Research Article

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On a nonlinear boundary value problems with impulse action

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Abstract: In this work, a boundary value problems for a system of nonlinear ordinary differential equations that incorporates impulsive actions is considered. This formulation is significant for modeling real-world phenomena in which abrupt changes occur at specific time instants. The study established sufficient conditions for the existence of isolated solutions to the proposed boundary value problems. This is crucial to ensure that the mathematical models accurately reflect the behavior of systems subject to impulsive actions. Algorithms were developed to find solutions to the boundary value problems. These algorithms leverage the parameterization method, which is effective in handling the discontinuities introduced by impulsive actions. The research includes a numerical implementation of the proposed algorithms, demonstrating their practicality and effectiveness in solving the boundary value problems with impulsive actions. The findings have implications in various fields, including mechanics, electrical engineering, and biology, where systems often experience sudden changes due to external influences. In general, the research contributes to the understanding and solution of nonlinear boundary value problems affected by impulsive actions, providing a framework for further exploration and application in scientific and engineering contexts.

Keywords: boundary value problems with impulsive action, isolated solution, parameterization method algorithms, convergence, sufficient solvability conditions

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This work is dedicated to the bright memory of our scientific advisor, D. S. Dzhumabaev, on the occasion of his 70th anniversary.

1 Introduction and problems statement

Investigating various problems in the natural sciences often involves dealing with evolutionary processes described by differential equations and subject to short-term perturbations. When mathematically modeling such processes, it is often convenient to neglect the duration of these perturbations, considering them to be of an impulse (shock) nature. Such idealization leads to the need to study systems of differential equations whose solutions undergo abrupt changes. Frequently, discontinuities in certain dependencies within the system studied are essential characteristics. Many specific problems whose mathematical models involve differential equations with discontinuous trajectories can be found in various areas of mathematical natural science:

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mechanics, electrical engineering, chemistry, biology and medicine, process control, dynamics of aircrafts, economics, and other branches of science and technology.

The growing interest in systems with discontinuous trajectories is primarily associated with the demands of modern technology, where impulse control systems, impulse computing systems, and neural networks have taken a prominent place and are rapidly developing, expanding their application scope in diverse technical problems varying in physical nature and functional purpose. A natural response to this has been a noticeable increase in the number of mathematical works dedicated to the study of differential equations with impulse effects. Classical monographs systematically addressing differential equations with impulse perturbations include the books by Samoilenko and Perestyuk [1] and Lakshmikantam et al. [2]. The results compiled in these monographs have served as a basis for further development of analytical and qualitative methods in the theory of impulse-disturbed systems [3–8].

The solvability of various types of boundary value problems using operator methods has been actively investigated by Samoilenko et al. [9], including problems with impulsive action [10].

In the work of Dzhumabaev, a parameterization method research and solving a linear two-point boundary value problems for a system of ordinary differential equations (ODEs), was developed [11]. The Dzhumabaev's parameterization method provides a constructive algorithm for finding solutions, which is particularly useful for nonlinear problems where analytical solutions may be difficult or unattainable [12]. The application of this method allows for the development of a systematic approach to obtaining solutions, without relying solely on existence theorems.

It is important to note that the use of the parameterization method to nonlinear simplifies the derivation of sufficient conditions for the existence of isolated solutions, which is crucial in the context of nonlinear problems. Furthermore, the algorithms of the method are amenable to numerical implementation. It enables the use of computational methods to obtain approximate solutions. This feature is particularly advantageous in practical applications where numerical solutions are often required.

The versatility of the parameterization method allows it to be applied to a wide range of boundary value problems, including those with complex boundary conditions and multiple impulsive actions. Such broad applicability makes it a valuable tool in mathematical modeling across various fields.

Thus, researchers obtained an effective tool for the constructive solving of boundary value problems for various classes of differential equations [13–27], such that it stands out due to its constructive nature, flexibility in handling discontinuities, ability to establish solvability conditions, and suitability for numerical implementation, making it a powerful tool for solving various classes of nonlinear boundary value problems for differential equations, including those subjected to impulsive actions.

The solving and investigation of boundary value problems with impulsive action at fixed time instants by the parameterization method are devoted to [28–38].

In these works, conditions for solvability were obtained, algorithms for finding solutions were constructed, and coefficient criteria for unique solvability were obtained.

We consider the boundary value problems with impulsive actions on $[a, b]$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x), \quad t \in (a, b)\backslash\{\theta_1, \theta_2, \ldots, \theta_m\}, \ x \in \mathbb{R}^n, \tag{1}$$

$$x(\theta_i) - x(\theta_i - 0) = p_i, \quad p_i \in \mathbb{R}^n, \ i = \overline{1, m}, \tag{2}$$

$$Bx(a) + Cx(b) = d, \quad d \in \mathbb{R}^n, \tag{3}$$

where $f : ([a, b]\backslash\{\theta_1, \theta_2, \ldots, \theta_m\}) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, B and C are the given $(n \times n)$ matrices and d and p_i $(i = \overline{1, m})$ are the given n vectors, $a = \theta_0 < \theta_1 < \ldots < \theta_m < \theta_{m+1} = b$, $\Theta = \{\theta_1, \theta_2, \ldots, \theta_m\}$.

Let $PC([a, b]\backslash\Theta, \mathbb{R}^n)$ be a space of piecewise continuous functions with the norm

$$||x||_1 = \max_{i=\overline{0,m}} \sup_{t\in[\theta_i,\theta_{i+1})} ||x(t)||.$$

A solution to problem (1)–(3) is a piecewise-continuously differentiable on $(a,b)\backslash\Theta$ function $x^*(t)\in PC([a,b]\backslash\Theta,\mathbb{R}^n)$ that satisfies

(1) differential equation (1); moreover, at the points $t=a, t=b$, equation (1) satisfies the one-sided derivatives $\frac{dx^*(t)}{dt}\Big|_{t=a}=\lim_{t\to a+0}\frac{x^*(t)-x^*(a)}{t-a}$, $\frac{dx^*(t)}{dt}\Big|_{t=b}=\lim_{t\to b-0}\frac{x^*(t)-x^*(b)}{t-b}$,

(2) the impulsive conditions (2) at the points of the set Θ, and

(3) the boundary condition (3).

2 Method for solving the problem (1)–(3)

We choose a natural number N. Denote by Δ_N the partition $[a,b)\backslash\Theta=\cup_{r=1}^{(m+1)N}[t_{r-1},t_r)$, where

$$t_r=\theta_s+(r-sN)\cdot\frac{\theta_{s+1}-\theta_s}{N},\quad r=\overline{sN,(s+1)N},\quad s=\overline{0,m}.$$

We introduce the space $C([a,b)\backslash\Theta,\Delta_N,\mathbb{R}^{n(m+1)N})$ consisting of all function systems $x[t]=(x_1(t), x_2(t),\dots,x_{(m+1)N}(t))$, where functions $x_r:[t_{r-1},t_r)\to\mathbb{R}^n$ are continuous and have finite limits $\lim_{t\to t_r-0}x_r(t)$ for all $r=\overline{1,(m+1)N}$, with the norm $\|x\|_2=\max_{r=\overline{1,(m+1)N}}\sup_{t\in[t_{r-1},t_r)}\|x_r(t)\|$.

Let us introduce the notations: $\lambda_r=x(t_{r-1}), u_r(t)=x(t)-\lambda_r, t\in[t_{r-1},t_r), r=\overline{1,(m+1)N}$. Then, we reduce problem (1)–(3) to the equivalent multipoint boundary-value problem with parameters

$$\frac{du_r}{dt}=f(t,\lambda_r+u_r),\quad t\in[t_{r-1},t_r),\ r=\overline{1,(m+1)N},\tag{4}$$

$$u_r(t_{r-1})=0,\quad r=\overline{1,(m+1)N},\tag{5}$$

$$\lambda_{iN+1}-\lambda_{iN}-\lim_{t\to t_{iN}-0}u_{iN}(t)-p_i=0,\quad i=\overline{1,m},\tag{6}$$

$$B\lambda_1+C\lambda_{(m+1)N}+C\lim_{t\to t_{(m+1)N}-0}u_{(m+1)N}(t)-d=0,\tag{7}$$

$$\lambda_r+\lim_{t\to t_r-0}u_r(t)-\lambda_{r+1}=0,\quad r=\overline{iN+1,(i+1)N-1},\ i=\overline{0,m},\tag{8}$$

where (8) are the gluing conditions at the points of partition of the intervals (θ_{i-1},θ_i), $i=\overline{1,m+1}$.

The solution of problem (4)–(8) is the pair $(\lambda^*,u^*[t])$, where $\lambda^*=(\lambda_1^*,\lambda_2^*,\dots,\lambda_{(m+1)N}^*)\in\mathbb{R}^{n(m+1)N}$, $u^*[t]=(u_1^*(t),u_2^*(t),\dots,u_{(m+1)N}^*(t))\in C([a,b)\backslash\Theta,\Delta_N,\mathbb{R}^{n(m+1)N})$.

If $(\lambda^*,u^*[t])$ is a solution problem (4)–(8), then the function

$$x^*(t)=\begin{cases}\lambda_r^*+u_r^*(t) & \text{for } t\in[t_{r-1},t_r),\ r=\overline{1,(m+1)N},\\ \lambda_{(m+1)N}^*+\lim_{t\to b-0}u_{(m+1)N}^*(t) & \text{for } t=b\end{cases}$$

is a solution to problem (1)–(3).

If $\tilde{x}(t)$ is a solution to problem (1)–(3), then, the pair $(\tilde{\lambda},\tilde{u}[t])$ with elements $\tilde{\lambda}=(\tilde{x}(t_0),\tilde{x}(t_1),\dots,\tilde{x}(t_{(m+1)N-1}))\in\mathbb{R}^{n(m+1)N}$, $\tilde{u}[t]=(\tilde{x}(t)-\tilde{x}(t_0),\tilde{x}(t)-\tilde{x}(t_1),\dots,\tilde{x}(t)-\tilde{x}(t_{(m+1)N-1}))$ is a solution to problem (4)–(8).

For fixed values of λ_r, the Cauchy problem (4), (5) is equivalent to the Volterra integral equation of the second kind

$$u_r(t)=\int_{t_{r-1}}^{t}f(\tau_1,\lambda_r+u_r(\tau_1))d\tau_1,\quad t\in[t_{r-1},t_r),\ r=\overline{1,(m+1)N}.\tag{9}$$

Using equality (9), we obtain the expression

$$u_r(t) = \int_{t_{r-1}}^{t} f\left(\tau_1, \lambda_r + \int_{t_{r-1}}^{\tau_1} f\left(\tau_2, \lambda_r + \ldots + \int_{t_{r-1}}^{\tau_{\nu-1}} f(\tau_\nu, \lambda_r + u_r(\tau_\nu))\mathrm{d}\tau_\nu \ldots\right)\mathrm{d}\tau_2\right)\mathrm{d}\tau_1,$$

$t \in [t_{r-1}, t_r)$, $r = \overline{1, (m+1)N}$.

Then

$$\lim_{t \to t_r - 0} u_r(t) = \int_{t_{r-1}}^{t_r} f\left(\tau_1, \lambda_r + \int_{t_{r-1}}^{\tau_1} f\left(\tau_2, \lambda_r + \ldots + \int_{t_{r-1}}^{\tau_{\nu-1}} f(\tau_\nu, \lambda_r + u_r(\tau_\nu))\mathrm{d}\tau_\nu \ldots\right)\mathrm{d}\tau_2\right)\mathrm{d}\tau_1, \tag{10}$$

$r = \overline{1, (m+1)N}$.

Substituting these limits into (6)–(8), we obtain a system of nonlinear equations

$$\begin{aligned} \frac{\theta_{i+1} - \theta_i}{N}\Bigg(\lambda_{iN+1} - \lambda_{iN} - \int_{t_{iN-1}}^{t_{iN}} f\Bigg(\tau_1, \lambda_{iN} + \int_{t_{iN-1}}^{\tau_1} f\Bigg(\tau_2, \lambda_{iN} \\ + \ldots + \int_{t_{iN-1}}^{\tau_{\nu-1}} f(\tau_\nu, \lambda_{iN} + u_{iN}(\tau_\nu))\mathrm{d}\tau_\nu \ldots\Bigg)\mathrm{d}\tau_2\Bigg)\mathrm{d}\tau_1 - p_i\Bigg) = 0, \quad i = \overline{1, m}, \end{aligned} \tag{11}$$

$$\begin{aligned} \frac{\theta_{m+1} - \theta_m}{N}\Bigg(B\lambda_1 + C\lambda_{(m+1)N} + C\int_{t_{(m+1)N-1}}^{t_{(m+1)N}} f\Bigg(\tau_1, \lambda_{(m+1)N} + \int_{t_{(m+1)N-1}}^{\tau_1} f\Bigg(\tau_2, \lambda_{(m+1)N} \\ + \ldots + \int_{t_{(m+1)N-1}}^{\tau_{\nu-1}} f(\tau_\nu, \lambda_{(m+1)N} + u_{(m+1)N}(\tau_\nu))\mathrm{d}\tau_\nu \ldots\Bigg)\mathrm{d}\tau_2\Bigg)\mathrm{d}\tau_1 - d\Bigg) = 0, \end{aligned} \tag{12}$$

$$\lambda_r + \int_{t_{r-1}}^{t_r} f\left(\tau_1, \lambda_r + \int_{t_{r-1}}^{\tau_1} f\left(\tau_2, \lambda_r + \ldots + \int_{t_{r-1}}^{\tau_{\nu-1}} f(\tau_\nu, \lambda_r + u_r(\tau_\nu))\mathrm{d}\tau_\nu \ldots\right)\mathrm{d}\tau_2\right)\mathrm{d}\tau_1 - \lambda_{r+1} = 0, \tag{13}$$

$$r = \overline{iN+1, (i+1)N-1}, \quad i = \overline{0, m}.$$

For known $u_r(t)$ ($r = \overline{1, (m+1)N}$), the system of equations (11)–(13) is a system of equations with respect to the parameters $(\lambda_1, \lambda_2, \ldots, \lambda_{(m+1)N})$. We write the system of equations (11)–(13) in the following form:

$$Q_{\nu,\Delta_N}(\lambda, u) = 0, \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{(m+1)N}) \in \mathbb{R}^{n(m+1)N}. \tag{14}$$

3 Algorithms of Dzhumabaev's parameterization method and convergence conditions

Condition A. There exists a partition Δ_N, a natural number ν such that the system of nonlinear equations $Q_{\nu,\Delta_N}(\lambda, 0) = 0$ have a solution $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_{(m+1)N}^{(0)}) \in \mathbb{R}^{n(m+1)N}$.

