Research Article

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Infinitely many solutions for a class of Kirchhoff-type equations

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Abstract: In this article, we consider a class of Kirchhoff-type equations:

$$\begin{cases} -\left(a + b\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x\right)\Delta u = f(x,u), & x \in \Omega,\\ u = 0, & x \in \partial\Omega. \end{cases}$$

It is a generalization of the classical wave equation and is often used to model wave propagation in various physical media. The nonlocal term $b(\int_{\Omega}|\nabla u|^2\mathrm{d}x)\Delta u$ in the equation causes the variational functional of the equation to have fundamentally different properties from the case of $b = 0$. As far as we know, there are relatively few conclusions regarding infinitely many solutions of this equation. Under weaker assumptions, we obtain that the equation has infinitely many high-energy solutions. And our results extend the conclusions of Mao-Zhang (2009) and Zhang-Perera (2006).

Keywords: Kirchhoff-type equation, multiple solutions, high energy solution

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1 Introduction and main results

This article is related to a class of Kirchhoff-type equations:

$$\begin{cases} -\left(a + b\int\limits_{\Omega}|\nabla u|^2\mathrm{d}x\right)\Delta u = f(x,u), & x \in \Omega,\\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N = 1, 2$, or 3), $a, b > 0$ are constants, and $f(x,u) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The equation is a generalization of the classical wave equation, capturing more complex wave phenomena such as dispersion and dissipation. It was initially formulated by Kirchhoff to address the limitations of the simple wave equation in describing wave propagation in acoustics [1]. It is connected with the stationary analogue of the equation:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int\limits_0^L\left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

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where ρ denotes the mass density, u is the displacement, P_0 is the tension at the initial moment, h represents the cross-sectional area of the elastic string, and E and L denote Young's modulus for the material and the length of the string, respectively [2]. Equation (1) is particularly important for understanding waves in media with varying properties, such as density and elasticity, which are common in seismic wave propagation and the vibration analysis of structures.

In physics, Kirchhoff-type equations find applications in various fields. In solid mechanics, they model the vibrations of elastic bodies, considering both the material's inertia and its elastic properties. In fluid dynamics, they describe the behavior of surface waves under the influence of gravity and surface tension. In addition, these equations are integral to the theory of general relativity, where they predict the propagation of gravitational waves. By accounting for additional physical effects, Kirchhoff-type equations enhance our predictive capabilities for wave phenomena across different physical contexts [3].

It was not until Lions [4] proposed the following: Kirchhoff equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \left(a + b\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(x, u), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

that equation (1) received increasing attention in mathematics, where $a, b > 0$ are constants and $f(x, u)$ describes the external force. For nontrivial solutions, Pei and Zhang [5] obtained a nontrivial solution by combining a variant version of the mountain pass theorem with the moser-trudinger inequality for $a = b = 1$. Song et al. [6] derived a nontrivial solution by applying the classical linking theorem and the G-linking theorem. By the morse theory and the minimax method, Sun et al. [7] computed the critical groups including the cases where zero is a mountain pass solution and the nonlinearity is resonant at zero. As an application, they demonstrated the multiplicity of nontrivial solutions for (1) when $f(x, u) = g(x, u) + |u|^{2^*-2}$, where $g(x, u)$, an additional critical nonlinear term, satisfies certain conditions. Some results about nontrivial solutions of (1) can be found in [8–10] and the references therein.

Cheng and Wu [11] used the variational method to prove the existence of two positive solutions of (1), one dealing with the asymptotic behaviors of $f(x, u)$ near zero and infinity, and the other addressing the 4-superlinear of $f(x, u)$ at infinity. Yang and Zhang [12] used invariant sets of descent flow to obtain two solutions of (1), a positive one and a negative one. Zhang and Perera [13] assumed

$$\nu F(x, t) \leq t f(x, t), \quad \text{for } x \in \Omega,\ |t| \geq L > 0,\ \nu > 4, \tag{2}$$

and obtained a positive, a negative and a sign-changing solution of (1) by invariant sets of descent flow. Later, Mao and Zhang [14] assumed $\tilde{F}(x, t) = \frac{1}{4} f(x, t)t - F(x, t) \to +\infty$, as $|t| \to +\infty$ uniformly in $x \in \Omega$, $|f(x, t)|^\sigma \leq C\tilde{F}(x, t)|t|^\sigma$, for $|t|$ large, $\sigma > \max\{1, \frac{N}{2}\}$, and obtained at least one positive, one negative, and one sign-changing solution of (1).

Moreover, Shuai [15] combined the constrained variational method with the deformation lemma to obtain a least energy sign-changing solution, whose energy is strictly larger than the ground state energy. By using analytical skills and a non-Nehari manifold method, Tang and Cheng [16] proved that (1) has a ground state sign-changing solution, whose energy is strictly larger than twice the ground state solutions. Liu et al. [17] discussed the existence of a positive and a negative solution by using an iterative technique and the mountain pass theorem, and a sign-changing solution by combining the iterative technique and the Nehari method.

