{\Phi,\phi,\infty}(w)$ to itself.

Remark 3.1. (1) Since Theorem 3.1 shows a sufficient condition of the boundedness of M on the weighted Orlicz-Lorentz-Morrey spaces, it can be considered a kind of generalization of [10, Theorem 1.7], where a characterization of the boundedness of the operator M on the weighted Lorentz spaces $\Lambda^p(w)$ is given.

(2) If $w = 1$ and $\Phi \in \Delta_2$, then $w \in B_{1,\infty}$ and (3.1) hold. Indeed, (3.1) establishes from the fact that $\widetilde{\Phi} \in \nabla_2$ which implies that there exists a $p_1 > 1$ such that $t^{p_1}/\widetilde{\Phi}(t)$ is almost decreasing and the following estimate:

$$\int_s^\infty \widetilde{\Phi}\left(\frac{s}{W(t)}\right)w(t)dt = s\int_0^1 \frac{\widetilde{\Phi}(h)}{h^2}dh = s\int_0^1 \frac{\widetilde{\Phi}(h)}{h^{p_1}}h^{p_1-2}dh \le C\frac{\widetilde{\Phi}(1)}{p_1 - 1}s = C\frac{\widetilde{\Phi}(1)}{p_1 - 1}W(s).$$

It can also be explained by [49, Theorem 2.1]. If $w = 1$ and $\Phi \in \nabla_2$, then (3.2) and (3.3) hold. Indeed, for the same reason, in light of the fact that $\Phi \in \nabla_2$ which yields that there exists a $p_2 > 1$ such that $t^{p_2}/\Phi(t)$ is almost decreasing, it follows that

$$\begin{aligned}\int_r^\infty W^{-1}\left(\frac{1}{\Phi(t)}\right)dt &= \int_r^\infty \frac{t^{p_2}}{\Phi(t)}\frac{1}{t^{p_2}}dt \\ &\le \int_r^\infty \frac{t^{p_2}}{\Phi(t)}\frac{1}{t^{p_2}}dt \\ &\le C\int_r^\infty \frac{r^{p_2}}{\Phi(r)}\frac{1}{t^{p_2}}dt \\ &\le C\frac{r}{\Phi(r)} = C\frac{r}{W(\Phi(r))},\end{aligned}$$

which implies (3.2). Thus, Theorem 3.1 extends [4, Theorem 3.3] from Orlicz-Morrey spaces and weak Orlicz-Morrey spaces to weighted Orlicz-Lorentz-Morrey spaces and weak weighted Orlicz-Lorentz-Morrey spaces, respectively. The aforementioned theorem is also an extension of [6, Theorem 6.1].

The next theorem discusses sufficient and necessary conditions of the boundedness of the operator I_ρ on weighted Orlicz-Lorentz-Morrey spaces and weak weighted Orlicz-Lorentz-Morrey spaces.

Theorem 3.2. *Let $\Phi, \Psi \in \bar{\Phi}_Y$, $\phi \in \mathcal{G}^{\mathrm{dec}}$. Assume that $\rho : (0, \infty) \to (0, \infty)$ satisfy* (1.2) *and* (1.3).

(*i*) *If $w \in B_{1,\infty}$, $w(0) < \infty$ and there exists a positive constant A such that, for all $r \in (0, \infty)$,*

$$\int_0^r \frac{\rho(t)}{t}dt\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right) + \int_r^\infty \frac{\rho(t)}{t}\phi(t)W\left(\frac{1}{\phi(t)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(t)}\right)}\right)dt \le A\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right), \tag{3.4}$$

then for any positive constant C_0, there exists a positive constant C_1 such that, for all $f \in \Lambda^{\Phi,\phi}(w)$ with $\|f\|_{\Lambda^{\Phi,\phi}(w)} \neq 0$,

$$\Psi\left(\frac{I_\rho f(x)}{C_1\|f\|_{\Lambda^{\Phi,\phi}(w)}}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{\Lambda^{\Phi,\phi}(w)}}\right), \quad x \in \mathbb{R}^n. \tag{3.5}$$

Thus, I_ρ is bounded from $\Lambda^{\Phi,\phi}(w)$ to $\Lambda^{\Psi,\phi,\infty}(w)$. Moreover, if $\Phi \in \nabla_2$ and there exists a positive constant B such that, for all $r \in (0,\infty)$,

$$\int_0^r \frac{\rho(t)}{t}\mathrm{d}t\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right) + \int_r^\infty \frac{\rho(t)}{t}\Phi^{-1}(W^{-1}(\phi(r)))\mathrm{d}t \le B\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right), \tag{3.6}$$

then for all $f \in \Lambda^{\Phi,\phi,\infty}(w)$ with $\|f\|_{\Lambda^{\Phi,\phi,\infty}(w)} \neq 0$,

$$\Psi\left(\frac{I_\rho f(x)}{C_1\|f\|_{\Lambda^{\Phi,\phi,\infty}(w)}}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{\Lambda^{\Phi,\phi,\infty}(w)}}\right), \quad x \in \mathbb{R}^n. \tag{3.7}$$

Hence, if $\Phi \in \nabla_2$ and (3.1) *holds, then the operator I_ρ is bounded from $\Lambda^{\Phi,\phi}(w)$ to $\Lambda^{\Psi,\phi}(w)$ by* (3.5), *and if $\Phi \in \nabla_2$,* (3.2) *and* (3.3) *hold, then I_ρ is bounded from $\Lambda^{\Phi,\phi,\infty}(w)$ to $\Lambda^{\Psi,\phi,\infty}(w)$ by* (3.7).

(ii) *If $W \in \Delta_2$ and I_ρ is bounded from $\Lambda^{\Phi,\phi}(w)$* to *$\Lambda^{\Psi,\phi,\infty}(w)$, then there exists a positive constant C such that, for all $r \in (0,\infty)$,*

$$\int_0^r \frac{\rho(t)}{t}\mathrm{d}t\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right) \le C\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right). \tag{3.8}$$

Furthermore, under the assumption that there exists a positive constant C such that, for all $r \in (0,\infty)$,

$$\int_0^r \Phi\left[\phi(s)W\left(\frac{1}{\phi(s)}\right)\Phi^{-1}\left(\frac{1}{W(\frac{1}{\phi(s)})}\right)\right]w\left(\frac{s^n}{\phi(r)r^n}\right)s^{n-1}\mathrm{d}s \le C\phi(r)r^n, \tag{3.9}$$

if I_ρ is bounded from $\Lambda^{\Phi,\phi}(w)$ to $\Lambda^{\Psi,\phi,\infty}(w)$, then (3.6) *holds.*

Remark 3.2. The aforementioned theorem is an extension of [4, Theorem 3.4], which is corresponding to the case of $w = 1$. It is also generalization of [6, Theorem 7.3].

The following theorem shows sufficient and necessary conditions of the boundedness of the operator M_ρ on weighted Orlicz-Lorentz-Morrey spaces and weak weighted Orlicz-Lorentz-Morrey spaces.