Let Condition A be satisfied. We assume a Cauchy problems

$$\frac{\mathrm{d}u_r}{\mathrm{d}t} = f(t, u_r + \lambda_r^{(0)}), \quad u_r(t_{r-1}) = 0 \tag{15}$$

having a solution $u_r^{(0)}(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, (m+1)N}$, and the system of functions $u^{(0)}[t]$ belongs to the space $C([a,b)\backslash\Theta, \Delta_N, \mathbb{R}^{n(m+1)N})$.

By using a pair $(\lambda^{(0)}, u^{(0)}[t])$, we specify a piecewise continuous function on $[a,b]$

$$x^{(0)}(t) = \begin{cases} \lambda_r^{(0)} + u_r^{(0)}(t) & \text{for } t \in [t_{r-1}, t_r),\ r = \overline{1,(m+1)N}, \\ \lambda_{(m+1)N}^{(0)} + \lim\limits_{t\to b-0} u_{(m+1)N}^{(0)}(t) & \text{for } t = b. \end{cases}$$

We take the numbers $\rho_\lambda > 0$, $\rho_u > 0$, $\rho_x > 0$ and define the sets

$$S(\lambda^{(0)}, \rho_\lambda) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{(m+1)N}) \in \mathbb{R}^{n(m+1)N} : \|\lambda - \lambda^{(0)}\| = \max_{r=\overline{1,(m+1)N}} \|\lambda_r - \lambda_r^{(0)}\| < \rho_\lambda\},$$

$$S(u^{(0)}[t], \rho_u) = \{u[t] \in C([a,b)\backslash\Theta, \Delta_N, \mathbb{R}^{n(m+1)N}) : \|u - u^{(0)}\|_2 < \rho_u\},$$

$$S(x^{(0)}(t), \rho_x) = \{x(t) \in PC([a,b]\backslash\Theta, \mathbb{R}^n) : \|x - x^{(0)}\|_1 < \rho_x\},$$

$$G_f(\rho_x) = \{(t,x) : t \in [a,b]\backslash\Theta, x \in S(x^{(0)}(t), \rho_x)\}.$$

Let Condition A hold, and let the components of the system of functions $u^{(0)}[t]$ be solutions to Cauchy problems (15).

We construct sequences $\{(\lambda^{(k)}, u^{(k)}(t))\}_{k=1}^{\infty}$ and $\{x^{(k)}(t)\}_{k=1}^{\infty}$ by performing the following sequence of steps:

Step k

(a) Solve the equation $Q_{\nu,\Delta_N}(\lambda, u^{(k-1)}) = 0$ and find $\lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_{(m+1)N}^{(k)}) \in \mathbb{R}^{n(m+1)N}$.

(b) Solve the Cauchy problems

$$\frac{du_r}{dt} = f(t, u_r + \lambda_r^{(k)}), \quad u_r(t_{r-1}) = 0,$$

and find the components of the system $u^{(k)}[t] = (u_1^{(k)}(t), \ldots, u_{(m+1)N}^{(k)}(t))$.

(c) Construct the function

$$x^{(k)}(t) = \begin{cases} \lambda_r^{(k)} + u_r^{(k)}(t), & \text{if } t \in [t_{r-1}, t_r),\ r = \overline{1,(m+1)N}, \\ \lambda_{(m+1)N}^{(k)} + \lim\limits_{t\to b-0} u_{(m+1)N}^{(k)}(t), & \text{if } t = t_{(m+1)N} = b. \end{cases}$$

where $k = 1, 2, \ldots$.

Condition B. The function $f(t,x)$ in $G_f(\rho_x)$ is continuous, has a uniformly continuous partial derivative $f'_x(t,x)$, and there exists a number $L > 0$ such that $\|f'_x(t,x)\| \leqslant L$ for all $(x,t) \in G_f(\rho_x)$.

Theorem 1. *Suppose that for some Δ_N with $N \in \mathbb{N}$, and for $\nu \in \mathbb{N}$, as well as positive constants $\rho_\lambda > 0$, $\rho_u > 0$, and $\rho_x > 0$, the following conditions are satisfied:*

(*i*) *conditions A and B,*

(*ii*) *Jacobi matrix* $\frac{\partial Q_{\nu,\Delta_N}(\lambda,u)}{\partial \lambda} : \mathbb{R}^{n(m+1)N} \to \mathbb{R}^{n(m+1)N}$ *is invertible for all* $(\lambda, u[t]) \in S(\lambda^{(0)}, \rho_\lambda) \times S(u^{(0)}[t], \rho_u)$,

(*iii*) $\left\| \left(\frac{\partial Q_{\nu,\Delta_N}(\lambda,u)}{\partial \lambda} \right)^{-1} \right\| \leqslant \gamma_\nu(\Delta_N)$, $\gamma_\nu(\Delta_N)$-const,

(*iv*) $q_\nu(\Delta_N) < 1$, *where*

$$q_\nu(\Delta_N) = \gamma_\nu(\Delta_N) \max\left\{1, \max_{i=\overline{0,m}} \frac{\theta_{i+1} - \theta_i}{N}, \frac{\theta_{m+1} - \theta_m}{N} \|C\|\right\} \max_{i=\overline{0,m}} \left(e^{L\frac{\theta_{i+1}-\theta_i}{N}} - \sum_{j=0}^{\nu} \frac{(L(\theta_{i+1} - \theta_i))^j}{j! N^j} \right),$$

(*v*) $\frac{\gamma_\nu(\Delta_N)}{1 - q_\nu(\Delta_N)} \|Q_{\nu,\Delta_N}(\lambda^{(0)}, u^{(0)})\| < \rho_\lambda$,

(*vi*) $\frac{\gamma_\nu(\Delta_N)}{1 - q_\nu(\Delta_N)} \max_{i=\overline{0,m}} \left(e^{L\frac{\theta_{i+1}-\theta_i}{N}} - 1 \right) \|Q_{\nu,\Delta_N}(\lambda^{(0)}, u^{(0)})\| < \rho_u$,

(*vii*) $\max_{p=\overline{1,\nu}} \left\{ \rho_\lambda \max_{i=\overline{0,m}} \sum_{j=0}^{p-1} \frac{(L(\theta_{i+1}-\theta_i))^j}{j! N^j} + \rho_u \max_{i=\overline{0,m}} \frac{(L(\theta_{i+1}-\theta_i))^{p-1}}{(p-1)! N^{p-1}} \right\} \leqslant \rho_x$.

Then, the sequence of pairs $(\lambda^{(k)}, u^{(k)}[t])$, $k \in \mathbb{N}$, *defined by the algorithm, belongs to the set* $S(\lambda^{(0)}, \rho_\lambda) \times S(u^{(0)}[t], \rho_u)$, *and converges to the pair* $(\lambda^*, u^*[t])$, *which is an isolated solution of problem* (4)–(8) *in the set* $S(\lambda^{(0)}, \rho_\lambda) \times S(u^{(0)}[t], \rho_u)$. *Moreover, the following estimates hold*:

$$\|\lambda^* - \lambda^{(k)}\| \leqslant \frac{(q_\nu(\Delta_N))^k}{1 - q_\nu(\Delta_N)} \gamma_\nu(\Delta_N) \|Q_{\nu,\Delta_N}(\lambda^{(0)}, u^{(0)})\|, \tag{16}$$

$$\|u_r^*(t) - u_r^{(k)}(t)\| \leqslant (e^{L(t-t_{r-1})} - 1)\|\lambda_r^* - \lambda_r^{(k)}\|, \tag{17}$$

$k = 0, 1, 2, \ldots, t \in [t_{r-1}, t_r)$, $r = \overline{1, (m+1)N}$.

Proof. By some number N, we perform a partition Δ_N of the interval $[a, b)$. Let us reduce problem (1), (2) to the equivalent multipoint boundary value problem with parameters (4)–(8).

Taking any pair $(\lambda, u[t]) \in S(\lambda^{(0)}, \rho_\lambda) \times S(u^{(0)}[t], \rho_u)$, then

$$\begin{gathered} \|\lambda_r - \lambda_r^{(0)} + u_r(t) - u_r^{(0)}(t)\| \leqslant \|\lambda_r - \lambda_r^{(0)}\| + \|u_r(t) - u_r^{(0)}(t)\| < \rho_\lambda + \rho_u \leqslant \rho_x, \\ t \in [t_{r-1}, t_r), \ r = \overline{1, (m+1)N}. \end{gathered} \tag{18}$$

By virtue of Condition B for all $r = \overline{1, (m+1)N}$, the following inequalities occur:

$$\begin{aligned} &\left\| \lambda_r + \int_{t_{r-1}}^{t} f(\tau_1, \lambda_r + u_r(\tau_1)) d\tau_1 - \lambda_r^{(0)} - u_r^{(0)}(t) \right\| \\ &\leqslant \|\lambda_r - \lambda_r^{(0)}\| + \left\| \int_{t_{r-1}}^{t} f(\tau_1, \lambda_r + u_r(\tau_1)) d\tau_1 - \int_{t_{r-1}}^{t} f(\tau_1, \lambda_r^{(0)} + u_r^{(0)}(\tau_1)) d\tau_1 \right\| \\ &\leqslant (1 + L(t - t_{r-1}))\|\lambda_r - \lambda_r^{(0)}\| + \int_{t_{r-1}}^{t} L\|u_r(\tau) - u_r^{(0)}(\tau)\| d\tau \\ &< (1 + L(t_r - t_{r-1}))\rho_\lambda + \rho_u L(t_r - t_{r-1}) \leqslant \rho_x, \quad t \in [t_{r-1}, t_r). \end{aligned} \tag{19}$$

Similarly, we obtain that

$$\begin{aligned} &\left\| \lambda_r + \int_{t_{r-1}}^{t} f\left(\tau_1, \lambda_r + \ldots + \int_{t_{r-1}}^{\tau_{\nu-2}} f(\tau_{\nu-1}, \lambda_r + u_r(\tau_{\nu-1})) d\tau_{\nu-1} \ldots\right) d\tau_1 - \lambda_r^{(0)} - u_r^{(0)}(t) \right\| \\ &\leqslant \|\lambda_r - \lambda_r^{(0)}\| + \left\| \int_{t_{r-1}}^{t} f\left(\tau_1, \lambda_r + \ldots + \int_{t_{r-1}}^{\tau_{\nu-2}} f(\tau_{\nu-1}, \lambda_r + u_r(\tau_{\nu-1})) d\tau_{\nu-1} \ldots\right) d\tau_1 \right. \\ &\left. - \int_{t_{r-1}}^{t} f\left(\tau_1, \lambda_r^{(0)} + \ldots + \int_{t_{r-1}}^{\tau_{\nu-2}} f(\tau_{\nu-1}, \lambda_r^{(0)} + +u_r^{(0)}(\tau_{\nu-1})) d\tau_{\nu-1} \ldots\right) d\tau_1 \right\| \\ &\leqslant \sum_{j=0}^{\nu-1} \frac{(L(t_r - t_{r-1}))^j}{j!} \|\lambda_r - \lambda_r^{(0)}\| + \int_{t_{r-1}}^{t} L \ldots \int_{t_{r-1}}^{\tau_{\nu-2}} L\|u_r(\tau) - u_r^{(0)}(\tau)\| d\tau_{\nu-1} \ldots d\tau_1 \\ &< \sum_{j=0}^{\nu-1} \frac{(L(t_r - t_{r-1}))^j}{j!} \rho_\lambda + \frac{(L(t_r - t_{r-1}))^{\nu-1}}{(\nu-1)!} \rho_u \leqslant \rho_x, \quad t \in [t_{r-1}, t_r). \end{aligned} \tag{20}$$

In view of (18)–(20) and inequality (vii) of the theorem, the pairs

$$(t, \lambda_r + u_r(t)), \quad \left(t, \lambda_r + \int_{t_{r-1}}^{t} f(\tau_1, \lambda_r + u_r(\tau_1)) d\tau_1\right), \ldots,$$

$$\left(t, \lambda_r + \int\limits_{t_{r-1}}^{t} f\left(\tau_1, \lambda_r + \ldots + \int\limits_{t_{r-1}}^{\tau_{\nu-2}} f(\tau_{\nu-1}, \lambda_r + u_r(\tau_{\nu-1}))\mathrm{d}\tau_{\nu-1} \ldots\right)\mathrm{d}\tau_1\right),$$

where $(\lambda, u[t]) \in S(\lambda^{(0)}, \rho_\lambda) \times S(u^{(0)}[t], \rho_u)$ with $t \in [t_{r-1}, t_r)$ for all $r = \overline{1, (m+1)N}$ belong to the set $G_f(\rho_x)$.