In the present article, we assume:

(f_1) $f(x, t) = o(|t|)$, as $|t| \to 0$ uniformly for any $x \in \Omega$.

(f_2) There exist constants $C > 0$ and $p \in (4, 2^*)$ such that

$$|f(x, t)| \leq C(1 + |t|^{p-1}),$$

where $2^* = +\infty$ for $N = 1, 2$, and $2^* = \frac{2N}{N-2}$ for $N \geq 3$.

(f_3) $f(x, -t) = -f(x, t)$, for $(x, t) \in \Omega \times \mathbb{R}$.

(f_4) $\frac{F(x,t)}{t^4} \to +\infty$, as $|t| \to +\infty$ uniformly for $x \in \Omega$, where $F(x, t) = \int_0^t f(x, s)\mathrm{d}s$.

(f_5) $\tilde{F}(x,t) := \frac{1}{4}f(x,t)t - F(x,t) \to +\infty$, as $|t| \to +\infty$ uniformly in $x \in \Omega$, and there exists $\tilde{C} > 0$, $\sigma > \max\{1, \frac{N}{2}\}$, such that $|f(x,t)|^\sigma \le \tilde{C}\tilde{F}(x,t)|t|^\sigma$, for $|t|$ large.

(f_6) $tf(x,t) \ge 4F(x,t)$, for $(x,t) \in \Omega \times \mathbb{R}$.

Our main results are as follows.

Theorem 1.1. *If the assumptions* (f_1)–(f_5) *hold, then equation* (1) *admits infinitely many high-energy solutions.*

Remark 1.2. Assuming $uf(x,u) \ge 0$, conditions (f_4) and (f_5) are weaker than those in [13] (specifically compared to (2)). Examples of functions satisfying (f_4) and (f_5) but not (2) include

$$f(x,u) = u^3 \ln(1+|u|) \text{ or } F(x,u) = |u|^\mu + (\mu-4)|u|^{\mu-\varepsilon}\sin^2\left(\frac{|u|^\varepsilon}{\varepsilon}\right),$$

where $4 < \mu < 2^*$, $0 < \varepsilon < 3 - \frac{\mu}{2}$.

Corollary 1.3. *Under the assumption of* (2) *and* $\inf_{x\in\Omega,|u|=L} F(x,u) > 0$, *we can certify that* (1) *has infinitely many high-energy solutions.*

Remark 1.4. Our results generalize the results of [13] and [14]. Our assumptions are weaker than those of [13], but the number of solutions we obtained is greater than those in [13]. In [14], Mao-Zhang obtained three solutions of (1), and we have infinitely many high-energy solutions.

Theorem 1.2. *If assumptions* (f_1)–(f_4), *and* (f_6) *hold, then equation* (1) *has infinitely many high-energy solutions.*

The article is organized as follows. Section 2 gives some lemmas to prove the theorems. Lemmas 2.1 and 2.2 are devoted to proving the functional corresponding to (1) satisfies the $(\mathrm{PS})_c$ condition. Section 3 proves Theorems 1.1 and 1.2.

2 Variational framework and lemmas

Let $H := H_0^1(\Omega)$ be a Hilbert space equipped with the inner product and norm

$$(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad \|u\| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

$L^p(\Omega)$ $(1 \le p < \infty)$ be the normal Banach space with the norm $|u|_p = \left(\int_\Omega |u|^p \mathrm{d}x\right)^{\frac{1}{p}}$. Since Ω is a bounded domain, according to the Sobolev embedding theorem, $H \hookrightarrow L^s(\Omega)$ continuously for $s \in [1, 2^*]$, compactly for $s \in [1, 2^*)$. Moreover, for any $u \in H$, there exists a constant $\eta_s > 0$ such that

$$|u|_s \le \eta_s \|u\|. \tag{3}$$

We choose a countable orthogonal basis $\{e_i\}$ of H with $\|e_i\| = 1$, $i = 1, 2, \ldots$. Set

$$X_i = \operatorname{span}\{e_i\},\ Y_k = \bigoplus_{i=1}^{k} X_i,\ Z_k = \overline{\bigoplus_{i=k}^{\infty} X_i}. \tag{4}$$

Meanwhile, a weak solution is the critical point of the functional

$$\Phi(u) = \frac{a}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{b}{4}\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^2 - \int_\Omega F(x,u)\mathrm{d}x,$$

and for any $u, v \in H$,

$$(\Phi'(u), v) = (a + b\|u\|^2)\int_\Omega \nabla u \nabla v \mathrm{d}x - \int_\Omega f(x,u)v \mathrm{d}x.$$

In this article, C and C_i in different positions denote different positive constants.