Theorem 3.3. *Let $\Phi, \Psi \in \bar{\Phi}_Y$, $\phi \in \mathcal{G}^{\mathrm{dec}}$, $\rho : (0,\infty) \to (0,\infty)$.*

(*i*) *Assume that $w \in B_{1,\infty}$, $w(0) < \infty$, and $\lim_{r\to 0}\phi(r) = 0$, or that $\frac{\Phi^{-1}(t)}{\Psi^{-1}(t)}$ is almost decreasing on* $(0,\infty)$. *If there exists a constant $C > 0$ such that, for all $r \in (0,\infty)$,*

$$\sup_{0<t\le r} \rho(t)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right) \le C\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right), \tag{3.10}$$

then, for any $C_0 > 0$, there exists $C_1 > 0$ such that for all $f \in \Lambda^{\Phi,\phi}(w)$ with $f \neq 0$,

$$\Psi\left(\frac{M_\rho f(x)}{C_1\|f\|_{\Lambda^{\Phi,\phi}(w)}}\right) \le \Phi\left(\frac{Mf(x)}{C_0\|f\|_{\Lambda^{\Phi,\phi}(w)}}\right), \quad x \in \mathbb{R}^n. \tag{3.11}$$

Hence, M_ρ is bounded from $\Lambda^{\Phi,\phi}(w)$ to $\Lambda^{\Psi,\phi,\infty}(w)$. If, moreover, $\Phi \in \nabla_2$ and for all $r \in (0,\infty)$,

$$\sup_{0<t\le r} \rho(t)\Phi^{-1}(W^{-1}(\phi(r))) \le C\Psi^{-1}(W^{-1}(\phi(r))), \tag{3.12}$$

then for all $f \in \Lambda^{\Phi,\phi,\infty}(w)$ *with* $f \neq 0$,

$$\Psi\left(\frac{M_\rho f(x)}{C_1\|f\|_{\Lambda^{\Phi,\phi,\infty}(w)}}\right) \leq \Phi\left(\frac{Mf(x)}{C_0\|f\|_{\Lambda^{\Phi,\phi,\infty}(w)}}\right), \quad x \in \mathbb{R}^n. \tag{3.13}$$

Consequently, if $\Phi \in \nabla_2$ *and* (3.1) *holds, then the operator* M_ρ *is bounded from* $\Lambda^{\Phi,\phi}(w)$ *to* $\Lambda^{\Psi,\phi}(w)$ *by* (3.11), *and if* $\Phi \in \nabla_2$, (3.2) *and* (3.3) *hold, then the operator* M_ρ *is bounded from* $\Lambda^{\Phi,\phi,\infty}(w)$ *to* $\Lambda^{\Psi,\phi,\infty}(w)$ *by* (3.13).

(*ii*) *If* M_ρ *is bounded from* $\Lambda^{\Phi,\phi}(w)$ *to* $\Lambda^{\Psi,\phi,\infty}(w)$, *then* (3.10) *holds.*

Remark 3.3. (1) According to (1.3) and (3.4), we have that

$$\sup_{0<t\leq r} \rho(t)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right) \leq \int_0^{K_2 r} \frac{\rho(t)}{t}\mathrm{d}t\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right) \leq C\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right),$$

which yields (3.10). If $\rho(t) = (\ln(1/r))^{-\alpha}$ for small $r > 0$ or $\rho(r) = (\ln r)^\alpha$ with $\alpha \geq 0$, then (3.10) is strictly weaker than (3.4).

(2) The aforementioned theorem is an extension from Orlicz-Morrey spaces [4, Theorem 3.5] to weighted Orlicz-Lorentz-Morrey spaces. It generalizes the results in [17,50,51] as well.

4 Some properties of the Orlicz-Lorentz-Morrey spaces

In this section, we give some properties of weighted Orlicz-Lorentz-Morrey and weak weighted Orlicz-Lorentz-Morrey spaces.

Lemma 4.1. *Let* $\Phi \in \Phi_Y$ *and* $\phi \in \mathcal{G}^{\mathrm{dec}}$. *Then, for any ball* $B = B(a, r)$,

$$\|\chi_{B(a,r)}\|_{\Lambda^{\Phi,\phi}(w)} \sim \|\chi_{B(a,r)}\|_{\Lambda^{\Phi,\phi,\infty}(w)} \sim \frac{1}{\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)}.$$

Proof. First, we claim that if $B \cap B(a, r) \neq \emptyset$ and $B \cap B(a, r)^c \neq \emptyset$, then

$$\|\chi_{B(a,r)}\|_{\Phi,\phi,w,B} \leq \max\left\{\sup_{B\subset B(a,r)} \|\chi_{B(a,r)}\|_{\Phi,\phi,w,B}, \sup_{B\supset B(a,r)} \|\chi_{B(a,r)}\|_{\Phi,\phi,w,B}\right\}. \tag{4.1}$$

Indeed, let $B = B(x, R)$ with $B \cap B(a, r) \neq \emptyset$ and $B \cap B(a, r)^c \neq \emptyset$. Let $B_1 = B(a, R)$, and thus, $|B| = |B_1|$. If $R \leq r$, then by (2.18),

$$\begin{aligned}\|\chi_{B(a,r)}\|_{\Phi,\phi,w,B} &= \inf\left\{\lambda > 0 : \int_0^\infty \Phi\left(\frac{(\chi_{B(a,r)\cap B})^*(\phi(r)|B|t)}{\lambda}\right)w(t)\mathrm{d}t \leq 1\right\} \\ &\leq \inf\left\{\lambda > 0 : \int_0^\infty \Phi\left(\frac{(\chi_{B(a,r)\cap B_1})^*(\phi(r)|B_1|t)}{\lambda}\right)w(t)\mathrm{d}t \leq 1\right\} \\ &\leq \|\chi_{B(a,r)}\|_{\Phi,\phi,w,B_1}.\end{aligned}$$

Analogously, if $R > r$, we also have

$$\|\chi_{B(a,r)}\|_{\Phi,\phi,w,B} \leq \|\chi_{B(a,r)}\|_{\Phi,\phi,w,B_1}.$$

So (4.1) follows. According to the definition of $\|\cdot\|_{\Lambda^{\Phi,\phi}(w)}$ and (4.1), we have

$$\begin{aligned}\|\chi_{B(a,r)}\|_{\Lambda^{\Phi,\phi}(w)} &= \sup_B \|f\|_{\Phi,\phi,w,B}\\ &= \max\left\{\sup_{B\subset B(a,r)} \|\chi_{B(a,r)}\|_{\Phi,\phi,w,B},\ \sup_{B\supset B(a,r)} \|\chi_{B(a,r)}\|_{\Phi,\phi,w,B}\right\}\\ &= \max\left\{\sup_{B\subset B(a,r)}\left\{\inf\left\{\lambda : \left\|\Phi\left(\frac{\chi_{B(a,r)\cap B}}{\lambda}\right)\right\|_{\Lambda^1_{\phi_B}(w)} \le 1\right\}\right\},\right.\\ &\quad\left.\times \sup_{B\supset B(a,r)}\left\{\inf\left\{\lambda : \left\|\Phi\left(\frac{\chi_{B(a,r)\cap B}}{\lambda}\right)\right\|_{\Lambda^1_{\phi_B}(w)} \le 1\right\}\right\}\right\}\\ &= \max\left\{\sup_{R<r}\frac{1}{\Phi^{-1}\{[W(\frac{1}{\phi(R)})]^{-1}\}},\ \sup_{R>r}\frac{1}{\Phi^{-1}\left(\frac{1}{W(\frac{|B(a,r)|}{\phi(R)\,|B(a,R)|})}\right)}\right\}.\end{aligned}$$