We will search for the solution of problem (4)–(8) using the proposed algorithm. Taking the pair $(\lambda^{(0)}, u^{(0)}[t])$ from Condition A as the initial approximation, we find the next approximation with respect to the parameter from the equation

$$Q_{\nu,\Delta_N}(\lambda, u^{(0)}) = 0, \quad \lambda \in \mathbb{R}^{n(m+1)N}. \tag{21}$$

By virtue of the conditions of the theorem, the operator $Q_{\nu,\Delta_N}(\lambda, u^{(0)})$ in $S(\lambda^{(0)}, \rho_\lambda)$ satisfies all the assumptions of Theorem A [12, p. 345]. Taking a number $\varepsilon_0 > 0$, satisfying the inequalities

$$\varepsilon_0\gamma_\nu(\Delta_N) \leqslant \frac{1}{2}, \quad \frac{\gamma_\nu(\Delta_N)}{1 - \varepsilon_0\gamma_\nu(\Delta_N)}\|Q_{\nu,\Delta_N}(\lambda^{(0)}, u^{(0)})\| < \rho_\lambda,$$

and using uniform continuity in $S(\lambda^{(0)}, \rho_\lambda)$ Jacobi matrices $\frac{\partial Q_{\nu,\Delta_N}(\lambda, u^{(0)})}{\partial\lambda}$, we find $\delta_0 \in (0, 0.5\rho_\lambda]$ such that for any $\lambda, \tilde{\lambda} \in S(\lambda^{(0)}, \rho_\lambda)$, satisfying inequality $\|\lambda - \tilde{\lambda}\| < \delta_0$ is true such that

$$\left\|\frac{\partial Q_{\nu,\Delta_N}(\lambda, u^{(0)})}{\partial\lambda} - \frac{\partial Q_{\nu,\Delta_N}(\tilde{\lambda}, u^{(0)})}{\partial\lambda}\right\| < \varepsilon_0.$$

Let us choose $\alpha \geqslant \alpha_0 = \max\left\{1, \frac{\gamma_\nu(\Delta_N)}{\delta_0}\|Q_{\nu,h}(\lambda^{(0)}, u^{(0)})\|\right\}$, build an iterative process: $\lambda^{(1,0)} = \lambda^{(0)}$,

$$\lambda^{(1,m+1)} = \lambda^{(1,m)} - \frac{1}{\alpha}\left(\frac{\partial Q_{\nu,\Delta_N}(\lambda^{(1,m)}, u^{(0)})}{\partial\lambda}\right)^{-1} \cdot Q_{\nu,\Delta_N}(\lambda^{(1,m)}, u^{(0)}), \quad m = 0, 1, 2, \ldots. \tag{22}$$

The iterative process (22) converges to $\lambda^{(1)} \in S(\lambda^{(0)}, \rho_\lambda)$-isolated solution of equation (21) and

$$\|\lambda^{(1)} - \lambda^{(0)}\| \leqslant \gamma_\nu(\Delta_N)\|Q_{\nu,\Delta_N}(\lambda^{(0)}, u^{(0)})\| < \rho_\lambda. \tag{23}$$

Under our assumptions, the Cauchy problem (4), (5) for $\lambda_r = \lambda_r^{(1)}$ on $[t_{r-1}, t_r)$ has a unique solution $u_r^{(1)}(t)$ and for it holds the following inequality:

$$\|u_r^{(1)}(t) - u_r^{(0)}(t)\| \leqslant \int\limits_{t_{r-1}}^{t} L(\|\lambda_r^{(1)} - \lambda_r^{(0)}\| + \|u_r^{(1)}(\tau) - u_r^{(0)}(\tau)\|)\mathrm{d}\tau.$$

Using the Gronwall-Bellman lemma, we obtain

$$\|u_r^{(1)}(t) - u_r^{(0)}(t)\| \leqslant (e^{L(t-t_{r-1})} - 1)\|\lambda_r^{(1)} - \lambda_r^{(0)}\|, \quad t \in [t_{r-1}, t_r),\ r = \overline{1, (m+1)N}. \tag{24}$$

From (23) and (24) we obtain $u^{(1)}[t] = (u_1^{(1)}(t), u_2^{(1)}(t), \ldots, u_N^{(1)}(t)) \in S(u^{(0)}[t], \rho_u)$.

From the structure of the operator $Q_{\nu,\Delta_N}(\lambda, u)$ and the equality $Q_{\nu,\Delta_N}(\lambda^{(1)}, u^{(0)}) = 0$, it follows that

$$\begin{aligned}\|Q_{\nu,\Delta_N}(\lambda^{(1)}, u^{(1)})\| &= \|Q_{\nu,\Delta_N}(\lambda^{(1)}, u^{(1)}) - Q_{\nu,\Delta_N}(\lambda^{(1)}, u^{(0)})\| \\ &\leqslant \max\left\{1, \max_{i=\overline{1,m}}\frac{\theta_{i+1} - \theta_i}{N}, \frac{\theta_{m+1} - \theta_m}{N}\|C\|\right\} \\ &\quad \times \max_{r=\overline{1,(m+1)N}} \int\limits_{t_{r-1}}^{t_r} L \ldots \int\limits_{t_{r-1}}^{\tau_{\nu-1}} L\|u_r^{(1)}(\tau_\nu) - u_r^{(0)}(\tau_\nu)\|\mathrm{d}\tau_\nu \ldots \mathrm{d}\tau_1.\end{aligned}$$

Substituting instead of $\|u_r^{(1)}(\tau_\nu) - u_r^{(0)}(\tau_\nu)\|$ the right-hand side of (24) and computing the repeated integrals, we have

$$\gamma_\nu(h)\|Q_{\nu,\Delta_N}(\lambda^{(1)}, u^{(1)})\| \leqslant q_\nu(\Delta_N)\|\lambda^{(1)} - \lambda^{(0)}\|. \tag{25}$$

Let us take $\rho_1 = \gamma_\nu(\Delta_N)\|Q_{\nu,\Delta_N}(\lambda^{(1)}, u^{(1)})\|$. If $\lambda \in S(\lambda^{(1)}, \rho_1)$, then, due to the inequalities (iv), (v) of the theorem and (23), (25), the following estimate holds

$$\|\lambda - \lambda^{(0)}\| \leqslant \|\lambda - \lambda^{(1)}\| + \|\lambda^{(1)} - \lambda^{(0)}\| < \gamma_\nu(\Delta_N)\|Q_{\nu,\Delta_N}(\lambda^{(1)}, u^{(1)})\| + \|\lambda^{(1)} - \lambda^{(0)}\|$$
$$\leqslant (q_\nu(\Delta_N) + 1)\|\lambda^{(1)} - \lambda^{(0)}\| < \frac{\gamma_\nu(\Delta_N)}{1 - q_\nu(\Delta_N)}\|Q_{\nu,\Delta_N}(\lambda^{(0)}, u^{(0)})\| < \rho_\lambda,$$

that is, $S(\lambda^{(1)}, \rho_1) \subset S(\lambda^{(0)}, \rho_\lambda)$.

The operator $Q_{\nu,\Delta_N}(\lambda, u^{(1)})$ in $S(\lambda^{(1)}, \rho_1)$ satisfies all conditions of Theorem A [12]. Therefore, the iterative process: $\lambda^{(2,0)} = \lambda^{(1)}$,

$$\lambda^{(2,m+1)} = \lambda^{(2,m)} - \frac{1}{\alpha}\left(\frac{\partial Q_{\nu,\Delta_N}(\lambda^{(2,m)}, u^{(1)})}{\partial \lambda}\right)^{-1} Q_{\nu,\Delta_N}(\lambda^{(2,m)}, u^{(1)}), \quad m = 0, 1, 2, \ldots,$$

converges to $\lambda^{(2)} = (\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_N^{(2)}) \in S(\lambda^{(1)}, \rho_1)$-isolated solution of the equation $Q_{\nu,\Delta_N}(\lambda, u^{(1)}) = 0$ and

$$\|\lambda^{(2)} - \lambda^{(1)}\| \leqslant \gamma_\nu(\Delta_N)\|Q_{\nu,\Delta_N}(\lambda^{(1)}, u^{(1)})\|. \tag{26}$$

From (26) and (25), it follows that

$$\|\lambda^{(2)} - \lambda^{(1)}\| \leqslant q_\nu(\Delta_N)\|\lambda^{(1)} - \lambda^{(0)}\|.$$

Assuming that the pair $(\lambda^{(k-1)}, u^{(k-1)}[t]) \in S(\lambda^{(0)}, \rho_\lambda) \times S(u^{(0)}[t], \rho_u)$ defined and established estimates

$$\|\lambda^{(k-1)} - \lambda^{(k-2)}\| \leqslant q_\nu^{k-2}(\Delta_N)\|\lambda^{(1)} - \lambda^{(0)}\|, \tag{27}$$

$$\gamma_\nu(\Delta_N)\|Q_{\nu,\Delta_N}(\lambda^{(k-1)}, u^{(k-1)})\| \leqslant q_\nu(\Delta_N)\|\lambda^{(k-1)} - \lambda^{(k-2)}\|, \tag{28}$$

kth approximation with respect to the parameter $\lambda^{(k)}$ can be found from the equation $Q_{\nu,\Delta_N}(\lambda, u^{(k-1)}) = 0$. Using (27), (28), and equality $Q_{\nu,h}(\lambda^{(k-1)}, u^{(k-2)}) = 0$, similar to (25), we establish the validity of the inequality

$$\gamma_\nu(\Delta_N)\|Q_{\nu,\Delta_N}(\lambda^{(k-1)}, u^{(k-1)})\| \leqslant q_\nu^{k-1}(\Delta_N)\|\lambda^{(1)} - \lambda^{(0)}\|. \tag{29}$$

Let us take $\rho_{k-1} = \gamma_\nu(\Delta_N)\|Q_{\nu,\Delta_N}(\lambda^{(k-1)}, u^{(k-1)})\|$ and show that $S(\lambda^{(k-1)}, \rho_{k-1}) \subset S(\lambda^{(0)}, \rho_\lambda)$. Indeed, in view of (27)–(29) and inequalities (v)

$$\|\lambda - \lambda^{(0)}\| \leqslant \|\lambda - \lambda^{(k-1)}\| + \|\lambda^{(k-1)} - \lambda^{(k-2)}\| + \ldots + \|\lambda^{(1)} - \lambda^{(0)}\|$$
$$< \rho_{k-1} + q_\nu^{k-2}(\Delta_N)\|\lambda^{(1)} - \lambda^{(0)}\| + \ldots + \|\lambda^{(1)} - \lambda^{(0)}\|$$
$$\leqslant \|\lambda^{(1)} - \lambda^{(0)}\| < \frac{\gamma_\nu(\Delta_N)}{1 - q_\nu(\Delta_N)}\|Q_{\nu,\Delta_N}(\lambda^{(0)}, u^{(0)})\| < \rho_\lambda.$$

Since $Q_{\nu,\Delta_N}(\lambda, u^{(k-1)})$ in $S(\lambda^{(k-1)}, \rho_{k-1})$ satisfies all the conditions of Theorem A [12], there exists $\lambda^{(k)} \in S(\lambda^{(k-1)}, \rho_{k-1})$ – solution of the equation $Q_{\nu,\Delta_N}(\lambda, u^{(k-1)}) = 0$ and the following estimate is valid

$$\|\lambda^{(k)} - \lambda^{(k-1)}\| \leqslant q_\nu(\Delta_N)\|Q_{\nu,\Delta_N}(\lambda^{(k-1)}, u^{(k-1)})\|. \tag{30}$$

Solving the Cauchy problems (4) and (5) for $\lambda_r = \lambda_r^{(k)}$, we find functions $u_r^{(k)}(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, (m+1)N}$. If $\rho_k = \gamma_\nu(\Delta_N)\|Q_{\nu,\Delta_N}(\lambda^{(k)}, u^{(k)})\| = 0$, then $Q_{\nu,\Delta_N}(\lambda^{(k)}, u^{(k)}) = 0$. Hence, taking into account that $u_r^{(k)}(t)$ is a solution to the Cauchy problem (4), (5) at $\lambda_r = \lambda_r^{(k)}$ on $[t_{r-1}, t_r)$, $r = \overline{1, (m+1)N}$, we obtain the equalities

$$\lambda_{iN+1}^{(k)} - \lambda_{iN}^{(k)} - \lim_{t \to t_{iN}-0} u_{iN}^{(k)}(t) - p_i = 0, \quad i = \overline{1, m},$$

$$B\lambda_1^{(k)} + C\lambda_{(m+1)N}^{(k)} + C \lim_{t \to t_{(m+1)N}-0} u_{(m+1)N}^{(k)}(t) - d = 0,$$

$$\lambda_r^{(k)} + \lim_{t \to t_r-0} u_r^{(k)}(t) - \lambda_{r+1}^{(k)} = 0, \quad r = \overline{iN+1, (i+1)N-1}, \ i = \overline{0, m},$$

i.e., the pair $(\lambda^{(k)}, u^{(k)}[t])$ is a solution of problem (4)–(8).

Using (29), (30), and the Gronwall-Bellman inequality, we set estimates

$$\|\lambda^{(k)} - \lambda^{(k-1)}\| \leqslant q_\nu(\Delta_N)\|\lambda^{(k-1)} - \lambda^{(k-2)}\|, \tag{31}$$

$$\|u_r^{(k)}(t) - u_r^{(k-1)}(t)\| \leqslant (e^{L(t-t_{r-1})} - 1)\|\lambda_r^{(k)} - \lambda_r^{(k-1)}\|, \quad t \in [t_{r-1}, t_r),\ r = \overline{1, (m+1)N}. \tag{32}$$

From inequalities (31), (32), and $q_\nu(\Delta_N) < 1$, it follows that the sequence of pairs $(\lambda^{(k)}, u^{(k)}[t])$ for $k \to \infty$ converges to $(\lambda^*, u^*[t])$ is a solution to problem (4)–(8). Moreover, due to inequalities (v) and (vi) of the theorem $(\lambda^{(k)}, u^{(k)}[t])$, $k \in \mathbb{N}$, and $(\lambda^*, u^*[t])$ belong to $S(\lambda^{(0)}, \rho_\lambda) \times S(u^{(0)}[t], \rho_u)$. In inequalities

$$\|\lambda^{(k+\ell)} - \lambda^{(k)}\| < \frac{(q_\nu(\Delta_N))^k}{1 - q_\nu(\Delta_N)}\gamma_\nu(\Delta_N)\|Q_{\nu,\Delta_N}(\lambda^{(0)}, u^{(0)})\|,$$

$$\|u_r^{(k+\ell)}(t) - u_r^{(k)}(t)\| \leqslant (e^{L(t-t_{r-1})} - 1)\|\lambda_r^{(k+\ell)} - \lambda_r^{(k)}\|, \quad t \in [t_{r-1}, t_r),\ r = \overline{1, (m+1)N},$$

passing to the limit as $\ell \to \infty$, we obtain estimates (16) and (17).