Theorem 2.1. ([18], Fountain theorem) *Let X be a Banach space, and let $\{X_i\}$ be a sequence of subspaces of X with $\dim X_i < \infty$ for all $i \in \mathbb{N}$. Moreover, let $X = \overline{\oplus_{i=1}^{\infty} X_i}$, $Y_k = \oplus_{i=1}^{k} X_i$ and $Z_k = \overline{\oplus_{i=k}^{\infty} X_i}$. Consider an even functional $I \in C^1(X, \mathbb{R})$, i.e., $I(u) = I(-u)$. Assume for every $k \in \mathbb{N}$, there exist $\rho_k > \gamma_k > 0$ such that*
(I_1) $a_k := \max_{u \in Y_k, \|u\| = \rho_k} I(u) \le 0$,
(I_2) $b_k := \inf_{u \in Z_k, \|u\| = \gamma_k} I(u) \to +\infty$, *as* $k \to \infty$,
(I_3) *I satisfies* $(\mathrm{PS})_c$ *condition for every* $c > 0$.

Then I has an unbounded sequence of critical values.

Lemma 2.1. *Assume that (f_1), (f_2), (f_4), and (f_5) hold, then $\Phi(u)$ satisfies the $(\mathrm{PS})_c$ condition for $c > 0$. That is, any sequence $\{u_n\} \subset H$ satisfying $\Phi(u_n) \to c > 0$ and $\Phi'(u_n) \to 0$ as $n \to +\infty$, has a convergent subsequence.*

Proof. Following the standard approach, we need to prove that $\{u_n\}$ is bounded and that it contains a convergent subsequence.

If $\{u_n\}$ is unbounded, assume $\|u_n\| \to +\infty$, as $n \to +\infty$. We have

$$\begin{aligned} c + 1 + \|u_n\| &\ge \Phi(u_n) - \frac{1}{4}(\Phi'(u_n), u_n) \\ &= \frac{a}{4}\int_\Omega |\nabla u_n|^2 \mathrm{d}x + \int_\Omega \left(\frac{1}{4} f(x,u_n)u_n - F(x,u_n)\right)\mathrm{d}x \\ &= \frac{a}{4}\|u_n\|^2 + \int_\Omega \tilde{F}(x,u_n)\mathrm{d}x. \end{aligned}$$

Then

$$c + 1 \ge \frac{a}{4}\|u_n\|^2 - \|u_n\| + \int_\Omega \tilde{F}(x,u_n)\mathrm{d}x \ge \int_\Omega \tilde{F}(x,u_n)\mathrm{d}x. \tag{5}$$

Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$, meanwhile by (3), $|v_n|_s \le \eta_s \|v_n\| = \eta_s$, for any $s \in [1, 2^*)$. Note that

$$\frac{(\Phi'(u_n), u_n)}{\|u_n\|^4} = \frac{a}{\|u_n\|^2} + b - \frac{\int_\Omega f(x,u_n)u_n \mathrm{d}x}{\|u_n\|^4},$$

then as $n \to +\infty$,

$$\lim_{n\to+\infty} \int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x = b. \tag{6}$$

Next we certify that $b = 0$, which will lead to a contradiction. For $0 \le \alpha < \beta \le +\infty$, let

$$O_n(\alpha,\beta) := \{x \in \Omega : \alpha \le |u_n(x)| < \beta\}.$$

For large n and $\alpha < \beta$, we have

$$\int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x = \int_{O_n(0,\alpha)} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x + \int_{O_n(\alpha,\beta)} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x + \int_{O_n(\beta,+\infty)} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x. \tag{7}$$

Subsequently we will prove $\int_{O_n(0,\alpha)} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \to 0$, $\int_{O_n(\alpha,\beta)} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \to 0$, and $\int_{O_n(\beta,+\infty)} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \to 0$, as $n \to +\infty$, respectively.

First, we prove $\int_{O_n(\beta,+\infty)} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \to 0$, as $n \to +\infty$. $\|u_n\| \to +\infty$ as $n \to +\infty$ shows that there exists an integer $N_0 > 0$ such that $\|u_n\| \geq 1$ if $n > N_0$. For $\sigma > \max\{1, \frac{N}{2}\}$, let $\tau = \frac{2\sigma}{\sigma-1}$, then $\tau \in (2, 2^*)$ and $\sigma = \frac{\tau}{\tau-2}$. By (f_5) and $\beta \to +\infty$,

$$\begin{aligned}\left|\int\limits_{O_n(\beta,+\infty)} \frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x\right| &\leq \int\limits_{O_n(\beta,+\infty)} \frac{|f(x,u_n)|}{|u_n|\|u_n\|}\frac{v_n^2}{\|u_n\|}\mathrm{d}x \\ &\leq \left(\frac{1}{\|u_n\|^2}\int\limits_{O_n(\beta,+\infty)} \frac{|f(x,u_n)|^\sigma}{|u_n|^\sigma}\mathrm{d}x\right)^{\frac{1}{\sigma}}\left(\frac{1}{\|u_n\|^2}\int\limits_{O_n(\beta,+\infty)} |v_n|^\tau \mathrm{d}x\right)^{\frac{2}{\tau}} \\ &\leq \left(\frac{\tilde{C}}{\|u_n\|^2}\int\limits_{O_n(\beta,+\infty)} \tilde{F}(x,u_n)\mathrm{d}x\right)^{\frac{1}{\sigma}}\left(\frac{1}{\|u_n\|^2}\int\limits_{O_n(\beta,+\infty)} |v_n|^\tau \mathrm{d}x\right)^{\frac{2}{\tau}}.\end{aligned} \tag{8}$$