Note that $\phi \in \mathcal{G}^{\mathrm{dec}}$, which implies

$$\phi(r)r^n \le C\phi(s)s^n, \quad r < s.$$

Hence, for $R > r$,

$$\frac{1}{\Phi^{-1}\left(\frac{1}{W(\frac{|B(0,r)|}{\phi(R)\,|B(a,R)|})}\right)} \le \frac{1}{\Phi^{-1}\left(\frac{1}{W(\frac{C\,|B(0,r)|}{\phi(r)r^n})}\right)} \sim \frac{1}{\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)}.$$

Therefore,

$$\|\chi_{B(a,r)}\|_{\Lambda^{\Phi,\phi}(w)} \sim \frac{1}{\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)}.$$

In view of (2.19), by the same token,

$$\|\chi_{B(a,r)}\|_{\Lambda^{\Phi,\phi,\infty}(w)} \sim \frac{1}{\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)}.$$

□

If $\|\cdot\| : \mathcal{M}(X,\mu) \to [0,+\infty]$ is a positively homogeneous functional and $E = \{f \in (\mathcal{M},\mu) : \|f\| < \infty\}$, we define the associate norm by

$$\|f\|_{E'} = \sup\left\{\int_X |f(x)g(x)|\,\mathrm{d}\mu : \|g\| \le 1, g \in E\right\}, \quad f \in \mathcal{M}(X,\mu).$$

The associate space of E is then $E' = \{f \in \mathcal{M}(X,\mu) : \|f\|_{E'} < \infty\}$ (see [8, Definition 2.4.1]).

Lemma 4.2. *Let $\phi : (0,\infty) \to (0,\infty)$ and (3.1) hold. Then, there exists a positive constant C such that for all $f \in \Lambda^{\Phi,\phi}(w)$ and for all $B = B(a,r)$,*

$$\frac{1}{|B|}\int_B |f(x)|\,\mathrm{d}x \le C\phi(r)W\left(\frac{1}{\phi(r)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\|f\|_{\Phi,\phi,B,w}.$$

Proof. By [52, Theorem 3.1], we know

$$(\Lambda_X^{\Phi}(w))' = M_{\widetilde{\Phi},w},$$

where $\|\cdot\|_{M_{\Phi_*,w}}$ is defined as

$$\|f\|_{M_{\widetilde{\Phi},w}} = \|S(f_\mu^*)\|_{L^{\widetilde{\phi}}(w)},$$

where for a weight w in $\mathbb{R}_+$, the operator S is defined on the nonnegative measurable functions on $\mathbb{R}_+$ as follows:

$$S(f)(x) = \frac{1}{W(t)}\int_0^t f(s)\mathrm{d}s.$$

If letting $(X,\mu) = (B, \frac{1}{\phi(r)\,|\,B\,|}\mathrm{d}x)$, we obtain

$$\frac{1}{|B|}\int_{B(a,r)} |f(x)|\mathrm{d}x = \phi(r)\int_B f(x)\frac{1}{\phi(r)|B|}\mathrm{d}x \le \phi(r)\|f\|_{\Lambda_X^{\Phi}(w)}\|\chi_B\|_{M_{\widetilde{\Phi},w}}. \tag{4.2}$$

Note that

$$(\chi_B)_\mu^*(s) = \begin{cases} 1, & \text{if } 0 < s < \dfrac{1}{\phi(r)}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, by (3.1),

$$\int_0^\infty \widetilde{\Phi}(S((\chi_B)_\mu^*)(t))w(t)\mathrm{d}t \le C\int_0^\infty \widetilde{\Phi}((\chi_B)_\mu^*(t))w(t)\mathrm{d}t,$$

for all balls $B \subset \mathbb{R}^n$. Since the modular inequality is stronger than the norm inequality, we obtain that

$$\|\chi_B\|_{M_{\widetilde{\phi},w}} \le C\|(\chi_B)_\mu^*\|_{L^{\widetilde{\Phi}}(w)},$$

for all balls $B \subset \mathbb{R}^n$. And thus by the fact that $\|f\|_{\Lambda_X^{\Phi}(w)} = \|f\|_{\Phi,\phi,B,w}$, we obtain that

$$\begin{aligned} \text{RHS of } (4.2) &\le C\phi(r)\|(\chi_B)_\mu^*\|_{L^{\widetilde{\Phi}}(w)}\|f\|_{\Lambda_X^{\Phi}(w)} \\ &\le C\phi(r)\frac{1}{\widetilde{\Phi}^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)}\|f\|_{\Lambda_X^{\Phi}(w)} \\ &\sim \phi(r)W\left(\frac{1}{\phi(r)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\|f\|_{\Phi,\phi,B,w}, \quad \text{by (2.12).} \end{aligned}$$

□

Lemma 4.3. *Let $\Phi \in \Phi_Y$, $\phi : (0,\infty) \to (0,\infty)$ and w satisfy (3.2). Then, there exists a positive constant C such that, for all $f \in \Lambda^{\Phi,\phi,\infty}(w)$ and for all $B = B(a,r)$,*

$$\frac{1}{|B|}\int_B |f(x)|\mathrm{d}x \le C\Phi^{-1}(W^{-1}(\phi(r)))\|f\|_{\Phi,\phi,w,B,\mathrm{weak}}.$$

Proof. First, let $\Phi \in Y^{(1)} \cup Y^{(2)}$. Assume that $\|f\|_{\Phi,\phi,w,B,\mathrm{weak}} = 1$. If $b(\Phi) < \infty$ and $t \in [b(\Phi),\infty)$, then $\lambda_f(t) = 0$. For a ball $B(a,r)$, let

$$t_0 = \Phi^{-1}(W^{-1}(\phi(r))).$$

Then, $\Phi(t_0) = W^{-1}(\phi(r)) \in (0, \infty)$. Thus, $t_0 \in (a(\Phi), b(\Phi))$. Hence,

$$\int_B |f(x)|\mathrm{d}x = \int_0^{b(\Phi)} \lambda_{f\chi_B}(t)\mathrm{d}t \le t_0|B| + \int_{t_0}^{b(\Phi)} \lambda_{f\chi_B}(t)\mathrm{d}t. \tag{4.3}$$