Note that

$$\|\lambda^{(k+\ell)} - \lambda^{(k)}\| \leqslant \frac{1 - (q_\nu(\Delta_N))^\ell}{1 - q_\nu(\Delta_N)}q_\nu(\Delta_N)\|\lambda^{(k)} - \lambda^{(k-1)}\|,$$

$$\|u_r^{(k+\ell)}(t) - u_r^{(k)}(t)\| \leqslant (e^{L(t-t_{r-1})} - 1)\frac{1 - (q_\nu(\Delta_N))^\ell}{1 - q_\nu(\Delta_N)}q_\nu(\Delta_N)\|\lambda^{(k)} - \lambda^{(k-1)}\|,$$

$t \in [t_{r-1}, t_r)$, $r = \overline{1, (m+1)N}$. Let us move on to the limit as $\ell \to \infty$ and obtain the estimates

$$\|\lambda^* - \lambda^{(k)}\| \leqslant \frac{q_\nu(\Delta_N)}{1 - q_\nu(\Delta_N)}\|\lambda^{(k)} - \lambda^{(k-1)}\|, \tag{33}$$

$$\|u_r^*(t) - u_r^{(k)}(t)\| \leqslant (e^{L(t-t_{r-1})} - 1)\frac{q_\nu(\Delta_N)}{1 - q_\nu(\Delta_N)}\|\lambda^{(k)} - \lambda^{(k-1)}\|, \quad t \in [t_{r-1}, t_r),\ r = \overline{1, (m+1)N}.$$

Let us show the isolation of the solution. Let the pair $(\widetilde{\lambda}, \widetilde{u}[t])$ be a solution of problem (4)–(8) belonging to $S(\lambda^{(0)}, \rho_\lambda) \times S(u^{(0)}[t], \rho_u)$. Then, there are numbers $\widetilde{\delta}_1 > 0$, $\widetilde{\delta}_2 > 0$ such that

$$\|\widetilde{\lambda} - \lambda^{(0)}\| + \widetilde{\delta}_1 < \rho_\lambda, \qquad \max_{r=\overline{1,(m+1)N}}(e^{L(t_r-t_{r-1})} - 1)\|\widetilde{\lambda} - \lambda^{(0)}\| + \widetilde{\delta}_2 < \rho_u.$$

Considering that the functions $\widetilde{u}_r(t)$, $u_r^{(0)}(t)$ are solutions of the Cauchy problem (4), (5) for $\lambda_r = \widetilde{\lambda}_r$, $\lambda_r = \lambda_r^{(0)}$, respectively, and again using the Gronwall-Bellman inequality we have

$$\|\widetilde{u}_r(t) - u_r^{(0)}(t)\| \leqslant (e^{L(t-t_{r-1})} - 1)\|\widetilde{\lambda}_r - \lambda_r^{(0)}\|, \quad t \in [t_{r-1}, t_r),\ r = \overline{1, (m+1)N}.$$

If $\lambda \in S(\widetilde{\lambda}, \widetilde{\delta}_1)$, $u[t] \in S(\widetilde{u}[t], \widetilde{\delta}_2)$, then due to the inequalities

$$\|\lambda - \lambda^{(0)}\| \leqslant \|\lambda - \widetilde{\lambda}\| + \|\widetilde{\lambda} - \lambda^{(0)}\| < \widetilde{\delta}_1 + \|\widetilde{\lambda} - \lambda^{(0)}\| < \rho_\lambda,$$

$$\|u_r(t) - u_r^{(0)}(t)\| \leqslant \|u_r(t) - \widetilde{u}_r(t)\| + \|\widetilde{u}_r(t) - u_r^{(0)}(t)\| < \widetilde{\delta}_2 + \|\widetilde{u}_r(t) - u_r^{(0)}(t)\| < \rho_u,$$

$t \in [t_{r-1}, t_r)$, $r = \overline{1, (m+1)N}$, $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$, $u[t] \in S(u^{(0)}[t], \rho_u)$, that is $S(\widetilde{\lambda}, \widetilde{\delta}_1) \subset S(\lambda^{(0)}, \rho_\lambda)$, $S(\widetilde{u}[t], \widetilde{\delta}_2) \subset S(u^{(0)}[t], \rho_u)$.

Let us take a number $\varepsilon > 0$ such that

$$\varepsilon\gamma_\nu(\Delta_N) < 1, \quad q_\nu(\Delta_N) < 1 - \varepsilon\gamma_\nu(\Delta_N). \tag{34}$$

From a uniform continuity $f'_x(t, x)$ in $G_f(\rho_x)$ and structure of the Jacobi matrix $\frac{\partial Q_{\nu,\Delta_N}(\lambda, u)}{\partial\lambda}$ follows from its uniform continuity in $S(\widetilde{\lambda}, \widetilde{\delta}_1) \times S(\widetilde{u}[t], \widetilde{\delta}_2)$. Therefore, there is $\delta \in (0, \min\{\widetilde{\delta}_1, \widetilde{\delta}_2\}]$, for which

$$\left\|\frac{\partial Q_{\nu,\Delta_N}(\lambda, u)}{\partial\lambda} - \frac{\partial Q_{\nu,\Delta_N}(\widetilde{\lambda}, \widetilde{u})}{\partial\lambda}\right\| < \varepsilon$$

for all $(\lambda, u) \in S(\widetilde{\lambda}, \delta) \times S(\widetilde{u}[t], \delta)$. Note that if $(\widetilde{\lambda}, \widetilde{u}[t])$ is the solution of problem (4)–(8), $Q_{\nu,\Delta_N}(\widetilde{\lambda}, \widetilde{u}) = 0$ for any $\nu \in \mathbb{N}$.

Let $(\widehat{\lambda}, \widehat{u}[t]) \in S(\widetilde{\lambda}, \delta) \times S(\widetilde{u}[t], \delta)$ be another solution to problem (4)–(8). Since $Q_{\nu,\Delta_N}(\widetilde{\lambda}, \widetilde{u}) = 0$ and $Q_{\nu,\Delta_N}(\widehat{\lambda}, \widehat{u}) = 0$, from the equalities

$$\widetilde{\lambda} = \widetilde{\lambda} - \left(\frac{\partial Q_{\nu,\Delta_N}(\widetilde{\lambda}, \widetilde{u})}{\partial \lambda}\right)^{-1} Q_{\nu,\Delta_N}(\widetilde{\lambda}, \widetilde{u}), \quad \widehat{\lambda} = \widehat{\lambda} - \left(\frac{\partial Q_{\nu,\Delta_N}(\widetilde{\lambda}, \widetilde{u})}{\partial \lambda}\right)^{-1} Q_{\nu,\Delta_N}(\widehat{\lambda}, \widehat{u}),$$

it follows that

$$\widetilde{\lambda} - \widehat{\lambda} = -\left(\frac{\partial Q_{\nu,\Delta_N}(\widetilde{\lambda}, \widetilde{u})}{\partial \lambda}\right)^{-1} \int_0^1 \left(\frac{\partial Q_{\nu,\Delta_N}(\widehat{\lambda} + \theta(\widetilde{\lambda} - \widehat{\lambda}), \widetilde{u})}{\partial \lambda} - \frac{\partial Q_{\nu,\Delta_N}(\widetilde{\lambda}, \widetilde{u})}{\partial \lambda}\right) d\theta (\widetilde{\lambda} - \widehat{\lambda})$$
$$- \left(\frac{\partial Q_{\nu,\Delta_N}(\widetilde{\lambda}, \widetilde{u})}{\partial \lambda}\right)^{-1} (Q_{\nu,\Delta_N}(\widehat{\lambda}, \widetilde{u}) - Q_{\nu,\Delta_N}(\widehat{\lambda}, \widehat{u})),$$

then

$$\begin{aligned} \|\widetilde{\lambda} - \widehat{\lambda}\| &\leqslant \frac{\gamma_\nu(\Delta_N)}{1 - \varepsilon\gamma_\nu(\Delta_N)} \|Q_{\nu,\Delta_N}(\widehat{\lambda}, \widetilde{u}) - Q_{\nu,\Delta_N}(\widehat{\lambda}, \widehat{u})\| \\ &\leqslant \frac{\gamma_\nu(\Delta_N)}{1 - \varepsilon\gamma_\nu(\Delta_N)} \max\left\{1, \max_{i=\overline{1,m}} \frac{\theta_{i+1} - \theta_i}{N}, \frac{\theta_{m+1} - \theta_m}{N} \|C\|\right\} \\ &\times \max_{r=\overline{1,(m+1)N}} \left\{\int_{t_{r-1}}^{t_r} L \ldots \int_{t_{r-1}}^{\tau_{\nu-1}} L\|\widetilde{u}_r(\tau_\nu) - \widehat{u}_r(\tau_\nu)\| d\tau_\nu \ldots d\tau_1\right\}. \end{aligned} \tag{35}$$

Since

$$\|\widetilde{u}_r(t) - \widehat{u}_r(t)\| \leqslant \int_{t_{r-1}}^{t} L(\|\widetilde{\lambda}_r - \widehat{\lambda}_r\| + \|\widetilde{u}_r(\tau) - \widehat{u}_r(\tau)\|) d\tau,$$

by the Gronwall-Bellman lemma

$$\|\widetilde{u}_r(t) - \widehat{u}_r(t)\| \leqslant (e^{L(t-t_{r-1})} - 1)\|\widetilde{\lambda}_r - \widehat{\lambda}_r\|. \tag{36}$$

Substituting (36) to the right part of (35), we have

$$\|\widetilde{\lambda} - \widehat{\lambda}\| \leqslant \frac{q_\nu(\Delta_N)}{1 - \varepsilon\gamma_\nu(\Delta_N)} \|\widetilde{\lambda} - \widehat{\lambda}\|. \tag{37}$$

Thus, due to inequalities (34), (36), and (37), we have the equalities $\widetilde{\lambda}_r = \widehat{\lambda}_r$, $\widetilde{u}_r(t) = \widehat{u}_r(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, (m+1)N}$. Theorem 1 is proved. □

Comments on the theorem:

(1) Theorem 1 provides sufficient conditions for the feasibility and convergence of the proposed algorithm to solve problem (4)–(8). Furthermore, Theorem 1 establishes sufficient conditions for the existence of an isolated solution to problem (4)–(8).
(2) The fulfillment of conditions (i), (ii), and (v) ensures the applicability of Theorem A [12] to find the solution of equation (14) with a given $u[t]$.
(3) The fulfillment of condition (iv) guarantees the convergence of the proposed algorithm.
(4) The fulfillment of conditions (v)–(vii) is required to ensure isolation of the solution.

The choice of numbers N, ν depends on the properties of the initial data of the problem (1)–(3). If the data of the problems allow, it is possible to do so without dividing the interval and using substitutions.

Since problem (4)–(8) and problem (1)–(3) are equivalent, the following assertion holds.

Corollary. *Let the conditions (i)–(vii) of Theorem 1 hold for certain values of Δ_N ($N \in \mathbb{N}$), ν ($\nu \in \mathbb{N}$), $\rho_\lambda > 0$, $\rho_u > 0$, and $\rho_x > 0$. Then, the sequence $\{x^{(k)}(t)\}_{k=0}^{\infty}$ belongs to the ball $S(x^{(0)}(t), \rho_x)$ and converges to an isolated solution $x^*(t)$ of problem (1)–(3) in $S(x^{(0)}(t), \rho_x)$.*

4 Illustrative examples

To demonstrate the accuracy and efficiency of the proposed algorithm, this section examines two numerical examples of solving the boundary-value problem of type (1)–(3). The method described in Section 2 is applied to both cases, and all computations are carried out using the MathCAD system.

Example 1. Find the numerical solution of the boundary value problem (38)–(40) with an accuracy of $\varepsilon = 10^{-4}$

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = f\left(t, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right), \quad t \in (0.95, 1.10)\backslash\{1\}, \tag{38}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(1) - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(1-0) = p_1, \tag{39}$$

$$B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0.95) + C\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(1.10) = d, \tag{40}$$

where $f\left(t, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_2 \\ -x_1 - x_2^2 + \ln(t) \end{pmatrix} + \eta(1-t)\begin{pmatrix} -1 \\ 2/t + 1 \end{pmatrix} + \eta(t-1)\begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$, $\eta(t) = \begin{cases} 0 & \text{for} \quad t < 0, \\ 1 & \text{for} \quad t > 0, \end{cases}$ $p_1 = \begin{pmatrix} 0.5 \\ -1 \end{pmatrix}$, $B = \begin{pmatrix} -0.5 & 0.75 \\ -1 & 0.2 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0.375 \\ -0.2 & -0.125 \end{pmatrix}$, $d = \begin{pmatrix} 1.5 \\ 0.2 \end{pmatrix}$.