For (8), we will certify $\left(\frac{\tilde{C}}{\|u_n\|^2}\int_{O_n(\beta,+\infty)}\tilde{F}(x,u_n)\mathrm{d}x\right)^{\frac{1}{\sigma}}$ is bounded and $\left(\frac{1}{\|u_n\|^2}\int_{O_n(\beta,+\infty)}|v_n|^\tau\mathrm{d}x\right)^{\frac{2}{\tau}} \to 0$, as $\beta \to +\infty$. By (5), then

$$\begin{aligned}\frac{c+1}{\|u_n\|^2} &\geq \frac{1}{\|u_n\|^2}\int\limits_{\Omega}\tilde{F}(x,u_n)\mathrm{d}x \\ &\geq \frac{1}{\|u_n\|^2}\int\limits_{O_n(0,\alpha)}\tilde{F}(x,u_n)\mathrm{d}x + \frac{1}{\|u_n\|^2}\int\limits_{O_n(\alpha,\beta)}\tilde{F}(x,u_n)\mathrm{d}x + \frac{1}{\|u_n\|^2}\int\limits_{O_n(\beta,+\infty)}\tilde{F}(x,u_n)\mathrm{d}x.\end{aligned} \tag{9}$$

By (9), $\frac{1}{\|u_n\|^2}\int_\Omega \tilde{F}(x,u_n)\mathrm{d}x$ is bounded. To establish that $\frac{1}{\|u_n\|^2}\int_{O_n(\beta,+\infty)}\tilde{F}(x,u_n)\mathrm{d}x$ is bounded, we must prove $\frac{1}{\|u_n\|^2}\int_{O_n(0,\alpha)}\tilde{F}(x,u_n)\mathrm{d}x$ and $\frac{1}{\|u_n\|^2}\int_{O_n(\alpha,\beta)}\tilde{F}(x,u_n)\mathrm{d}x$ are bounded.

By (f_1), for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x,u_n)| \leq \varepsilon|u_n|, \tag{10}$$

as $|u_n| < \delta$. By (f_2),

$$|f(x,u_n)| \leq C + C|u_n|^{p-1} \leq \left(C + C\frac{1}{\delta^{p-1}}\right)|u_n|^{p-1}, \tag{11}$$

as $|u_n| \geq \delta$. By combining (10) with (11), we obtain

$$|f(x,u_n)| \leq \varepsilon|u_n| + \left(\frac{C}{\delta^{p-1}} + C\right)|u_n|^{p-1} = \varepsilon|u_n| + C_1|u_n|^{p-1}, \tag{12}$$

where $C_1 = \frac{C}{\delta^{p-1}} + C$. Then

$$|F(x,u_n)| \leq \frac{\varepsilon}{2}|u_n|^2 + \frac{C_1}{p}|u_n|^p. \tag{13}$$

Therefore, for $|u_n| \in [0, \alpha)$,

$$\begin{aligned}|\tilde{F}(x,u_n)| &\leq \frac{1}{4}|f(x,u_n)||u_n| + |F(x,u_n)| \\ &\leq \frac{1}{4}\varepsilon|u_n|^2 + \frac{C_1}{4}|u_n|^p + \frac{\varepsilon}{2}|u_n|^2 + \frac{C_1}{p}|u_n|^p \\ &\leq C_2|u_n|^2 + C_2|u_n|^p \\ &\leq C_3|u_n|^2,\end{aligned} \tag{14}$$

where $C_2 = \max\left\{\frac{3}{4}\varepsilon, \frac{4+p}{4p}C_1\right\}$, $C_3 = C_2(1+\alpha^{p-2})$.

By (14) and (3),

$$\frac{1}{\|u_n\|^2}\left|\int\limits_{O_n(0,\alpha)}\tilde{F}(x,u_n)\mathrm{d}x\right| \le \frac{C_3}{\|u_n\|^2}\int\limits_{O_n(0,\alpha)}|u_n|^2\mathrm{d}x = C_3\int\limits_{O_n(0,\alpha)}|v_n|^2\mathrm{d}x$$
$$\le C_3\int\limits_{\Omega}|v_n|^2\mathrm{d}x \le C_3\eta_2^2\|v_n\|^2 = C_3\eta_2^2,$$

then $\frac{1}{\|u_n\|^2}\int_{O_n(0,\alpha)}\tilde{F}(x,u_n)\mathrm{d}x$ is bounded. The boundedness of $\frac{1}{\|u_n\|^2}\int_{O_n(\alpha,\beta)}\tilde{F}(x,u_n)\mathrm{d}x$ can be proved in the same way. Therefore, $\frac{1}{\|u_n\|^2}\int_{O_n(\beta,+\infty)}\tilde{F}(x,u_n)\mathrm{d}x$ is bounded.