But

$$\begin{aligned}\int_{t_0}^{b(\Phi)} \lambda_{f\chi_B}(t)\mathrm{d}t &\le \int_{t_0}^{b(\Phi)} W^{-1}\left(\frac{1}{\Phi(t)}\right)\phi(r)|B|\mathrm{d}t, \quad \text{since } \|f\|_{\Lambda^{\Phi,\phi,\infty}(w)} = 1\\ &\le C|B|\phi(r)\frac{t_0}{W(\Phi(t_0))}, \quad \text{by (3.2)}\\ &\le C|B|t_0.\end{aligned}$$

Combining this and (4.3) yields that

$$\int_B |f(x)|\mathrm{d}x \le t_0|B|.$$

Second, let $\Phi \in Y^{(3)}$. By [4, Remark 4.2 (ii)], for any $0 < \delta < 1$, there exists $\Phi_1 \in Y^{(2)}$ such that

$$\Phi_1(\delta t) \le \Phi(t) \le \Phi_1(t), \quad t \in [0, \infty).$$

Therefore,

$$\delta\Phi^{-1}(u) \le \Phi_1^{-1}(u) \le \Phi^{-1}(u),$$

and according to the definition of $\|\cdot\|_{\Phi,\phi,w,B,\mathrm{weak}}$, we know

$$\delta\|f\|_{\Phi_1,\phi,w,B,\mathrm{weak}} \le \|f\|_{\Phi,\phi,w,B,\mathrm{weak}} \le \|f\|_{\Phi_1,\phi,w,B,\ \mathrm{weak}}.$$

By the first case, we obtain

$$\frac{1}{|B|}\int_B |f(x)|\mathrm{d}x \le C\Phi_1^{-1}(W^{-1}(\phi(r)))\|f\|_{\Phi_1,\phi,w,B,\ \mathrm{weak}} \le C\Phi^{-1}(W^{-1}(\phi(r)))\|f\|_{\Phi,\phi,w,B,\ \mathrm{weak}}/\delta,$$

which induces the conclusion by letting $\delta \to 1$. □

Lemma 4.4. *Let $\phi \in \mathcal{G}^{\mathrm{dec}}$, $w \in B_{1,\infty}$, and* (3.1) *hold. If $f \in \Lambda^{\Phi,\phi}(w)$,* $\operatorname{supp} f \cap 2B = \emptyset$, *and $B = B(a, r)$, then*

$$Mf(x) \le C\phi(r)W\left(\frac{1}{\phi(r)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\|f\|_{\Lambda^{\Phi,\phi}(w)}, \quad \text{for } x \in B,$$

where C only depends on Φ, ϕ, and w.

Proof. Let $x \in B'$ with a radius r'. If $r' < \frac{r}{2}$, then $\frac{1}{|B'|}\int_{B'}|f(y)|\mathrm{d}y = 0$. Since $w \in B_{1,\infty}$, the function

$$tW(1/t)\Phi^{-1}\left(\frac{1}{W(\frac{1}{t})}\right)$$

is almost increasing and satisfies the doubling condition. Thus, if $r' > r/2$, by Lemma 4.2 and $\phi \in \mathcal{G}^{\mathrm{dec}}$,

$$\frac{1}{|B'|}\int_{B'} |f(y)|\mathrm{d}y \le C\phi(r')\|f\|_{\Phi,\phi,B',w} W\left(\frac{1}{\phi(r')}\right)\Phi^{-1}\left(\frac{1}{W(\frac{1}{\phi(r')})}\right)$$

$$\leq C\phi(r)W\left(\frac{1}{\phi(r)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\|f\|_{\Phi,\phi,B',w}.$$

Hence,

$$Mf(x) \leq C\phi(r)W\left(\frac{1}{\phi(r)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\|f\|_{\Lambda^{\Phi,\phi}(w)}.$$

The lemma is proved. □

Lemma 4.5. *Let $\Phi \in \Phi_Y$, $\phi \in \mathcal{G}^{\mathrm{dec}}$, and w satisfy (3.2) and $f \in \Lambda^{\Phi,\phi,\infty}(w)$. For a ball $B = B(a,r)$, if supp $f \cap 2B = \emptyset$, then*

$$Mf(x) \leq C\Phi^{-1}(W^{-1}(\phi(r)))\|f\|_{\Lambda^{\Phi,\phi,\infty}(w)}, \quad \text{for } x \in B,$$

where the constant C_3 depends only on Φ, ϕ, and w.

Proof. For any ball $B' \in x$ whose radius is s, if $s \leq r/2$, then $\int_{B'}|f(x)|\mathrm{d}x = 0$. If $s > r/2$, then by Lemma 4.3 and $\phi \in \mathcal{G}^{\mathrm{dec}}$,

$$\int_{B'}|f(y)|\mathrm{d}y \leq C\Phi^{-1}(W^{-1}(\phi(s)))\|f\|_{\Lambda^{\Phi,\phi,\infty}(w)} \leq C\Phi^{-1}(W^{-1}(\phi(r)))\|f\|_{\Lambda^{\Phi,\phi,\infty}(w)},$$

since $\Phi^{-1}(W^{-1}(\phi(r)))$ is almost decreasing and satisfies the doubling condition. □

5 Proofs of main results

5.1 Proof of Theorem 3.1

Without loss of generality, we may assume that $\Phi \in \Phi_Y$. Let $f \in \Lambda^{\Phi,\phi}(w)$ and $\|f\|_{\Lambda^{\Phi,\phi}(w)} = 1$. Let $B = B(a,r)$, $f = f_1 + f_2$, $f_1 = f\chi_{2B}$, and

$$a_{kB} = \frac{1}{|kB|\phi(r)}, \quad k \in \mathbb{N}.$$

For the weak case, we need to prove that

$$\|Mf\|_{\Lambda^{\Phi,\phi,\infty}(w)} \leq C,$$

i.e., for any ball $B = B(a,r)$,

$$\|Mf\|_{\Phi,\phi,w,B,\,\mathrm{weak}} \leq C.$$

First, prove that

$$\|Mf_1\|_{\Phi,\phi,w,B,\ \mathrm{weak}} \leq C. \tag{5.1}$$

Since Φ is convex, by Jensen's inequality, we have

$$\Phi(Mf_1(x)) \leq M(\Phi(f_1))(x),$$

and thus,

$$\|\Phi(Mf_1)\|_{\Lambda^{1,\infty}_{\phi_B}(w)} \le \|M(\Phi(f_1))\|_{\Lambda^{1,\infty}_{\phi_B}(w)} \le C\left\|[A(\Phi(f_1))^*]\left(\frac{\cdot}{a_B}\right)\right\|_{L^{1,\infty}(w)}.$$