◀ We will formulate equation

$$Q_{1,\Delta_2}(\lambda, u) = 0, \quad \lambda \in \mathbb{R}^4,$$

where $Q_{1,\Delta_2}(\lambda, u) = ((Q_{1,\Delta_2}(\lambda, u))_1, (Q_{1,\Delta_2}(\lambda, u))_2, (Q_{1,\Delta_2}(\lambda, u))_3, (Q_{1,\Delta_2}(\lambda, u))_4)^{\mathrm{T}}$ is an operator with components

$$(Q_{1,\Delta_2}(\lambda, u))_1 = \begin{pmatrix} \lambda_{11} \\ \lambda_{12} \end{pmatrix} + \int_{0.95}^{0.975} \begin{pmatrix} \lambda_{12} + u_{12}(t) - 1 \\ -\lambda_{11} - u_{11}(t) - (\lambda_{12} + u_{12}(t))^2 + \ln(t) + 1 + \frac{2}{t} \end{pmatrix} dt - \begin{pmatrix} \lambda_{21} \\ \lambda_{22} \end{pmatrix},$$

$$\begin{aligned} (Q_{1,\Delta_2}(\lambda, u))_2 = {}& \frac{1}{40} \cdot \left(\begin{pmatrix} \lambda_{31} \\ \lambda_{32} \end{pmatrix} - \begin{pmatrix} \lambda_{21} \\ \lambda_{22} \end{pmatrix} - \begin{pmatrix} 0.5 \\ -1 \end{pmatrix}\right) \\ & - \frac{1}{40} \cdot \int_{0.975}^{1.0} \begin{pmatrix} \lambda_{22} + u_{22}(t) - 1 \\ -\lambda_{21} - u_{21}(t) - (\lambda_{22} + u_{22}(t))^2 + \ln(t) + 1 + \frac{2}{t} \end{pmatrix} dt, \end{aligned}$$

$$(Q_{1,\Delta_2}(\lambda, u))_3 = \begin{pmatrix} \lambda_{31} \\ \lambda_{32} \end{pmatrix} + \int_{1.0}^{1.05} \begin{pmatrix} \lambda_{32} + u_{32}(t) \\ -\lambda_{31} - u_{31}(t) - (\lambda_{32} + u_{32}(t))^2 + \ln(t) + \frac{1}{2} \end{pmatrix} dt - \begin{pmatrix} \lambda_{41} \\ \lambda_{42} \end{pmatrix},$$

$$\begin{aligned} (Q_{1,\Delta_2}(\lambda, u))_4 = {}& \frac{1}{20} \cdot \left(B\begin{pmatrix} \lambda_{11} \\ \lambda_{12} \end{pmatrix} + C \cdot \begin{pmatrix} \lambda_{41} \\ \lambda_{42} \end{pmatrix} - \begin{pmatrix} 1.5 \\ 0.2 \end{pmatrix}\right) \\ & + \frac{1}{20} \cdot C \cdot \int_{1.05}^{1.1} \begin{pmatrix} \lambda_{42} + u_{42}(t) \\ -\lambda_{41} - u_{41}(t) - (\lambda_{42} + u_{42}(t))^2 + \ln(t) + \frac{1}{2} \end{pmatrix} dt. \end{aligned}$$

Condition A is satisfied for $\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}, \lambda_4^{(0)}) = \left(\begin{pmatrix}\lambda_{11}^{(0)}\\ \lambda_{12}^{(0)}\end{pmatrix}, \begin{pmatrix}\lambda_{21}^{(0)}\\ \lambda_{22}^{(0)}\end{pmatrix}, \begin{pmatrix}\lambda_{31}^{(0)}\\ \lambda_{32}^{(0)}\end{pmatrix}, \begin{pmatrix}\lambda_{41}^{(0)}\\ \lambda_{42}^{(0)}\end{pmatrix}\right) = \left(\begin{pmatrix}-0.057146\\ 1.647708\end{pmatrix}, \begin{pmatrix}-0.040953\\ 1.657188\end{pmatrix}, \begin{pmatrix}0.475469\\ 0.664826\end{pmatrix}, \begin{pmatrix}0.508709\\ 0.645427\end{pmatrix}\right)$ with a precision of 10^{-5}, since there is an inequality $\|Q_{1,\Delta_2}(\lambda^{(0)}, 0)\| < 0.000009 < 10^{-5}$.

Solutions to the Cauchy problems of kind (15) were found using the Runge-Kutta method of the fourth order of accuracy. We have listed the values of the components of the functions system $u^{(0)}[t] = (u_1^{(0)}(t), u_2^{(0)}(t), u_3^{(0)}(t), u_4^{(0)}(t)) = \left(\begin{pmatrix}u_{11}^{(0)}(t)\\ u_{12}^{(0)}(t)\end{pmatrix}, \begin{pmatrix}u_{21}^{(0)}(t)\\ u_{22}^{(0)}(t)\end{pmatrix}, \begin{pmatrix}u_{31}^{(0)}(t)\\ u_{32}^{(0)}(t)\end{pmatrix}, \begin{pmatrix}u_{41}^{(0)}(t)\\ u_{42}^{(0)}(t)\end{pmatrix}\right)$ in Table 1.

Note that

$$\begin{aligned} q_1(\Delta_2) &= \gamma_1(\Delta_2)\max\{1,0.025,0.05,0.05^*\|C\|\}\cdot\max\{e^{0.025L}-1-0.025L, e^{0.05L}-1-0.05L\}\\ &\approx 68.142^*\{1,0.025,0.05,0.05^*0.375\}\\ &\quad\times\max\{e^{0.025*4.33}-1-0.025^*4.33, e^{0.05*4.33}-1-0.05^*4.33\}\approx 1.72>1. \end{aligned}$$

So, let us take $N = 4$ and introduce the notation $t_r = \begin{cases}0.95 + r/80, & \text{if } r = \overline{0,4},\\ 1.00 + (r-4)/40, & \text{if } r = \overline{5,8}.\end{cases}$ We will construct equation $Q_{1,\Delta_4}(\lambda, u) = 0$, $\lambda \in \mathbb{R}^8$, where $Q_{1,\Delta_4}(\lambda, u) = ((Q_{1,\Delta_4}(\lambda, u))_1, (Q_{1,\Delta_4}(\lambda, u))_2, \ldots,(Q_{1,\Delta_4}(\lambda, u))_8)^{\mathrm{T}}$ is an operator with components

$$(Q_{1,\Delta_4}(\lambda,u))_r = \begin{pmatrix}\lambda_{r1}\\ \lambda_{r2}\end{pmatrix} - \begin{pmatrix}\lambda_{r+1,1}\\ \lambda_{r+1,2}\end{pmatrix} + \int_{t_{r-1}}^{t_r}\begin{pmatrix}\lambda_{r2}+u_{r2}(t)-1\\ -\lambda_{r1}-u_{r1}(t)-(\lambda_{r2}+u_{r2}(t))^2+\ln(t)+1+\dfrac{2}{t}\end{pmatrix}\mathrm{d}t, \quad r = 1, 2, 3,$$

$$\begin{aligned}(Q_{1,\Delta_2}(\lambda,u))_4 &= \frac{1}{80}\cdot\left(\begin{pmatrix}\lambda_{51}\\ \lambda_{52}\end{pmatrix} - \begin{pmatrix}\lambda_{41}\\ \lambda_{42}\end{pmatrix} - \begin{pmatrix}0.5\\ -1\end{pmatrix}\right)\\ &\quad - \frac{1}{80}\cdot\int_{t_3}^{t_4}\begin{pmatrix}\lambda_{42}+u_{42}(t)-1\\ -\lambda_{41}-u_{41}(t)-(\lambda_{42}+u_{42}(t))^2+\ln(t)+1+\dfrac{2}{t}\end{pmatrix}\mathrm{d}t,\end{aligned}$$

$$(Q_{1,\Delta_2}(\lambda,u))_r = \begin{pmatrix}\lambda_{r1}\\ \lambda_{r2}\end{pmatrix} - \begin{pmatrix}\lambda_{r+1,1}\\ \lambda_{r+1,2}\end{pmatrix} + \int_{t_{r-1}}^{t_r}\begin{pmatrix}\lambda_{r2}+u_{r2}(t)\\ -\lambda_{r1}-u_{r1}(t)-(\lambda_{r2}+u_{r2}(t))^2+\ln(t)+\dfrac{1}{2}\end{pmatrix}\mathrm{d}t, \quad r = 5, 6, 7,$$

$$\begin{aligned}(Q_{1,\Delta_2}(\lambda,u))_8 &= \frac{1}{40}\cdot\left(B\cdot\begin{pmatrix}\lambda_{11}\\ \lambda_{12}\end{pmatrix} + C\cdot\begin{pmatrix}\lambda_{81}\\ \lambda_{82}\end{pmatrix} - \begin{pmatrix}1.5\\ 0.2\end{pmatrix}\right)\\ &\quad + \frac{1}{40}\cdot C\cdot\int_{t_7}^{t_8}\begin{pmatrix}\lambda_{82}+u_{82}(t)\\ -\lambda_{81}-u_{81}(t)-(\lambda_{82}+u_{82}(t))^2+\ln(t)+\dfrac{1}{2}\end{pmatrix}\mathrm{d}t.\end{aligned}$$

Condition A is satisfied for

$$\begin{aligned}\lambda^{(0)} = (\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots,\lambda_8^{(0)}) &= \left(\begin{pmatrix}\lambda_{11}^{(0)}\\ \lambda_{12}^{(0)}\end{pmatrix}, \begin{pmatrix}\lambda_{21}^{(0)}\\ \lambda_{22}^{(0)}\end{pmatrix}, \ldots, \begin{pmatrix}\lambda_{81}^{(0)}\\ \lambda_{82}^{(0)}\end{pmatrix}\right) = \left(\begin{pmatrix}-0.056962\\ 1.648151\end{pmatrix}, \begin{pmatrix}-0.048867\\ 1.652971\end{pmatrix},\right.\\ &\left.\begin{pmatrix}-0.040705\\ 1.657316\end{pmatrix}, \begin{pmatrix}-0.032488\\ 1.661212\end{pmatrix}, \begin{pmatrix}0.475773\\ 0.664690\end{pmatrix}, \begin{pmatrix}0.492390\\ 0.654637\end{pmatrix}, \begin{pmatrix}0.508756\\ 0.645109\end{pmatrix}, \begin{pmatrix}0.524884\\ 0.636074\end{pmatrix}\right)\end{aligned}$$

with a precision of 10^{-5}: $\|Q_{1,\Delta_4}(\lambda^{(0)}, 0)\| \leqslant 0.000007 < 10^{-5}$.

Table 1: Values of the components of the functions system $u^{(0)}[t]$ (Ex. 1, $N = 2$)

t	$u_1^{(0)}(t)$	t	$u_2^{(0)}(t)$	t	$u_3^{(0)}(t)$	t	$u_4^{(0)}(t)$
0.950	$\begin{pmatrix}0.000000\\0.000000\end{pmatrix}$	0.975	$\begin{pmatrix}0.000000\\0.000000\end{pmatrix}$	1.000	$\begin{pmatrix}0.000000\\0.000000\end{pmatrix}$	1.050	$\begin{pmatrix}0.000000\\0.000000\end{pmatrix}$
0.955	$\begin{pmatrix}0.003244\\0.001941\end{pmatrix}$	0.980	$\begin{pmatrix}0.003290\\0.001568\end{pmatrix}$	1.010	$\begin{pmatrix}0.006628\\-0.004131\end{pmatrix}$	1.060	$\begin{pmatrix}0.006436\\-0.003726\end{pmatrix}$
0.960	$\begin{pmatrix}0.006497\\0.003806\end{pmatrix}$	0.985	$\begin{pmatrix}0.006588\\0.003067\end{pmatrix}$	1.020	$\begin{pmatrix}0.013215\\-0.008175\end{pmatrix}$	1.070	$\begin{pmatrix}0.012835\\-0.007375\end{pmatrix}$
0.965	$\begin{pmatrix}0.009759\\0.005597\end{pmatrix}$	0.990	$\begin{pmatrix}0.009893\\0.004499\end{pmatrix}$	1.030	$\begin{pmatrix}0.019762\\-0.012134\end{pmatrix}$	1.080	$\begin{pmatrix}0.019198\\-0.010947\end{pmatrix}$
0.970	$\begin{pmatrix}0.013030\\0.007314\end{pmatrix}$	0.995	$\begin{pmatrix}0.013205\\0.005866\end{pmatrix}$	1.040	$\begin{pmatrix}0.026270\\-0.016010\end{pmatrix}$	1.090	$\begin{pmatrix}0.025525\\-0.014446\end{pmatrix}$
0.975	$\begin{pmatrix}0.016310\\0.008960\end{pmatrix}$	1.000	$\begin{pmatrix}0.016524\\0.007169\end{pmatrix}$	1.050	$\begin{pmatrix}0.032739\\-0.019805\end{pmatrix}$	1.100	$\begin{pmatrix}0.031818\\-0.017873\end{pmatrix}$

Solutions to the Cauchy problems of kind (15) were found using the Runge-Kutta method of the fourth order of accuracy. We have listed the values of the components of the function system $u^{(0)}[t] = (u_1^{(0)}(t), u_2^{(0)}(t), \ldots, u_8^{(0)}(t))$ in Tables 2–9.

Note that $\|Q_{1,\Delta_4}(\lambda^{(0)}, u^{(0)})\| \approx 0.000336$.

For $N = 4$, $\nu = 1$, $(\lambda^{(0)}, u^{(0)}[t])$, $\rho_\lambda = 0.2976$, $\rho_u = 0.03403$, $\rho_x = 0.33163$ we have that $\gamma_1(\Delta_4) < 138.82$,

$$\begin{aligned} q_1(\Delta_4) &= \gamma_1(\Delta_2)\max\{1, 0.0125, 0.025, 0.025 * \|C\|\}\max\{e^{0.0125L} - 1 - 0.0125L, e^{0.025L} - 1 - 0.025L\} \\ &< 138.82 * \max\{1, 0.0125, 0.025, 0.025 * 0.375\} \\ &\times \max\{e^{0.0125*4.33} - 1 - 0.0125 * 4.33, e^{0.025*4.33} - 1 - 0.025 * 4.33\} \approx 0.843511 < 1, \end{aligned}$$

and

$$\frac{(q_1(\Delta_4))^{47}}{1 - q_1(\Delta_4)} \cdot \gamma_1(\Delta_4) \cdot \|Q_{1,\Delta_4}(\lambda^{(0)}, u^{(0)})\| < 0.000099998 < \varepsilon. \tag{41}$$

From inequality (41), it follows that to obtain an approximate solution of problem (38)–(40) with the required accuracy of $\varepsilon = 10^{-4}$, no more than **47** steps of the proposed algorithm are needed.