In the following, we will prove $\left(\frac{1}{\|u_n\|^2}\int_{O_n(\beta,+\infty)}|v_n|^\tau\mathrm{d}x\right)^{\frac{2}{\tau}} \to 0$, as $\beta \to +\infty$.

According to the Hölder inequality, for any $\tau \in (2, 2^*)$, $\beta \to +\infty$, let $q > 0$ be a constant such that $\frac{\tau}{q} \in (0,1)$, by (3), then

$$\begin{aligned}\frac{1}{\|u_n\|^2}\int\limits_{O_n(\beta,+\infty)}|v_n|^\tau\mathrm{d}x &\le \frac{1}{\|u_n\|^2}\left(\int\limits_{O_n(\beta,+\infty)}(|v_n|^\tau)^{\frac{q}{\tau}}\mathrm{d}x\right)^{\frac{\tau}{q}}\left(\int\limits_{O_n(\beta,+\infty)}1^{\frac{q}{q-\tau}}\mathrm{d}x\right)^{\frac{q-\tau}{q}}\\ &\le \|u_n\|^{-2}\eta_q^\tau\|v_n\|^\tau|O_n(\beta,+\infty)|^{\frac{q-\tau}{q}}\\ &= \eta_q^\tau\frac{1}{\|u_n\|^{\frac{2\tau}{q}}}\left|\frac{O_n(\beta,+\infty)}{\|u_n\|^2}\right|^{\frac{q-\tau}{q}}.\end{aligned} \tag{15}$$

Let $r \ge 0$,

$$g(r) := \inf\{\tilde{F}(x,u) : x \in \Omega, u \in \mathbb{R}, |u| \ge r\},$$
$$S_\alpha^\beta := \inf\left\{\frac{\tilde{F}(x,u)}{|u|^2} : x \in \Omega, u \in \mathbb{R}, \alpha \le |u| < \beta\right\}.$$

(f_5) implies that $g(r) \to +\infty$ as $r \to +\infty$, and $g(\alpha) > 0$, $S_\alpha^\beta > 0$ for large $\alpha > 0$. Moreover, for any $x \in O_n(\alpha,\beta)$,

$$\tilde{F}(x,u_n) \ge S_\alpha^\beta\,|u_n|^2. \tag{16}$$

Meantime, $\frac{1}{\|u_n\|^2}\int_{O_n(\beta,+\infty)}\tilde{F}(x,u_n)\mathrm{d}x \ge g(\beta)\frac{|O_n(\beta,+\infty)|}{\|u_n\|^2}$. $g(\beta) \to +\infty$ as $\beta \to +\infty$ and the proven boundedness of $\frac{1}{\|u_n\|^2}\int_{O_n(\beta,+\infty)}\tilde{F}(x,u_n)\mathrm{d}x$, imply $\frac{|O_n(\beta,+\infty)|}{\|u_n\|^2} \to 0$ as $\beta \to +\infty$. By (15),

$$\frac{1}{\|u_n\|^2}\int\limits_{O_n(\beta,+\infty)}|v_n|^\tau\mathrm{d}x \to 0, \quad \text{as } \beta \to +\infty,$$

then $\int_{O_n(\beta,+\infty)}\frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x$ in (8) converges to 0, as $n \to +\infty$.

Second, we will prove $\int_{O_n(\alpha,\beta)}\frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \to 0$, as $n \to +\infty$. By (12), for $|u_n| \in [\alpha,\beta)$, $|f(x,u_n)||u_n| \le \varepsilon\,|u_n|^2 + C_1(\varepsilon)|u_n|^p \le C_4\,|u_n|^2$. By (16),

$$\left|\int\limits_{O_n(\alpha,\beta)}\frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x\right| \le \int\limits_{O_n(\alpha,\beta)}\frac{C_4\,|u_n|^2}{\|u_n\|^4}\mathrm{d}x \le \frac{1}{\|u_n\|^4S_\alpha^\beta}\int\limits_{O_n(\alpha,\beta)}\tilde{F}(x,u_n)\mathrm{d}x.$$

As the estimation of (14), for $|u_n| \in [\alpha,\beta)$, $\tilde{F}(x,u) \le C_5\,|u_n|^2$, then

$$\int_{O_n(\alpha,\beta)}\tilde{F}(x,u_n)\mathrm{d}x \le \int\limits_{O_n(\alpha,\beta)}C_5\,|u_n|^2\mathrm{d}x \le \int\limits_{\Omega}C_5\,|u_n|^2\mathrm{d}x \le C_6,$$

that is, $\int_{O_n(\alpha,\beta)}\tilde{F}(x,u_n)\mathrm{d}x$ is bounded. Then $\frac{1}{\|u_n\|^4 S_\alpha^\beta}\int\limits_{O_n(\alpha,\beta)}\tilde{F}(x,u_n)\mathrm{d}x \to 0$, as $n \to +\infty$. So as $n \to +\infty$, $\int_{O_n(\alpha,\beta)}\frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \to 0$.