The condition $w \in B_{1,\infty}$ implies that

$$A : L^1_{\mathrm{dec}}(w) \to L^{1,\infty},$$

and we have

$$\begin{aligned}
\left\|[A(\Phi(f_1))^*]\left(\frac{\cdot}{a_B}\right)\right\|_{L^{1,\infty}(w)} &\le C\left\|(\Phi(f_1))^*\left(\frac{\cdot}{a_B}\right)\right\|_{L^1(w)}\\
&= C\left\|(\Phi(f\chi_{2B}))^*\left(\frac{\cdot}{a_B}\right)\right\|_{L^1(w)}\\
&\le C\left\|(\Phi(f\chi_{2B})\|f\|_{\Phi,\phi,w,2B})^*\left(\frac{\cdot}{a_B}\right)\right\|_{L^1(w)}\\
&\le C\left\|(\Phi(f\chi_{2B}\|f\|_{\Phi,\phi,w,2B}))^*\left(\frac{\cdot}{a_{2B}}\right)\right\|_{L^1(w)}, \quad \text{since } w \in B_{1,\infty} \text{ yields that } W \in \Delta_2\\
&\le C\left\|(\Phi(f\chi_{2B}))^*\left(\frac{\cdot}{a_{2B}}\right)\right\|_{L^1(w)}\\
&\le C.
\end{aligned}$$

Thus, by convexity of Φ for $\overline{C} > 1$,

$$\left\|\Phi\left(\frac{Mf_1\chi_B}{\overline{C}}\right)\right\|_{\Lambda^{1,\infty}_{\phi_B}(w)} \le \frac{C}{\overline{C}} \le 1,$$

if

$$\overline{C} \ge C,$$

which deduces (5.1). Let

$$A_r = \phi(r)W\left(\frac{1}{\phi(r)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right).$$

Next, prove that

$$\|Mf_2\|_{\Phi,\phi,w,B,\mathrm{weak}} \le C. \tag{5.2}$$

Since $w \in B_{1,\infty}$ and $w(0) < \infty$, we obtain

$$\frac{W(r)}{r} \le C_1,\ 0 < r < \infty. \tag{5.3}$$

Thus,

$$\Phi\left(\frac{A_r}{C_1}\right)W\left(\frac{1}{\phi(r)}\right) = \Phi\left(\frac{\phi(r)}{C_1}W\left(\frac{1}{\phi(r)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\right)W\left(\frac{1}{\phi(r)}\right)$$

$$\begin{aligned}&\le \Phi\left(\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\right)W\left(\frac{1}{\phi(r)}\right), \quad \text{by (5.3)}\\ &\le \frac{1}{W\left(\frac{1}{\phi(r)}\right)}W\left(\frac{1}{\phi(r)}\right), \quad \text{by (2.8)}\\ &= 1.\end{aligned} \tag{5.4}$$

Then, by Lemma 4.4, the fact $\|f_2\|_{\Lambda^{\Phi,\phi}(w)} \le 1$ and (5.4),

$$\begin{aligned}\left\|\Phi\left(\frac{Mf_2\chi_B}{CC_1}\right)\right\|_{\Lambda^{1,\infty}_{\phi_B}(w)} &\le \left\|\Phi\left(\frac{CA_r\chi_B}{CC_1}\right)\right\|_{\Lambda^{1,\infty}_{\phi_B}(w)}\\ &= \sup_{t>0}\Phi(t)W\left(\frac{\lambda_{\frac{A_r\chi_B}{C_1}}(t)}{\phi(r)|B|}\right)\\ &\le \Phi\left(\frac{A_r}{C_1}\right)W\left(\frac{1}{\phi(r)}\right)\\ &\le 1.\end{aligned}$$

Thus, by (5.1) and (5.2),

$$\|Mf\|_{\Phi,\phi,w,B,\text{weak}} \le C(\|Mf_1\|_{\Phi,\phi,w,B,\text{weak}} + \|Mf_2\|_{\Phi,\phi,w,B,\text{weak}}) \le C.$$

For the strong case, let $f \in \Lambda^{\Phi,\phi}(w)$ and $\|f\|_{\Lambda^{\Phi,\phi}(w)} = 1$. We need to prove that

$$\|Mf\|_{\Lambda^{\Phi,\phi}(w)} \le C,$$

i.e., for any ball $B = B(a,r)$,

$$\|Mf\|_{\Phi,\phi,w,B} \le C. \tag{5.5}$$

First, prove

$$\|Mf_1\|_{\Phi,\phi,w,B} \le C. \tag{5.6}$$

Indeed, since $\Phi \in \nabla_2$, by [53, Theorem 1.2.1], there exists $0 < \alpha < 1$ such that Φ^α is quasi-convex. Thus,

$$\begin{aligned}\|\Phi(Mf_1\chi_B)\|_{\Lambda^1_{\phi_B}(w)} &= \|(\Phi^\alpha(Mf_1\chi_B))^{\frac{1}{\alpha}}\|_{\Lambda^1_{\phi_B}(w)}\\ &= \|\Phi^\alpha(Mf_1\chi_B)\|^{\frac{1}{\alpha}}_{\Lambda^{\frac{1}{\alpha}}_{\phi_B}(w)}\\ &\le C\,\|M(\Phi^\alpha(f_1\chi_B))\|^{\frac{1}{\alpha}}_{\Lambda^{1/\alpha}_{\phi_B}(w)}\\ &\le C\left\|[A(\Phi^\alpha(f_1))^*]\left(\frac{\cdot}{a_B}\right)\right\|^{1/\alpha}_{L^{1/\alpha}(w)}.\end{aligned} \tag{5.7}$$

Noting that $w \in B_{1,\infty}$ which implies $w \in B_{1/\alpha}$, we obtain

$$A : L^{1/\alpha}_{\text{dec}}(w) \to L^{1/\alpha}(w).$$

Thus, we have

$$\text{RHS of (5.7)} \le C\left\|(\Phi^\alpha(f_1))^*\left(\frac{\cdot}{a_B}\right)\right\|^{1/\alpha}_{L^{1/\alpha}(w)}$$

$$\begin{aligned}
&\le C\left\|(\Phi(f_1))^*\left(\frac{\cdot}{a_B}\right)\right\|_{L^1(w)}\\
&= C\left\|(\Phi(f\chi_{2B}))^*\left(\frac{\cdot}{a_B}\right)\right\|_{L^1(w)}\\
&\le C\left\|(\Phi(f\chi_{2B})\|f\|_{\Phi,\phi,w,2B})^*\left(\frac{\cdot}{a_B}\right)\right\|_{L^1(w)}\\
&\le C\left\|(\Phi(f\chi_{2B}\|f\|_{\Phi,\phi,w,2B}))^*\left(\frac{\cdot}{a_{2B}}\right)\right\|_{L^1(w)}, \quad \text{since } w\in B_{1,\infty} \text{ yields that } W\in\Delta_2\\
&\le C\left\|(\Phi(f\chi_{2B}))^*\left(\frac{\cdot}{a_{2B}}\right)\right\|_{L^1(w)}\\
&\le C.
\end{aligned} \tag{5.8}$$

Thus,

$$\left\|\Phi\left(\frac{Mf_1\chi_B}{\widetilde{C}}\right)\right\|_{\Lambda^1_{\phi_B}(w)} \le \frac{C}{\widetilde{C}} \le 1,$$

if

$$\widetilde{C} \ge C,$$

which deduces (5.6). Similarly, to prove (5.2), we may verify

$$\|Mf_2\|_{\Phi,\phi,w,B} \le C. \tag{5.9}$$

Hence, by (5.6) and (5.9),

$$\|Mf\|_{\Phi,\phi,w,B} \le C(\|Mf_1\|_{\Phi,\phi,w,B} + \|Mf_2\|_{\Phi,\phi,w,B}) \le C,$$

which implies (5.5).