Next, we will find the solution to equation $Q_{1,\Delta_4}(\lambda, u^{(0)}) = 0$. To do this, we use the iterative process:

$$\lambda^{(1,0)} = \lambda^{(0)}, \qquad \lambda^{(1,m+1)} = \lambda^{(1,m)} - \frac{1}{2}\left(\frac{\partial Q_{1,\Delta_4}(\lambda^{(1,m)}, u^{(0)})}{\partial\lambda}\right)^{-1} \cdot Q_{1,\Delta_4}(\lambda^{(1,m)}, u^{(0)}), \; m = 0, 1, 2, \ldots.$$

Table 2: Values of the components $u_1^{(k)}(t)$ of the functions system $u^{(k)}[t]$ for $k = 0, 1, 2, 3$ (Ex. 1, $N = 4$)

t	$u_1^{(0)}(t)$	$u_1^{(1)}(t)$	$u_1^{(2)}(t)$	$u_1^{(3)}(t)$
0.950000	$\begin{pmatrix}0.00000000\\0.00000000\end{pmatrix}$	$\begin{pmatrix}0.00000000\\0.00000000\end{pmatrix}$	$\begin{pmatrix}0.00000000\\0.00000000\end{pmatrix}$	$\begin{pmatrix}0.00000000\\0.00000000\end{pmatrix}$
0.953125	$\begin{pmatrix}0.00202738\\0.00121777\end{pmatrix}$	$\begin{pmatrix}0.00202859\\0.00121332\end{pmatrix}$	$\begin{pmatrix}0.00202861\\0.00121322\end{pmatrix}$	$\begin{pmatrix}0.00202863\\0.00121315\end{pmatrix}$
0.956250	$\begin{pmatrix}0.00405852\\0.00240555\end{pmatrix}$	$\begin{pmatrix}0.00406092\\0.00239669\end{pmatrix}$	$\begin{pmatrix}0.00406097\\0.00239648\end{pmatrix}$	$\begin{pmatrix}0.00406101\\0.00239635\end{pmatrix}$
0.959375	$\begin{pmatrix}0.00609333\\0.00356373\end{pmatrix}$	$\begin{pmatrix}0.00609690\\0.00355049\end{pmatrix}$	$\begin{pmatrix}0.00609699\\0.00355018\end{pmatrix}$	$\begin{pmatrix}0.00609703\\0.00355000\end{pmatrix}$
0.962500	$\begin{pmatrix}0.00813171\\0.00469270\end{pmatrix}$	$\begin{pmatrix}0.00813645\\0.00467513\end{pmatrix}$	$\begin{pmatrix}0.00813656\\0.00467471\end{pmatrix}$	$\begin{pmatrix}0.00813662\\0.00467447\end{pmatrix}$

Table 3: Values of the components $u_2^{(k)}(t)$ of the functions system $u^{(k)}[t]$ for $k = 0, 1, 2, 3$ (Ex. 1, $N = 4$)

t	$u_2^{(0)}(t)$	$u_2^{(1)}(t)$	$u_2^{(2)}(t)$	$u_2^{(3)}(t)$
0.962500	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$
0.965625	$\begin{pmatrix}0.0020423\\0.0010990\end{pmatrix}$	$\begin{pmatrix}0.0020430\\0.0010961\end{pmatrix}$	$\begin{pmatrix}0.0020430\\0.0010960\end{pmatrix}$	$\begin{pmatrix}0.0020430\\0.0010959\end{pmatrix}$
0.968750	$\begin{pmatrix}0.0040879\\0.0021695\end{pmatrix}$	$\begin{pmatrix}0.0040893\\0.0021637\end{pmatrix}$	$\begin{pmatrix}0.0040893\\0.0021635\end{pmatrix}$	$\begin{pmatrix}0.0040894\\0.0021634\end{pmatrix}$
0.971875	$\begin{pmatrix}0.0061369\\0.0032119\end{pmatrix}$	$\begin{pmatrix}0.0061389\\0.0032033\end{pmatrix}$	$\begin{pmatrix}0.0061390\\0.0032030\end{pmatrix}$	$\begin{pmatrix}0.0061390\\0.0032029\end{pmatrix}$
0.975000	$\begin{pmatrix}0.0081890\\0.0042267\end{pmatrix}$	$\begin{pmatrix}0.0081917\\0.0042153\end{pmatrix}$	$\begin{pmatrix}0.0081918\\0.0042149\end{pmatrix}$	$\begin{pmatrix}0.0081919\\0.0042147\end{pmatrix}$

We will take $\lambda^{(1,6)}$ as $\lambda^{(1)}$ to an accuracy of 10^{-5}, since $\|Q_{1,\Delta_4}(\lambda^{(1,6)}, u^{(0)})\| \leqslant 0.000005$. Thus,

$$\lambda^{(1)} = (\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_8^{(1)}) = \left(\begin{pmatrix}-0.056809\\1.648539\end{pmatrix}, \begin{pmatrix}-0.048673\\1.653194\end{pmatrix}, \begin{pmatrix}-0.040481\\1.657390\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}-0.032240\\1.661150\end{pmatrix}, \begin{pmatrix}0.476045\\0.664499\end{pmatrix}, \begin{pmatrix}0.492532\\0.654409\end{pmatrix}, \begin{pmatrix}0.508772\\0.644840\end{pmatrix}, \begin{pmatrix}0.524779\\0.635762\end{pmatrix}\right).$$

Let us find numerical solutions of Cauchy problems of the form (4) with $\lambda_r = \lambda_r^{(1)}$, $r = \overline{1,8}$. We have listed the values of the components of the functions system $u^{(1)}[t] = (u_1^{(1)}(t), u_2^{(1)}(t), \ldots, u_8^{(1)}(t))$ in Tables 2–9.

Let us introduce the notation $t_{r,j} = \begin{cases} t_r + j/640, & \text{if } r = \overline{0,4}, \\ t_r + j/320, & \text{if } r = \overline{5,8}, \end{cases}$ $j = \overline{0,8}$. The following inequalities hold:

$$\|\delta_1^{(1)}(1,4,8)\| < 0.00004, \quad \|\delta_2^{(1)}(1,4,8)\| < 0.00002, \quad \|\delta_3^{(1)}(1,4,8)\| < 0.000044.$$

However, since there is an inequality $\frac{q_1(\Delta_4)}{1-q_1(\Delta_4)}\|\lambda^{(1)} - \lambda^{(0)}\| \approx 0.004654 > \varepsilon$, we will look for $\lambda^{(2)}$ by solving the equation $Q_{1,\Delta_4}(\lambda, u^{(1)}) = 0$. We will use the iterative the process

$$\lambda^{(2,0)} = \lambda^{(1)}, \quad \lambda^{(2,m+1)} = \lambda^{(2,m)} - \frac{1}{2}\left(\frac{\partial Q_{1,\Delta_4}(\lambda^{(2,m)}, u^{(1)})}{\partial \lambda}\right)^{-1} \cdot Q_{1,\Delta_4}(\lambda^{(2,m)}, u^{(1)}), \quad m = 0, 1, 2, \ldots.$$

Table 4: Values of the components $u_3^{(k)}(t)$ of the functions system $u^{(k)}[t]$ for $k = 0, 1, 2, 3$ (Ex. 1, $N = 4$)

t	$u_3^{(0)}(t)$	$u_3^{(1)}(t)$	$u_3^{(2)}(t)$	$u_3^{(3)}(t)$
0.975000	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$
0.978125	$\begin{pmatrix}0.0020557\\0.0009863\end{pmatrix}$	$\begin{pmatrix}0.0020559\\0.0009849\end{pmatrix}$	$\begin{pmatrix}0.0020559\\0.0009848\end{pmatrix}$	$\begin{pmatrix}0.0020559\\0.0009847\end{pmatrix}$
0.981250	$\begin{pmatrix}0.0041144\\0.0019457\end{pmatrix}$	$\begin{pmatrix}0.0041148\\0.0019428\end{pmatrix}$	$\begin{pmatrix}0.0041148\\0.0019426\end{pmatrix}$	$\begin{pmatrix}0.0041149\\0.0019426\end{pmatrix}$
0.984375	$\begin{pmatrix}0.0061760\\0.0028785\end{pmatrix}$	$\begin{pmatrix}0.0061767\\0.0028742\end{pmatrix}$	$\begin{pmatrix}0.0061767\\0.0028739\end{pmatrix}$	$\begin{pmatrix}0.0061768\\0.0028738\end{pmatrix}$
0.987500	$\begin{pmatrix}0.0082406\\0.0037851\end{pmatrix}$	$\begin{pmatrix}0.0082414\\0.0037793\end{pmatrix}$	$\begin{pmatrix}0.0082415\\0.0037790\end{pmatrix}$	$\begin{pmatrix}0.0082416\\0.0037788\end{pmatrix}$

Table 5: Values of the components $u_4^{(k)}(t)$ of the functions system $u^{(k)}[t]$ for $k = 0, 1, 2, 3$ (Ex. 1, $N = 4$)

t	$u_4^{(0)}(t)$	$u_4^{(1)}(t)$	$u_4^{(2)}(t)$	$u_4^{(3)}(t)$
0.987500	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$
0.990625	$\begin{pmatrix}0.0020677\\0.0008796\end{pmatrix}$	$\begin{pmatrix}0.0020675\\0.0008795\end{pmatrix}$	$\begin{pmatrix}0.0020675\\0.0008794\end{pmatrix}$	$\begin{pmatrix}0.0020675\\0.0008794\end{pmatrix}$
0.993750	$\begin{pmatrix}0.0041380\\0.0017337\end{pmatrix}$	$\begin{pmatrix}0.0041377\\0.0017335\end{pmatrix}$	$\begin{pmatrix}0.0041377\\0.0017333\end{pmatrix}$	$\begin{pmatrix}0.0041377\\0.0017333\end{pmatrix}$
0.996875	$\begin{pmatrix}0.0062111\\0.0025627\end{pmatrix}$	$\begin{pmatrix}0.0062105\\0.0025623\end{pmatrix}$	$\begin{pmatrix}0.0062105\\0.0025621\end{pmatrix}$	$\begin{pmatrix}0.0062105\\0.0025620\end{pmatrix}$
1.000000	$\begin{pmatrix}0.0082866\\0.0033668\end{pmatrix}$	$\begin{pmatrix}0.0082858\\0.0033663\end{pmatrix}$	$\begin{pmatrix}0.0082859\\0.0033660\end{pmatrix}$	$\begin{pmatrix}0.0082859\\0.0033658\end{pmatrix}$

We will take $\lambda^{(2,1)}$ as $\lambda^{(2)}$ to an accuracy of 10^{-5}, since $\|Q_{1,\Delta_4}(\lambda^{(2,1)}, u^{(1)})\| \leqslant 0.000002 < 10^{-5}$. So $\lambda^{(2)}$ is

$$\lambda^{(2)} = \left(\begin{pmatrix}-0.056803\\1.648548\end{pmatrix}, \begin{pmatrix}-0.048667\\1.653201\end{pmatrix}, \begin{pmatrix}-0.040475\\1.657396\end{pmatrix}, \begin{pmatrix}-0.032234\\1.661155\end{pmatrix}, \begin{pmatrix}0.476052\\0.664502\end{pmatrix}, \begin{pmatrix}0.492537\\0.654412\end{pmatrix}, \begin{pmatrix}0.508776\\0.644842\end{pmatrix}, \begin{pmatrix}0.524783\\0.635764\end{pmatrix}\right).$$

Let us find numerical solutions of Cauchy problems of the form (4) with $\lambda_r = \lambda_r^{(2)}$, $r = \overline{1,8}$ (see Tables 2–9).

The following inequalities hold:

$$\|\delta_1^{(2)}(1,4,8)\| < 0.00004, \quad \|\delta_2^{(2)}(1,4,8)\| < 0.00002, \quad \|\delta_3^{(2)}(1,4,8)\| < 0.000038.$$

Since there is an inequality $\frac{q_1(\Delta_4)}{1-q_1(\Delta_4)}\|\lambda^{(2)} - \lambda^{(1)}\| \approx 0.000113 > \varepsilon$, we will look for $\lambda^{(3)}$ by solving the equation $Q_{1,\Delta_4}(\lambda, u^{(2)}) = 0$. We will use the iterative process:

$$\lambda^{(3,0)} = \lambda^{(2)}, \quad \lambda^{(3,m+1)} = \lambda^{(3,m)} - \frac{1}{2}\left(\frac{\partial Q_{1,\Delta_4}(\lambda^{(3,m)}, u^{(2)})}{\partial\lambda}\right)^{-1} \cdot Q_{1,\Delta_4}(\lambda^{(3,m)}, u^{(2)}), \quad m = 0, 1, 2, \ldots.$$

Since $\|Q_{1,\Delta_4}(\lambda^{(3,1)}, u^{(2)})\| \leqslant 0.000001$, we will take $\lambda^{(3,1)}$ as $\lambda^{(3)}$ to an accuracy of 10^{-5}

$$\lambda^{(3)} = \left(\begin{pmatrix}-0.056801\\1.648553\end{pmatrix}, \begin{pmatrix}-0.048664\\1.653206\end{pmatrix}, \begin{pmatrix}-0.040472\\1.657399\end{pmatrix}, \begin{pmatrix}-0.032231\\1.661158\end{pmatrix}, \begin{pmatrix}0.476055\\0.664504\end{pmatrix}, \begin{pmatrix}0.492540\\0.654414\end{pmatrix}, \begin{pmatrix}0.508779\\0.644844\end{pmatrix}, \begin{pmatrix}0.524785\\0.635765\end{pmatrix}\right).$$

Table 6: The values of the components $u_5^{(k)}(t)$ of the functions system $u^{(k)}[t]$ for $k = 0, 1, 2, 3$ (Ex. 1, $N = 4$)

t	$u_5^{(0)}(t)$	$u_5^{(1)}(t)$	$u_5^{(2)}(t)$	$u_5^{(3)}(t)$
1.00000	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$
1.00625	$\begin{pmatrix}0.0041462\\-0.0025926\end{pmatrix}$	$\begin{pmatrix}0.0041450\\-0.0025927\end{pmatrix}$	$\begin{pmatrix}0.0041450\\-0.0025928\end{pmatrix}$	$\begin{pmatrix}0.0041450\\-0.0025928\end{pmatrix}$
1.01250	$\begin{pmatrix}0.0082763\\-0.0051509\end{pmatrix}$	$\begin{pmatrix}0.0082739\\-0.0051512\end{pmatrix}$	$\begin{pmatrix}0.0082739\\-0.0051513\end{pmatrix}$	$\begin{pmatrix}0.0082740\\-0.0051514\end{pmatrix}$
1.01875	$\begin{pmatrix}0.0123905\\-0.0076755\end{pmatrix}$	$\begin{pmatrix}0.0123869\\-0.0076758\end{pmatrix}$	$\begin{pmatrix}0.0123870\\-0.0076760\end{pmatrix}$	$\begin{pmatrix}0.0123870\\-0.0076762\end{pmatrix}$
1.02500	$\begin{pmatrix}0.0164890\\-0.0101668\end{pmatrix}$	$\begin{pmatrix}0.0164843\\-0.0101672\end{pmatrix}$	$\begin{pmatrix}0.0164843\\-0.0101675\end{pmatrix}$	$\begin{pmatrix}0.0164844\\-0.0101677\end{pmatrix}$