Finally, we will demonstrate $\int_{O_n(0,\alpha)}\frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x \to 0$, as $n \to +\infty$. By (f_1), $|f(x,u)| \le C_7|u|$, and by (3),

$$\left|\int\limits_{O_n(0,\alpha)}\frac{f(x,u_n)u_n}{\|u_n\|^4}\mathrm{d}x\right| \le \int\limits_{O_n(0,\alpha)}\frac{|f(x,u_n)||u_n|}{\|u_n\|^4}\mathrm{d}x \le \int\limits_{O_n(0,\alpha)}\frac{C_7\,|u_n|^2}{\|u_n\|^4}\frac{C_7\eta_2^2}{\|u_n\|^2}\mathrm{d}x \to 0,$$

as $n \to +\infty$. Hence, by (6) and (7), we have $b = 0$. This is a contradiction with $b > 0$. Therefore, the assumption $\|u_n\| \to +\infty$ is invalid, $\{u_n\}$ is bounded.

Subsequently, we will prove $\{u_n\}$ has a convergent subsequence. Taking a subsequence of $\{u_n\}$, as well called $\{u_n\}$, such that $u_n \rightharpoonup u$ in H, $u_n \to u$ in $L^s(\Omega)$ for $s \in [1, 2^*)$, $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$, as $n \to +\infty$. According to $(\Phi'(u_n) - \Phi'(u), u_n - u) \to 0$ and

$$\begin{aligned}(\Phi'(u_n) - \Phi'(u), u_n - u) &= \left(a + b\int\limits_\Omega|\nabla u_n|^2\mathrm{d}x\right)\int\limits_\Omega\nabla u_n\nabla(u_n - u)\mathrm{d}x - \left(a + b\int\limits_\Omega|\nabla u|^2\mathrm{d}x\right)\int\limits_\Omega\nabla u\nabla(u_n - u)\mathrm{d}x\\ &\quad - \int\limits_\Omega(f(x,u_n) - f(x,u))(u_n - u)\mathrm{d}x\\ &= (a + b\|u_n\|^2)\|u_n - u\|^2 - b\left(\int\limits_\Omega|\nabla u|^2\mathrm{d}x - \int\limits_\Omega|\nabla u_n|^2\mathrm{d}x\right)\int\limits_\Omega\nabla u\nabla(u_n - u)\mathrm{d}x\\ &\quad - \int\limits_\Omega(f(x,u_n) - f(x,u))(u_n - u)\mathrm{d}x,\end{aligned}$$

then

$$\begin{aligned}(a + b\|u_n\|^2)\|u_n - u\|^2 &= (\Phi'(u_n) - \Phi'(u), u_n - u) + b\left(\int\limits_\Omega|\nabla u|^2\mathrm{d}x - \int\limits_\Omega|\nabla u_n|^2\mathrm{d}x\right)\int\limits_\Omega\nabla u\nabla(u_n - u)\mathrm{d}x\\ &\quad + \int\limits_\Omega(f(x,u_n) - f(x,u))(u_n - u)\mathrm{d}x.\end{aligned} \tag{17}$$

Since $\{u_n\}$ is bounded and $u_n \rightharpoonup u$, then

$$b\left(\int\limits_\Omega|\nabla u|^2\mathrm{d}x - \int\limits_\Omega|\nabla u_n|^2\mathrm{d}x\right)\int\limits_\Omega\nabla u\nabla(u_n - u)\mathrm{d}x = b(\|u\|^2 - \|u_n\|^2)(u, u_n - u) \to 0, \tag{18}$$

for $n \to +\infty$. In addition, by (12),

$$\begin{aligned}&\int\limits_\Omega(f(x,u_n) - f(x,u))(u_n - u)\mathrm{d}x\\ &\quad\le \int\limits_\Omega(|f(x,u_n)| + |f(x,u)|)|u_n - u|\mathrm{d}x\\ &\quad\le \int\limits_\Omega(\varepsilon(|u_n| + |u|) + C_1(\varepsilon)(|u_n|^{p-1} + |u|^{p-1}))|u_n - u|\mathrm{d}x\\ &\quad\le \varepsilon\left(\int\limits_\Omega|u_n|^2\mathrm{d}x\right)^{\frac12}\left(\int\limits_\Omega|u_n - u|^2\mathrm{d}x\right)^{\frac12} + \varepsilon\left(\int\limits_\Omega|u|^2\mathrm{d}x\right)^{\frac12}\left(\int\limits_\Omega|u_n - u|^2\mathrm{d}x\right)^{\frac12}\\ &\qquad + C_1(\varepsilon)\left(\int\limits_\Omega|u_n|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int\limits_\Omega|u_n - u|^p\mathrm{d}x\right)^{\frac1p} + C_1(\varepsilon)\left(\int\limits_\Omega|u|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int\limits_\Omega|u_n - u|^p\mathrm{d}x\right)^{\frac1p}\\ &\quad= \varepsilon(|u_n|_2 + |u|_2)|u_n - u|_2 + C_1(\varepsilon)(|u_n|_p^{p-1} + |u|_p^{p-1})|u_n - u|_p.\end{aligned} \tag{19}$$