Next, assume that $f\in\Lambda^{\Phi,\phi,\infty}(w)$ and $\|f\|_{\Lambda^{\Phi,\phi,\infty}(w)} = 1$. Subsequently, we verify that the the norm inequality

$$\|Mf\|_{\Lambda^{\Phi,\phi,\infty}(w)} \le C,$$

i.e., for any ball $B = B(a,r)$,

$$\|Mf\|_{\Phi,\phi,w,B,\,\mathrm{weak}} \le C. \tag{5.10}$$

Let $f = f_1 + f_2$, $f_1 = f\chi_{2B}$. Since $\Phi\in\nabla_2$, there exists $0<\alpha<1$ such that Φ^α is quasi-convex. Thus,

$$\begin{aligned}
\|\Phi(Mf_1\chi_B)\|_{\Lambda^{1,\infty}_{\phi_B}(w)} &= \left\|(\Phi^\alpha(Mf_1\chi_B))^{\frac{1}{\alpha}}\right\|_{\Lambda^{1,\infty}_{\phi_B}(w)}\\
&= \|\Phi^\alpha(Mf_1\chi_B)\|^{\frac{1}{\alpha}}_{\Lambda^{\frac{1}{\alpha},\infty}_{\phi_B}(w)}\\
&\le C\|M(\Phi^\alpha(f_1\chi_B))\|^{\frac{1}{\alpha}}_{\Lambda^{\frac{1}{\alpha},\infty}_{\phi_B}(w)}\\
&\le C\left\|[A(\Phi^\alpha(f_1))^*]\left(\frac{\cdot}{a_B}\right)\right\|^{1/\alpha}_{L^{1/\alpha,\infty}(w)}.
\end{aligned} \tag{5.11}$$

Noting that $w\in B_{1,\infty}$, which implies $w\in B^\infty_{1/\alpha,\infty}$ [12], we obtain

$$A : L^{1/\alpha,\infty}_{\mathrm{dec}}(w) \to L^{1/\alpha,\infty}(w).$$

Thus, similar to the estimate of (5.8), we have

$$\text{RHS of (5.11)} \le C\left\|\left(\Phi\left(\frac{f_1}{C_\Phi}\right)\right)^*\left(\frac{\cdot}{a_B}\right)\right\|_{L^{1,\infty}(w)} \le C,$$

which implies that

$$\|Mf_1\|_{\Phi,\phi,w,B,\mathrm{weak}} \le C. \tag{5.12}$$

Next, consider Mf_2. Let

$$D_r = \Phi^{-1}(W^{-1}(\phi(r))).$$

By Lemma 4.5,

$$\begin{aligned}
\left\|\Phi\left(\frac{Mf_2\chi_B}{CI}\right)\right\|_{\Lambda^{1,\infty}_{\phi_B}(w)} &\le \left\|\Phi\left(\frac{CD_r\chi_B}{CI}\right)\right\|_{\Lambda^{1,\infty}_{\phi_B}(w)}\\
&= \Phi\left(\frac{D_r}{I}\right)W\left(\frac{1}{\phi(r)}\right)\\
&\le \frac{1}{I}\Phi(\Phi^{-1}(W^{-1}(\phi(r))))W\left(\frac{1}{\phi(r)}\right)\\
&\le \frac{1}{I}W^{-1}(\phi(r))W\left(\frac{1}{\phi(r)}\right)\\
&\le \frac{J}{I}, \quad \text{by (3.3)}\\
&\le 1,
\end{aligned}$$

if $I \ge J$. Thus,

$$\|Mf_2\|_{\Phi,\phi,w,B,\mathrm{weak}} \le C_3 J,$$

which combining (5.12) concludes the proof. □

5.2 Proof of Theorem 3.2

We need the following lemmas to prove Theorem 3.2 (i).

Lemma 5.1. [43, Proposition 1] *Let $\rho, \tau : (0,\infty) \to (0,\infty)$. Assume that ρ satisfies* (1.3) *and τ satisfies the doubling condition* (2.15). *Define*

$$\tilde{\rho}(r) = \int_{K_1 r}^{K_2 r} \frac{\rho(t)}{t}\,\mathrm{d}t, \quad r \in (0,\infty).$$

Then, there exists a positive constant C such that, for all $r \in (0,\infty)$,

$$\sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) \le C \int_0^{K_2 r} \frac{\rho(t)}{t}\,\mathrm{d}t \tag{5.13}$$

$$\sum_{j=0}^{\infty} \tilde{\rho}(2^j r)\tau(2^j r) \le C \int_{K_1 r}^{\infty} \frac{\rho(t)\tau(t)}{t}. \tag{5.14}$$

Proof of Theorem 3.2. (i) By assumption (2.17), we may assume that ϕ is bijective from $(0,\infty)$ to itself. By (3.4) and the fact that $\lim_{t\to\infty}\phi(t) = 0$, there holds that

$$0 < \int_0^{\infty} \frac{\rho(t)}{t}\,\mathrm{d}t\,\Phi^{-1}(0) \le \Psi^{-1}(0). \tag{5.15}$$

Let $\|f\|_{\Lambda^{\Phi,\phi}(w)} = 1$ and $x \in \mathbb{R}^n$. We assume that

$$0 < \frac{Mf(x)}{C_0} < \infty \quad \text{and} \quad 0 \le \Phi\left(\frac{Mf(x)}{C_0}\right) < \infty.$$

Otherwise, there is nothing to prove. If $\Phi(\frac{Mf(x)}{C_0}) = 0$, then by (2.8) it follows that

$$\frac{Mf(x)}{c_0} \le \Phi^{-1}(0) = \sup\{u \ge 0 : \Phi(u) = 0\}.$$

Then, taking use of (5.13), (5.14), and (5.15), we obtain

$$\begin{aligned}|I_\rho(f)| &\le \sum_{j=-\infty}^{\infty} \int_{2^j \le |x-y| \le 2^{j+1}} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\mathrm{d}y \\ &\le C\frac{\tilde{\rho}(2^j)}{2^{jn}} \sum_{j=-\infty}^{\infty} \int_{|x-y|\le 2^{j+1}} |f(y)|\mathrm{d}y \\ &\le C\int_0^\infty \frac{\rho(t)}{t}\mathrm{d}tMf(x) \\ &\le C\int_0^\infty \frac{\rho(t)}{t}\mathrm{d}t\Phi^{-1}(0) \le C\Psi^{-1}(0) \le C\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right),\end{aligned}$$

which yields (3.5). If $\Phi(\frac{Mf(x)}{c_0}) > 0$, we can choose $0 < r < \infty$ such that

$$\frac{1}{W\left(\frac{1}{\phi(r)}\right)} = \Phi\left(\frac{Mf(x)}{c_0}\right). \tag{5.16}$$