Table 7: The values of the components $u_6^{(k)}(t)$ of the functions system $u^{(k)}[t]$ for $k = 0, 1, 2, 3$ (Ex. 1, $N = 4$)

t	$u_6^{(0)}(t)$	$u_6^{(1)}(t)$	$u_6^{(2)}(t)$	$u_6^{(3)}(t)$
1.02500	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$
1.03125	$\begin{pmatrix}0.0040838\\-0.0024602\end{pmatrix}$	$\begin{pmatrix}0.0040824\\-0.0024592\end{pmatrix}$	$\begin{pmatrix}0.0040824\\-0.0024593\end{pmatrix}$	$\begin{pmatrix}0.0040824\\-0.0024593\end{pmatrix}$
1.03750	$\begin{pmatrix}0.0081523\\-0.0048881\end{pmatrix}$	$\begin{pmatrix}0.0081494\\-0.0048862\end{pmatrix}$	$\begin{pmatrix}0.0081495\\-0.0048863\end{pmatrix}$	$\begin{pmatrix}0.0081495\\-0.0048864\end{pmatrix}$
1.04375	$\begin{pmatrix}0.0122057\\-0.0072842\end{pmatrix}$	$\begin{pmatrix}0.0122015\\-0.0072812\end{pmatrix}$	$\begin{pmatrix}0.0122015\\-0.0072814\end{pmatrix}$	$\begin{pmatrix}0.0122015\\-0.0072815\end{pmatrix}$
1.05000	$\begin{pmatrix}0.0162443\\-0.0096488\end{pmatrix}$	$\begin{pmatrix}0.0162386\\-0.0096449\end{pmatrix}$	$\begin{pmatrix}0.0162387\\-0.0096451\end{pmatrix}$	$\begin{pmatrix}0.0162387\\-0.0096453\end{pmatrix}$

Let us find numerical solutions of Cauchy problems of the form (4) with $\lambda_r = \lambda_r^{(3)}$, $r = \overline{1,8}$. We have listed the values of the components of the function system $u^{(3)}[t] = (u_1^{(3)}(t), u_2^{(3)}(t), \ldots, u_8^{(3)}(t))$ Tables 2–9.

Note that there are estimates

$$\frac{q_1(\Delta_4)}{1 - q_1(\Delta_4)} ||\lambda^{(3)} - \lambda^{(2)}|| < 0.00007 < \varepsilon,$$

$$\frac{q_1(\Delta_4)}{1 - q_1(\Delta_4)} \max\{e^{0.0125*4.33} - 1, e^{0.025*4.33} - 1\} ||\lambda^{(3)} - \lambda^{(2)}|| < 0.000008 < \varepsilon.$$

$$||\delta_1^{(3)}(1, 4, 8)|| < 0.000038 < \varepsilon, \quad ||\delta_2^{(3)}(1, 4, 8)|| < 0.00002 < \varepsilon, \quad ||\delta_3^{(3)}(1, 4, 8)|| < 0.0000333 < \varepsilon.$$

As can be seen from these inequalities, only **three** steps of the algorithm were needed to obtain the approximate solution of problem (38)–(40)

$$x^*(t) \approx x^{(3)}(t) = \begin{cases} \lambda_r^{(3)} + u_r^{(3)}(t), & \text{if } t \in [t_{r-1}, t_r), \quad r = \overline{1,8}, \\ \lambda_8^{(3)} + u_8^{(3)}(t_8), & \text{if } t = t_8. \end{cases}$$

The graph of the approximate solution to problem (38)–(40) is shown in Figure 1. ▶

Table 8: The values of the components $u_7^{(k)}(t)$ of the functions system $u^{(k)}[t]$ for $k = 0, 1, 2, 3$ (Ex. 1, $N = 4$)

t	$u_7^{(0)}(t)$	$u_7^{(1)}(t)$	$u_7^{(2)}(t)$	$u_7^{(3)}(t)$
1.05000	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$
1.05625	$\begin{pmatrix}0.0040246\\-0.0023354\end{pmatrix}$	$\begin{pmatrix}0.0040229\\-0.0023334\end{pmatrix}$	$\begin{pmatrix}0.0040230\\-0.0023334\end{pmatrix}$	$\begin{pmatrix}0.0040230\\-0.0023334\end{pmatrix}$
1.06250	$\begin{pmatrix}0.0080347\\-0.0046403\end{pmatrix}$	$\begin{pmatrix}0.0080314\\-0.0046362\end{pmatrix}$	$\begin{pmatrix}0.0080314\\-0.0046363\end{pmatrix}$	$\begin{pmatrix}0.0080314\\-0.0046364\end{pmatrix}$
1.06875	$\begin{pmatrix}0.0120305\\-0.0069151\end{pmatrix}$	$\begin{pmatrix}0.0120256\\-0.0069090\end{pmatrix}$	$\begin{pmatrix}0.0120256\\-0.0069091\end{pmatrix}$	$\begin{pmatrix}0.0120256\\-0.0069092\end{pmatrix}$
1.07500	$\begin{pmatrix}0.0160122\\-0.0091603\end{pmatrix}$	$\begin{pmatrix}0.0160056\\-0.0091521\end{pmatrix}$	$\begin{pmatrix}0.0160057\\-0.0091523\end{pmatrix}$	$\begin{pmatrix}0.0160057\\-0.0091525\end{pmatrix}$

Table 9: The values of the components $u_8^{(k)}(t)$ of the functions system $u^{(k)}[t]$ for $k = 0, 1, 2, 3$ (Ex. 1, $N = 4$)

t	$u_8^{(0)}(t)$	$u_8^{(1)}(t)$	$u_8^{(2)}(t)$	$u_8^{(3)}(t)$
1.07500	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.0000000\\0.0000000\end{pmatrix}$
1.08125	$\begin{pmatrix}0.0039685\\-0.0022177\end{pmatrix}$	$\begin{pmatrix}0.0039666\\-0.0022145\end{pmatrix}$	$\begin{pmatrix}0.0039666\\-0.0022146\end{pmatrix}$	$\begin{pmatrix}0.0039666\\-0.0022146\end{pmatrix}$
1.08750	$\begin{pmatrix}0.0079233\\-0.0044065\end{pmatrix}$	$\begin{pmatrix}0.0079194\\-0.0044003\end{pmatrix}$	$\begin{pmatrix}0.0079194\\-0.0044003\end{pmatrix}$	$\begin{pmatrix}0.0079194\\-0.0044004\end{pmatrix}$
1.09375	$\begin{pmatrix}0.0118644\\-0.0065669\end{pmatrix}$	$\begin{pmatrix}0.0118587\\-0.0065576\end{pmatrix}$	$\begin{pmatrix}0.0118587\\-0.0065577\end{pmatrix}$	$\begin{pmatrix}0.0118587\\-0.0065578\end{pmatrix}$
1.10000	$\begin{pmatrix}0.0157922\\-0.0086993\end{pmatrix}$	$\begin{pmatrix}0.0157845\\-0.0086870\end{pmatrix}$	$\begin{pmatrix}0.0157846\\-0.0086871\end{pmatrix}$	$\begin{pmatrix}0.0157846\\-0.0086872\end{pmatrix}$

Remarks.

(1) The parameters $\lambda^{(0)} = \lambda^{(0,17)} \in \mathbb{R}^{2N}$ were determined using an iterative process

$$\lambda^{(0,0)} = \lambda^0, \quad \lambda^{(0,m+1)} = \lambda^{(0,m)} - \frac{1}{2}\left(\frac{\partial Q_{1,\Delta_N}(\lambda^{(0,m)}, 0)}{\partial \lambda}\right)^{-1} \cdot Q_{1,\Delta_N}(\lambda^{(0,m)}, 0), \quad m = 0, 1, 2, \ldots,$$

where $\lambda^0 = \Bigg(\underbrace{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \ldots, \begin{pmatrix}0\\0\end{pmatrix}}_{N}\Bigg) \in \mathbb{R}^{2N}$ $(N = 2, 4)$.

(2) The fourth order Runge-Kutta method is used to solve the Cauchy problems for ODEs, and the Simpson rules (ship stability) are used to calculate definite integrals.

(3) To compute the derivatives, we used the following formulas:

(a) at the point $t_{r,0}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\lambda_{r1}^{(k)} + u_{r1}^{(k)}(t)\\ \lambda_{r2}^{(k)} + u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,0}} = \frac{1}{2(t_{r,1} - t_{r,0})}\left(-3\begin{pmatrix}u_{r1}^{(k)}(t)\\ u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,0}} + 4\begin{pmatrix}u_{r1}^{(k)}(t)\\ u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,1}} - \begin{pmatrix}u_{r1}^{(k)}(t)\\ u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,2}}\right),$$

(b) at the point $t_{r,j}$ for $j = \overline{1, K-1}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\lambda_{r1}^{(k)} + u_{r1}^{(k)}(t)\\ \lambda_{r2}^{(k)} + u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,j}} = \frac{1}{2(t_{r,1} - t_{r,0})}\left(\begin{pmatrix}u_{r1}^{(k)}(t)\\ u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,j+1}} - \begin{pmatrix}u_{r1}^{(k)}(t)\\ u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,j-1}}\right),$$

(c) at the point $t_{r,K}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\lambda_{r1}^{(k)} + u_{r1}^{(k)}(t)\\ \lambda_{r2}^{(k)} + u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,K}} = \frac{1}{2(t_{r,1} - t_{r,0})} \times \left(\begin{pmatrix}u_{r1}^{(k)}(t)\\ u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,K-2}} - 4\begin{pmatrix}u_{r1}^{(k)}(t)\\ u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,K-1}} + 3\begin{pmatrix}u_{r1}^{(k)}(t)\\ u_{r2}^{(k)}(t)\end{pmatrix}\Bigg|_{t=t_{r,K}}\right).$$

(4) In Example 1, the following designations were used:

$$\delta_2^{(k)}(m, N, K) = \begin{pmatrix}\lambda_{N+1,1}^{(k)}\\ \lambda_{N+1,2}^{(k)}\end{pmatrix} - \begin{pmatrix}\lambda_{N1}^{(k)} + u_{N1}^{(k)}(t_{N,K})\\ \lambda_{N2}^{(k)} + u_{N2}^{(k)}(t_{N,K})\end{pmatrix} - p_m,$$

$$\delta_3^{(k)}(m, N, K) = B \cdot \begin{pmatrix}\lambda_{11}^{(k)}\\ \lambda_{12}^{(k)}\end{pmatrix} + C \cdot \begin{pmatrix}\lambda_{(m+1)N,1}^{(k)} + u_{(m+1)N,1}^{(k)}(t_{(m+1)N,K})\\ \lambda_{(m+1)N,2}^{(k)} + u_{(m+1)N,2}^{(k)}(t_{(m+1)N,K})\end{pmatrix} - d,$$

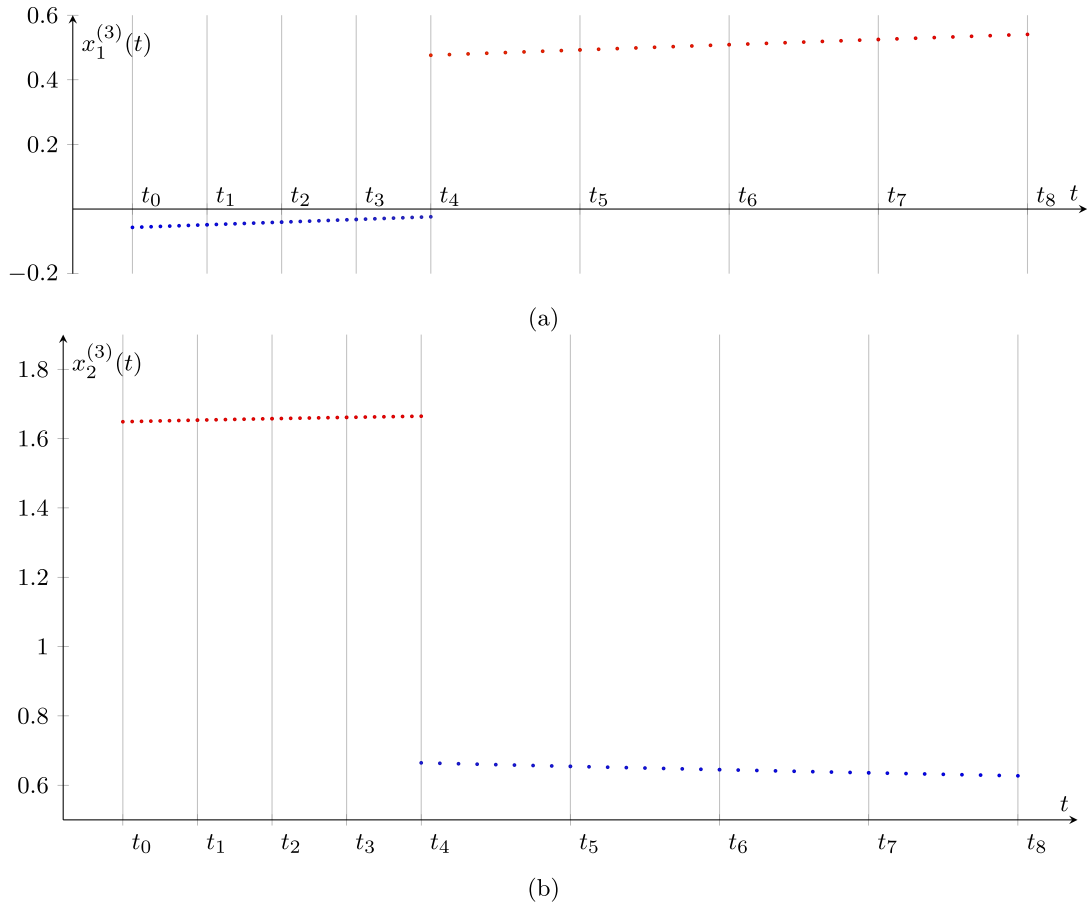

Figure 1: Approximate solution $x^{(3)}(t)$ of the problem (38)–(40): (a) first component $x_1^{(3)}(t)$; (b) second component $x_2^{(3)}(t)$.