Since $H \hookrightarrow L^s(\Omega)$ is compact for $s \in [1, 2^*)$, then $|u_n - u|_2 \to 0$, $|u_n - u|_p \to 0$ in $L^s(\Omega)$, as $n \to +\infty$. Therefore, $\int_\Omega (f(x, u_n) - f(x, u))(u_n - u)\mathrm{d}x \to 0$ as $n \to +\infty$. By combining (17), (18), and (19), we conclude that $\|u_n - u\| \to 0$ as $n \to \infty$. This implies that $\{u_n\}$ contains a convergent subsequence. □

Lemma 2.2. *Assume that* (f_1), (f_2), (f_4), *and* (f_6) *hold, then* $\Phi(u)$ *satisfies the* $(\mathrm{PS})_c$ *condition for* $c > 0$.

Proof. By (f_6),

$$\begin{aligned} c + 1 + \|u_n\| &\geq \Phi(u_n) - \frac{1}{4}(\Phi'(u_n), u_n) \\ &= \frac{a}{4}\|u_n\|^2 + \frac{1}{4}\int_\Omega (f(x, u_n)u_n - 4F(x, u_n))\mathrm{d}x \\ &\geq \frac{a}{4}\|u_n\|^2. \end{aligned}$$

Since $a > 0$, then $\{u_n\}$ is bounded in H.

Next we will prove $\{u_n\}$ has a convergent subsequence. Taking a subsequence of $\{u_n\}$, as well called $\{u_n\}$, such that $u_n \rightharpoonup u \mathrm{in} H$, $u_n \to u \quad$ in $L^s(\Omega)$ for $s \in [1, 2^*)$, $u_n \to u \quad$ a.e. $x \in \mathbb{R}^N$, as $n \to +\infty$. By computation,

$$\begin{aligned} &(a + b\|u_n\|^2)((u_n, u) - (u_n, u_n)) \\ &= (a + b\|u_n\|^2)\int_\Omega \nabla u_n \nabla(u - u_n)\mathrm{d}x - \int_\Omega f(x, u_n)(u - u_n)\mathrm{d}x + \int_\Omega f(x, u_n)(u - u_n)\mathrm{d}x \\ &= (\Phi'(u_n), (u - u_n)) + \int_\Omega f(x, u_n)(u - u_n)\mathrm{d}x. \end{aligned} \tag{20}$$

For $\{u_n\}$ is bounded in H and $\Phi'(u_n) \to 0$, then

$$(\Phi'(u_n), (u - u_n)) \leq \|\Phi'(u_n)\|\|u - u_n\| \leq \|\Phi'(u_n)\|(\|u\| + \|u_n\|) \to 0, \tag{21}$$

as $n \to +\infty$. Due to (12) and $u_n \to u$ in $L^s(\Omega)$ for $s \in [1, 2^*)$, we know that

$$\begin{aligned} \int_\Omega |f(x, u_n)(u - u_n)|\mathrm{d}x &\leq \int_\Omega (\varepsilon|u_n| + C_1(\varepsilon)|u_n|^{p-1})|u - u_n|\mathrm{d}x \\ &\leq \varepsilon\left(\int_\Omega |u_n|^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega |u - u_n|^2\mathrm{d}x\right)^{\frac{1}{2}} + C_1(\varepsilon)\left(\int_\Omega |u_n|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_\Omega |u - u_n|^p\mathrm{d}x\right)^{\frac{1}{p}} \\ &= \varepsilon\, |u_n|_2|u - u_n|_2 + C_1(\varepsilon)|u_n|_p^{p-1}|u - u_n|_p \to 0, \end{aligned} \tag{22}$$

as $n \to +\infty$. By combining (20), (21), and (22), we obtain $(u_n, u) - (u_n, u_n) \to 0$. Meanwhile by $u_n \rightharpoonup u$ in H and $(u_n, u) - (u_n, u_n) = (u_n - u, u) + (u, u) - (u_n, u_n)$, then $(u_n, u_n) \to (u, u)$, as $n \to +\infty$. In summary, $u_n \to u$ in H, $n \to +\infty$. Therefore, $\Phi(u)$ satisfies the $(\mathrm{PS})_c$ condition. □

Remark 2.2. The method of proving that $\{u_n\}$ has a convergent subsequence in Lemma 2.2 can be applied to prove Lemma 2.1.