Let

$$\begin{aligned}J_1 &= \sum_{j=-\infty}^{-1} \frac{\tilde{\rho}(2^j r)}{(2^j r)^n} \int_{|x-y|<2^{j+1}r} |f(y)|\mathrm{d}y, \\ J_2 &= \sum_{j=0}^{\infty} \frac{\tilde{\rho}(2^j r)}{(2^j r)^n} \int_{|x-y|<2^{j+1}r} |f(y)|\mathrm{d}y.\end{aligned}$$

Then,

$$|I_\rho f(x)| \le J_1 + J_2.$$

By (5.16) and (2.8), we have that

$$Mf(x) \le C_0\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right).$$

Then, using (5.13), we have

$$J_1 \le \int_0^{K_2 r} \frac{\rho(t)}{t}\mathrm{d}tMf(x) \le \int_0^{K_2 r} \frac{\rho(t)}{t}\mathrm{d}t\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right).$$

Thus, by Lemma 4.2,

$$J_2 \le C\sum_{j=0}^{\infty} \tilde{\rho}(2^j r)\phi(2^{j+1}r)W\left(\frac{1}{\phi(2^{j+1}r)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(2^{j+1}r)}\right)}\right).$$

In view of $\phi \in \mathcal{G}^{\mathrm{dec}}$, $w \in B_{1,\infty}$, and $\Phi \in \Phi_Y$, it follows that the function

$$F(t) = \phi(t)W\left(\frac{1}{\phi(t)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(t)}\right)}\right)$$

satisfies

$$\frac{1}{C} \le \frac{F(r)}{F(s)} \le C,$$

if

$$1/2 \le r/s \le 2.$$

Thus, by (5.14),

$$J_2 \le \int_{K_1 r}^{\infty} \frac{\rho(t)}{t}\phi(t)W\left(\frac{1}{\phi(t)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(t)}\right)}\right)\mathrm{d}t.$$

Therefore, by (3.4), the doubling condition of $\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)$ and $\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)$ and (5.16),

$$\begin{aligned}
J_1 &\le C\frac{\Psi^{-1}\left(\frac{1}{W(\frac{1}{\phi(K_2 r)})}\right)}{\Phi^{-1}\left(\frac{1}{W(\frac{1}{\phi(K_2 r)})}\right)} \cdot \Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\\
&\le C\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\\
&\le C\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
J_2 &\le C\int_{K_1 r}^{\infty} \frac{\rho(t)}{t}\phi(t)W\left(\frac{1}{\phi(t)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(t)}\right)}\right)\mathrm{d}t\\
&\le C\Psi^{-1}\left(\frac{1}{W(\frac{1}{\phi(K_1 r)})}\right)\\
&\le C\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\\
&\le C\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right).
\end{aligned}$$

Combining the estimate of J_1 and J_2 with (2.8), we have (3.5).

The proof of (3.7) is similar to the one of (3.5) except that we use Lemma 4.3 instead of Lemma 4.2 to the evaluation of J_2. We omit the details. □

To prove Theorem 3.2 (ii), we need the following three lemmas.

Lemma 5.2. [54, Lemma 2.1] *There exists a positive constant C such that, for all $R > 0$,*

$$\int_0^{R/2} \frac{\rho(t)}{t} dt \chi_{B(0,R/2)}(x) \leq C I_\rho(\chi_{B(0,R)})(x), \quad x \in \mathbb{R}^n.$$

Lemma 5.3. *For $\Phi \in \Phi_Y$, $\phi \in \mathcal{G}^{\mathrm{dec}}$, and $W \in \Delta_2$, let*

$$g(r) = \phi(r) W\left(\frac{1}{\phi(r)}\right) \Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right), \quad r > 0. \tag{5.17}$$

If (3.9) *holds, then*

$$g(|\cdot|) \in \Lambda^{\Phi,\phi}(w).$$

Remark 5.1. When $w = 1$, (3.9) reduces to

$$\int_0^r \Phi[\Phi^{-1}(\phi(s))] s^{n-1} ds < C\phi(r) r^n, \quad 0 < r < \infty. \tag{5.18}$$

Since $\Phi(\Phi^{-1}(s)) \leq s$, for $s > 0$,Lemma 5.3 improves [4, Lemma 5.4].

Proof. For any ball $B = B(a, r)$,

$$\begin{aligned}
\int_0^\infty \Phi\left(\frac{(g(|\cdot|)\chi_B)^*(\phi(r)|B|t)}{\delta}\right) w(t) dt &\leq \int_0^\infty \Phi\left(\frac{(g(\cdot)\chi_{B(0,r)})^*(\phi(r)|B|t)}{\delta}\right) w(t) dt \\
&= \int_0^\infty \Phi\left(\frac{g(\phi(r)^{1/n} r t^{1/n}) \chi_{(0,r)}(\phi(r)^{1/n} r t^{1/n})}{\delta}\right) w(t) dt \\
&\leq \frac{1}{\delta} \int_0^r \Phi(g(s)) w\left(\frac{s^n}{\phi(r) r^n}\right) \frac{n s^{n-1}}{\phi(r) r^n} ds \\
&\leq \frac{n}{\delta \phi(r) r^n} \int_0^r \Phi(g(s)) w\left(\frac{s^n}{\phi(r) r^n}\right) s^{n-1} ds \\
&\leq \frac{n C_0}{\delta} \leq 1,
\end{aligned}$$

if

$$\delta > n C_0.$$

Thus,

$$g(|\cdot|) \in \Lambda^{\Phi,\phi}(w). \qquad \square$$

Lemma 5.4. *Let $\Phi \in \Phi_Y$, $\phi \in \mathcal{G}^{\mathrm{dec}}$, and g be defined as in* (5.17)*. Then, there exists a positive constant C such that, for all $R > 0$,*

$$\int_{2R}^\infty \frac{\rho(t) g(t)}{t} dt \chi_{B(0,R)}(x) \leq C I_\rho(g(|\cdot|))(x), \quad x \in \mathbb{R}^n.$$