$$\delta_1^{(k)}(m,N,K)=\left|\begin{array}{l} Z\leftarrow \dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\lambda_{11}^{(k)}+u_{11}^{(k)}(t_{1,0})\\ \lambda_{12}^{(k)}+u_{12}^{(k)}(t_{1,0})\end{pmatrix}-f\left(t,\begin{pmatrix}\lambda_{11}^{(k)}+u_{11}^{(k)}(t_{1,0})\\ \lambda_{12}^{(k)}+u_{12}^{(k)}(t_{1,0})\end{pmatrix}\right)\\ \text{for } j\in 1\ldots K\\ Z\leftarrow \text{stack}\left(Z,\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\lambda_{11}^{(k)}+u_{11}^{(k)}(t_{1,j})\\ \lambda_{12}^{(k)}+u_{12}^{(k)}(t_{1,j})\end{pmatrix}-f\left(t,\begin{pmatrix}\lambda_{11}^{(k)}+u_{11}^{(k)}(t_{1,j})\\ \lambda_{12}^{(k)}+u_{12}^{(k)}(t_{1,j})\end{pmatrix}\right)\right)\\ \text{for } r\in 2\ldots(m+1)N\\ \quad \text{for } j\in 0\ldots K\\ \quad Z\leftarrow \text{stack}\left(Z,\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\lambda_{r1}^{(k)}+u_{r1}^{(k)}(t_{r,j})\\ \lambda_{r2}^{(k)}+u_{r2}^{(k)}(t_{r,j})\end{pmatrix}-f\left(t,\begin{pmatrix}\lambda_{r1}^{(k)}+u_{r1}^{(k)}(t_{r,j})\\ \lambda_{r2}^{(k)}+u_{r2}^{(k)}(t_{r,j})\end{pmatrix}\right)\right)\\ Z. \end{array}\right.$$

Here, k is the algorithm step number, m is the count of impulse action points, N is the count of subdivisions of the intervals between the impulse action points, K is the count of nodes in solving the Cauchy problems, and $r=\overline{1,N}$.

Example 2. Consider a two-point boundary value problems for a system of two nonlinear differential equations subjected to impulsive action at one point

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}x_1\\x_2\end{pmatrix} = f\left(t, \begin{pmatrix}x_1\\x_2\end{pmatrix}\right), \quad t \in (0.5, 1.5)\backslash\{1\}, \tag{42}$$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}(1) - \begin{pmatrix}x_1\\x_2\end{pmatrix}(1-0) = p_1, \tag{43}$$

$$B\begin{pmatrix}x_1\\x_2\end{pmatrix}(0.5) + C\begin{pmatrix}x_1\\x_2\end{pmatrix}(1.5) = d, \tag{44}$$

where $f\left(t, \begin{pmatrix}x_1\\x_2\end{pmatrix}\right) = \begin{pmatrix}x_2\\ -x_1 - x_2^2 + \ln(t)\end{pmatrix} + \eta(1-t)\begin{pmatrix}-1\\ 2/t+1\end{pmatrix} + \eta(t-1)\begin{pmatrix}0\\0.5\end{pmatrix}$, $\eta(t) = \begin{cases}0 & \text{for } t < 0,\\ 1 & \text{for } t > 0,\end{cases}$ $p_1 = \begin{pmatrix}0.5\\-1\end{pmatrix}$, $B = \begin{pmatrix}-0.5 & 0.75\\ -1 & 0.2\end{pmatrix}$, $C = \begin{pmatrix}0 & 0.375\\ -0.2 & -0.125\end{pmatrix}$, $d = \begin{pmatrix}0.5\ln 2 + 2.5\\ 1.2\ln 2 - 0.2\ln 3 + 5/12\end{pmatrix}$.

Find the numerical solution of the boundary value problem (42)–(44) with an accuracy of $\varepsilon = 10^{-3}$, and compare the results with the exact solution $x^*(t) = \begin{pmatrix}x_1^*(t)\\x_2^*(t)\end{pmatrix}$, where

$$x_1^*(t) = \begin{cases}\ln t & \text{for } t \in [0.5, 1),\\ 0.5 + \ln t & \text{for } t \in [1.0, 1.5],\end{cases} \quad x_2^*(t) = \begin{cases}1 + 1/t & \text{for } t \in [0.5, 1),\\ 1/t & \text{for } t \in [1.0, 1.5].\end{cases}$$

◀ We will use the notations

$$\delta_N(k) = \frac{(q_1(\sigma_N))^k}{1 - q_1(\sigma_N)} \cdot \gamma_1(\sigma_N) \cdot \|Q_{1,\sigma_N}(\lambda^{(0)}, u^{(0)})\|_\infty,$$

$$t_r = \begin{cases}0.5 + r/(2N), & \text{if } r = \overline{0, N},\\ 1.00 + (r-N)/(2N), & \text{if } r = \overline{N+1, 2N},\end{cases}$$

$$\mu(N) = (x^*(t_0), x^*(t_1), \ldots, x^*(t_{2N-1})),$$

and choose the number N based on the data from Table 10.

From Table 10, it is evident that the number $N \geqslant 52$ can be chosen. The conditions of Theorem 1 are satisfied for $N = 64$, $\nu = 1$, $\rho_\lambda = 1.63234$, $\rho_u = 0.0918$, $\rho_x = 1.72414$, $\gamma_1(\Delta_{64}) \approx 526.715$, $q_1(\Delta_{64}) \approx 0.8022 < 1$.

From these estimates (Table 11), it follows that the approximate solution at the 6th step (Figures 2 and 3) is found with an accuracy not exceeding $\varepsilon = 0.001\,10^{-3}$. The third column of Table 11 is based on the equation (33) inequality.

Table 10: Selection of the number of partitions for the intervals [0.5, 1.0) and (1.0, 1.5] (Ex. 2)

N	$\gamma_1(\Delta_N) \approx$	$q_1(\Delta_N) \approx$	$\|Q_{1,\Delta_N}(\lambda^{(0)}, u^{(0)})\|_\infty \approx$	$k : \sigma_N(k) < \varepsilon = 10^{-3}$
51	419.3803	1.0106 > 1		
52	427.64	0.991 < 1	0.0009362	1,184
64	526.715	0.8022 < 1	0.000613	34
128	1055.11	0.3981 < 1	0.0009432	9

Table 11: Selection of the number of partitions for the intervals [0.5, 1.0) and (1.0, 1.5] (Ex. 2)

k	$\|\mu(64) - \lambda^{(k)}\|_\infty \approx$	$\frac{q_1(\Delta_{64})}{1-q_1(\Delta_{64})}\|\lambda^{(k)} - \lambda^{(k-1)}\|_\infty \approx$
0	0.009825	
1	0.004	0.026
2	0.0022	0.0055
3	0.00141	0.0033
4	$0.0009724 < \varepsilon$	$0.002 > \varepsilon$
5	$0.000744 < \varepsilon$	$0.0012003 > \varepsilon$
6	$0.00063 < \varepsilon$	$0.0007112015 < \varepsilon$

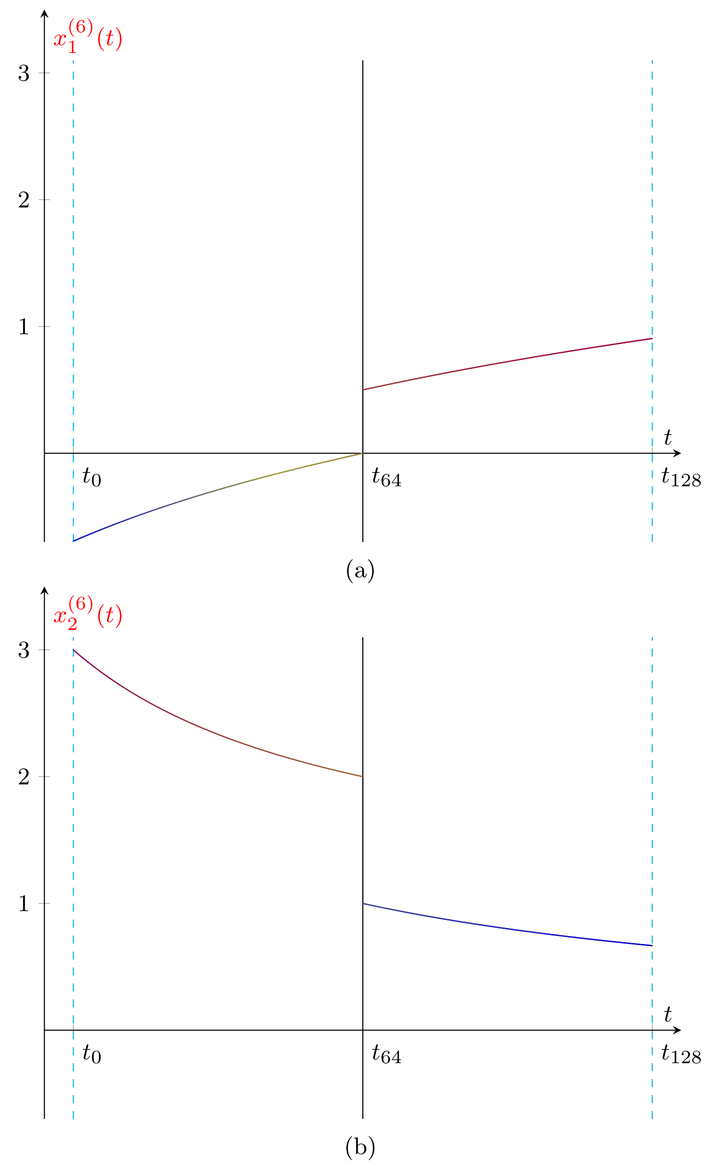

Figure 2: Approximate solution $x^{(6)}(t)$ of the problem (42)–(44): (a) first component $x_1^{(6)}(t)$; (b) second component $x_2^{(6)}(t)$.

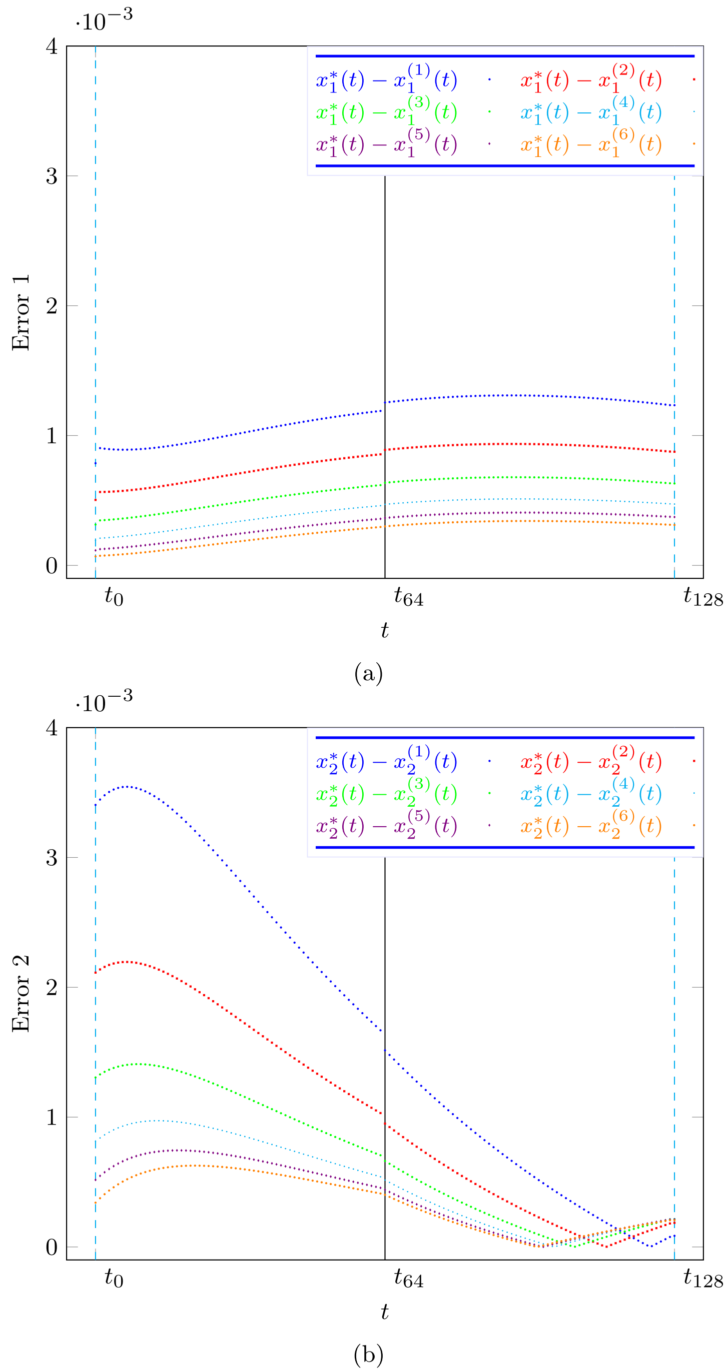

Figure 3: Error rectangles for the numerical solution of the problem (42)–(44): (a) plot of absolute errors in the numerical approximation $x_1(t)$; (b) plot of absolute errors in the numerical approximation of $x_2(t)$.

5 Conclusion

This work was devoted to obtaining sufficient conditions for the existence of an isolated solution within a certain ball for a two-point boundary value problems of a system of nonlinear ODEs subjected to impulsive actions. The ideas of the parameterization method were employed to determine the discontinuous trajectory. By leveraging the concepts of the parameterization method, the authors managed to develop an algorithm for finding a solution to the given problems. The authors plan to consider most of the problems from studies [28–38] for nonlinear systems of differential equations, as well as apply the parametrization method to the problems from study [39].

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