3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Firstly, we prove $a_k \coloneqq \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leq 0$, where Y_k is defined in (4). According to (f_4), for any $M > 0$, there exist $\delta > 0$, such that, for $|u_n| \geq \delta$,

$$F(x, u_n) \geq M\,|u_n|^4. \tag{23}$$

(f_1) implies that there exists $\bar{\delta} > 0$ such that $\left|\frac{f(x,u_n)}{u_n}\right| \le 1$ for $0 < |u_n| < \bar{\delta}$. By (f_2), for $\bar{\delta} \le |u_n| \le \delta$, there exists $M_1 > 0$ such that

$$\left|\frac{f(x,u_n)}{u_n}\right| \le \frac{C(1+|u_n|^{p-1})}{|u_n|} \le M_1.$$

Then $f(x,u_n) \ge -(M_1+1)|u_n|$, for $0 \le |u_n| \le \delta$. By the definition of $F(x,u_n) = \int_0^{u_n} f(x,s)\mathrm{d}s$,

$$F(x,u_n) \ge -C_8\,|u_n|^2, \tag{24}$$

for $0 \le |u_n| \le \delta$, where $C_8 = \frac{1}{2}(M_1+1)$. By combining (23) with (24), we obtain $F(x,u) \ge M\,|u|^4 - C_8\,|u|^2$. Since all norms are equivalent in Y_k, a finite-dimensional space, there exist constants $d_1, d_2 > 0$ such that in Y_k:

$$\begin{aligned}\Phi(u) &\le \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - M\,|u|_4^4 + C_8\,|u|_2^2\\ &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - Md_1\|u\|^4 + C_8 d_2\|u\|^2\\ &= \left(\frac{b}{4} - Md_1\right)\|u\|^4 + \left(\frac{a}{2} + C_8 d_2\right)\|u\|^2.\end{aligned}$$

As $\frac{b}{4} - Md_1 < 0$, that is, $M > \frac{b}{4d_1}$, by the aforementioned inequality, let $\rho_k > 0$ is big enough such that $a_k \le 0$. In fact, $\rho_k > \gamma_k > 0$, where γ_k will be defined in (25).

Next, we will prove $b_k \coloneqq \inf_{u\in Z_k,\|u\|=\gamma_k}\Phi(u) \to +\infty$, as $k \to \infty$, where Z_k is defined in (4).

By the conclusion of Lemma 3.8 in [18], we have $\beta_k \coloneqq \sup_{u\in Z_k,\|u\|=1}|u|_s \to 0$, as $k \to \infty$, for $s \in [2, 2^*)$. Let $\varepsilon \in \left(0, \frac{a}{\beta_k^2}\right)$, for $u \in Z_k$, $\|u\| = \gamma_k$, by $\beta_k \to 0$ as $k \to \infty$, we can see that $\beta_k = \sup_{u\in Z_k,\|u\|=\gamma_k}\left|\frac{u}{\|u\|}\right|_s$. Therefore, $|u|_s \le \beta_k\|u\|$. By (3) and (13),

$$\begin{aligned}\Phi(u) &\ge \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\varepsilon}{2}\int_\Omega |u|^2\mathrm{d}x - \frac{C_1}{p}\int_\Omega |u|^p\mathrm{d}x\\ &\ge \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\varepsilon}{2}\beta_k^2\|u\|^2 - \frac{C_1}{p}\beta_k^p\|u\|^p\\ &\ge \left(\frac{a-\varepsilon\beta_k^2}{2} - \frac{C_1\beta_k^p}{p}\|u\|^{p-2}\right)\|u\|^2.\end{aligned}$$

Let

$$\frac{a-\varepsilon\beta_k^2}{2} - \frac{C_1\beta_k^p}{p}\|u\|^{p-2} = \frac{a-\varepsilon\beta_k^2}{4},$$

then

$$\gamma_k = \|u\| = \left(\frac{(a-\varepsilon\beta_k^2)p}{4C_1\beta_k^p}\right)^{\frac{1}{p-2}}. \tag{25}$$

Therefore,

$$b_k \ge \frac{a-\varepsilon\beta_k^2}{4}\left(\frac{(a-\varepsilon\beta_k^2)p}{4C_1\beta_k^p}\right)^{\frac{2}{p-2}}.$$

According to $p \in (4, 2^*)$ and $\beta_k \to 0$ as $k \to \infty$, then $b_k \to +\infty$ as $k \to \infty$. Lemma 2.1 establishes that $\Phi(u)$ satisfies the $(PS)_c$ condition. Furthermore, from (f_3), it is immediate that $\Phi(u) = \Phi(-u)$.

Hence, by Theorem 2.1, $\Phi(u)$ has an unbounded sequence of critical values. Therefore, (1) has infinitely many high-energy solutions. □

Proof of Theorem 1.2. The proofs that $a_k \leq 0$ and $b_k \to +\infty$ as $k \to \infty$ follow the same method as in Theorem 1.1. Lemma 2.2 proves that $\Phi(u)$ satisfies the $(PS)_c$ condition; moreover, $\Phi(u) = \Phi(-u)$, and hence, $\Phi(u)$ has an unbounded sequence of critical values. Therefore, (1) has infinitely many high-energy solutions. □

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