Proof. Note that $g(2\,\cdot) \sim g(\cdot)$ and if $x \in B(0,R)$, then $|x-y| \sim |y|$ with $y \notin B(0,2R)$. Therefore,

$$\begin{aligned} I_\rho(g(|\cdot|))(x) &\geq \int_{\mathbb{R}^n\setminus B(x,2R)} \frac{\rho(|x-y|)g(|y|)}{|x-y|^n}\mathrm{d}y \\ &= \int_{\mathbb{R}^n\setminus B(0,2R)} \frac{\rho(y)g(|x-y|)}{|y|^n}\mathrm{d}y \\ &\sim \int_{\mathbb{R}^n\setminus B(0,2R)} \frac{\rho(|y|)g(|y|)}{|y|^n}\mathrm{d}y \\ &\sim \int_{2R}^{\infty} \frac{\rho(t)g(t)}{t^n}\mathrm{d}t, \end{aligned}$$

which completes the proof. □

We continue to prove Theorem 3.2 (ii). On the one hand, by Lemma 5.2 and the boundedness of I_ρ, we obtain

$$\int_0^r \frac{\rho(t)}{t}\mathrm{d}t\|\chi_{B(0,r)}\|_{\Lambda^{\Psi,\phi,\infty}(w)} \leq \|I_\rho\chi_{B(0,2r)}\|_{\Lambda^{\Psi,\phi,\infty}(w)} \leq \|\chi_{B(0,2r)}\|_{\Lambda^{\Phi,\phi}(w)}.$$

By Lemma 4.1 and the doubling condition of $\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)$, we obtain

$$\int_0^r \frac{\rho(t)}{t}\mathrm{d}t\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right) \leq C\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right).$$

On the other hand, by Lemma 5.4, the boundedness of I_ρ, and Lemma 5.3, we achieve that

$$\int_r^\infty \frac{\rho(t)g(t)}{t^n}\mathrm{d}t\|\chi_{B(0,r/2)}\|_{\Lambda^{\Psi,\phi,\infty}} \leq \|I_\rho g\|_{\Lambda^{\Psi,\phi,\infty}} \leq C\|g\|_{\Lambda^{\Phi,\phi}} \leq C.$$

Taking into account Lemma 4.1, the doubling condition of $\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)$ and (3.9), we have

$$\int_r^\infty \frac{\rho(t)}{t}\phi(t)W\left(\frac{1}{\phi(t)}\right)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(t)}\right)}\right)\mathrm{d}t \leq C\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right).$$

We complete the proof. □

5.3 Proof of Theorem 3.3

(i) Let $\|f\|_{\Lambda^{\Psi,\phi}(w)} = 1$ and $0 < Mf(x) < \infty$. To prove (3.11), we only need to show

$$\rho(r)\frac{1}{|B(a,r)|}\int_{B(a,r)} |f(y)|\mathrm{d}y \leq C_1\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right), \quad x \in B(a,r).$$

We consider two cases:

$$\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right) \geq \frac{1}{A}\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\phi(r)W\left(\frac{1}{\phi(r)}\right) \tag{5.19}$$

or

$$\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right) < \frac{1}{A}\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\phi(r)W\left(\frac{1}{\phi(r)}\right). \tag{5.20}$$

For (5.19), by Lemma 4.2, $\|f\|_{\Lambda^{\Psi,\phi}(w)} = 1$, and condition (3.11), we obtain

$$\begin{aligned}\rho(r)\frac{1}{|B(a,r)|}\int_{B(a,r)}|f(y)|dy &\le \|f\|_{\Phi,\phi,B,w}\phi(r)W\left(\frac{1}{\phi(r)}\right)\rho(r)\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\\ &\le \phi(r)W\left(\frac{1}{\phi(r)}\right)\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)\\ &\le \frac{1}{A}\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right).\end{aligned}$$

For (5.20), since $w(0) < \infty$ and $w \in B_{1,\infty}$, we know $W(t)/t \le C$. Thus,

$$\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right) \le \Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right),$$

i.e.,

$$\Phi\left(\frac{Mf(x)}{C_0}\right) \le \frac{1}{W\left(\frac{1}{\phi(r)}\right)}. \tag{5.21}$$

If $\lim_{r\to 0}\phi(r) = 0$, then since

$$\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)$$

is almost decreasing with respect to $r \in (0,\infty)$, by (5.21), there exists $t_0 \in (r,\infty)$ such that

$$\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right) = \Psi^{-1}\left(\frac{1}{W(\frac{1}{\phi(t_0)})}\right). \tag{5.22}$$

Hence,

$$\rho(r) \le \sup_{0<s\le t_0}\rho(s) \le A\frac{\Psi^{-1}\left(\frac{1}{W(\frac{1}{\phi(t_0)})}\right)}{\Phi^{-1}\left(\frac{1}{W(\frac{1}{\phi(t_0)})}\right)} = A\frac{\Psi^{-1}(\Phi(\frac{Mf(x)}{C_0}))}{\Phi^{-1}(\Phi(\frac{Mf(x)}{C_0}))} \le A\frac{\Psi^{-1}(\Phi(\frac{Mf(x)}{C_0}))}{\frac{Mf(x)}{C_0}}.$$

If $\frac{\Psi^{-1}(t)}{\Phi^{-1}(t)}$ is almost decreasing, then by (3.10),

$$\rho(r) \le A \frac{\Psi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)}{\Phi^{-1}\left(\frac{1}{W\left(\frac{1}{\phi(r)}\right)}\right)} \le A \frac{\Psi^{-1}(\Phi(\frac{Mf(x)}{C_0}))}{\Phi^{-1}(\Phi(\frac{Mf(x)}{C_0}))} \le A \frac{\Psi^{-1}(\Phi(\frac{Mf(x)}{C_0}))}{\frac{Mf(x)}{C_0}}.$$

Thus,

$$\rho(r)\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y \le AC_0\Psi^{-1}\left(\Phi\left(\frac{Mf(x)}{C_0}\right)\right).$$

To verify (3.13), it is enough to use Lemma 4.3 instead of Lemma 4.2 since other parts are similar to the proof of (3.11).

To prove Theorem 3.3 (ii), we state the following lemma.

Lemma 5.5. [17, Lemma 5.1] *Let* $\rho : (0,\infty) \to (0,\infty)$. *Then,*

$$\sup_{0<t\le r} \rho(t)\chi_{B(0,r)}(x) \le M_\rho(\chi_{B(0,r)})(x), \quad x \in \mathbb{R}^n, r > 0.$$

Proof of Theorem 3.3 (ii). By Lemma 5.5 and the boundedness of M_ρ from $\Lambda^{\Phi,\phi}(w)$ to $\Lambda^{\Psi,\phi,\infty}(w)$, we obtain

$$\left(\sup_{0<t\le r}\rho(t)\right)\|\chi_{B(0,r)}\|_{\Lambda^{\Psi,\phi,\infty}(w)} \le \|M_\rho\chi_{B(0,r)}\|_{\Lambda^{\Psi,\phi,\infty}(w)} \le \|\chi_{B(0,r)}\|_{\Lambda^{\Phi,\phi}(w)}.$$

Thus, Lemma 4.1 yields (3.10). □

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