Research Article

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Joint approximation of analytic functions by the shifts of Hurwitz zeta-functions in short intervals

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Abstract: In the article, we obtain that, for algebraically independent over $\mathbb{Q}$ parameters $\alpha_1, \ldots, \alpha_r$, there are infinitely many shifts $(\zeta(s+i\tau, \alpha_1), \ldots, \zeta(s+i\tau, \alpha_r))$ of Hurwitz zeta-functions with $\tau \in [T, T+H]$, $T^{27/82} \leqslant H \leqslant T^{1/2}$, that approximate any r-tuple of analytic functions on the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. More precisely, the latter set of shifts has a positive density. For the proof, a probabilistic approach is applied.

Keywords: Hurwitz zeta-function, joint universality, space of analytic functions, weak convergence of probability measures

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1 Introduction

Denote, as usual, by $\mathbb{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of all prime, positive integer, non-negative integer, integer, rational, real and complex numbers, respectively. Let $s = \sigma + it$, $\sigma, t \in \mathbb{R}$, $i^2 = -1$, be a complex variable, and $0 < \alpha \leqslant 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$ was introduced in [1], and, for $\sigma > 1$, is defined by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}.$$

Moreover, $\zeta(s, \alpha)$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1.

The function $\zeta(s, \alpha)$ is a generalization of the Riemann zeta-function $\zeta(s)$ because, for $\sigma > 1$,

$$\zeta(s, 1) = \sum_{m=0}^{\infty} \frac{1}{(m+1)^s} = \sum_{m=1}^{\infty} \frac{1}{m^s} = \zeta(s)$$

and

$$\zeta\left(s, \frac{1}{2}\right) = 2^s \sum_{m=0}^{\infty} \frac{1}{(2m+1)^s} = 2^s \left(\sum_{m=1}^{\infty} \frac{1}{m^s} - \sum_{m=1}^{\infty} \frac{1}{(2m)^s} \right) = (2^s - 1)\zeta(s).$$

On the other hand, the functions $\zeta(s)$ and $\zeta(s, \alpha)$ have one essential difference: the function $\zeta(s)$, for $\sigma > 1$,

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has the representation by the Euler product:

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

while the function $\zeta(s, \alpha)$, except for the cases $\alpha = 1$ and $\alpha = 1/2$, has no such representation. Moreover, it is well known that $\zeta(s)$ satisfies the symmetric functional equation:

$$\xi(s) = \xi(1 - s), \quad \xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \quad s \in \mathbb{C},$$

where $\Gamma(s)$ is the Euler gamma-function, while, for $\zeta(s, \alpha)$, the following nonsymmetric equations connecting s and $1 - s$ are true:

$$\zeta(1 - s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s}\left(\mathrm{e}^{-\pi i s/2}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{2\pi i m \alpha}}{m^s} + \mathrm{e}^{\pi i s/2}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2\pi i m \alpha}}{m^s}\right), \quad \sigma > 1$$

or

$$\zeta(s, \alpha) = \frac{2\Gamma(1 - s)}{(2\pi)^{1-s}}\left(\sin\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\cos(2\pi m \alpha)}{m^{1-s}} + \cos\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\sin(2\pi m \alpha)}{m^{1-s}}\right), \quad \sigma < 0.$$

This is one of the causes of differences in value distribution of $\zeta(s)$ and $\zeta(s, \alpha)$ and also reflects in the approximate functional equation for $\zeta(s, \alpha)$, which is the main ingredient for the proof of the mean square estimate in short intervals [2]. Regardless of that difference, the Hurwitz zeta-function is an important analytic object of number theory having a wide field of applications. Since Dirichlet L-functions have a representation by Hurwitz zeta-functions with rational parameters, the functions $\zeta(s, \alpha)$ play an important role in the investigation of prime numbers in arithmetic progressions. In addition, Hurwitz zeta-functions have deep applications in algebraic number theory and special function theory.

One of the most interesting analytic properties of the Hurwitz zeta-function is its universality, i.e., the ability to approximate analytic functions defined in the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ by shifts $\zeta(s + i\tau, \alpha)$, $\tau \in \mathbb{R}$. Universality is a common feature of the functions $\zeta(s)$ and $\zeta(s, \alpha)$; however, the shifts $\zeta(s + i\tau)$ approximate only nonvanishing on D functions.

Universality of the function $\zeta(s)$ and Dirichlet L-functions was proved by Voronin [3], see [4–8]. His discovery is a certain infinite-dimensional generalization of the Bohr-Courant theorem [9] on the denseness of the set:

$$\{\zeta(\sigma + it) : t \in \mathbb{R}\},$$

with fixed $1/2 < \sigma < 1$.

Value distribution of the function $\zeta(s, \alpha)$, including universality, depends on the arithmetic of the parameter α. The final universality results are known only for transcendental and rational α. Let $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Moreover, let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then the following statement is known.

Proposition 1. *Suppose that the parameter α is transcendental or rational $\neq 1$ or 1/2. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau \in [0, T] : \sup_{s \in K}|\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, the lower limit can be replaced by the limit for all but at most countably many $\varepsilon > 0$.

The first part of the proposition was obtained by S.M.Gonek in his thesis [10]. In the case of rational $\alpha = a/q$, $(a, q) = 1$, the representation

$$\zeta\left(s, \frac{a}{q}\right) = \frac{q^s}{\varphi(q)}\sum_{\chi (\mathrm{mod}\ q)}\overline{\chi}(a)L(s, \chi),$$

where $L(s,\chi)$ denotes the Dirichet L-function, and $\varphi(q)$ is the Euler totient function, and the joint hybrid universality of Dirichlet L-functions that, if $0 \leqslant \theta_p < 1$, $p \in \mathbb{P}$, $p|q$, then for every $\varepsilon > 0$, there exists $\tau \in \mathbb{R}$ such that

$$\left\| \frac{\tau \log p}{2\pi} - \theta_p \right\| < \varepsilon, \quad p|q$$

and

$$\sup_{s\in K} |L(s + i\tau, \chi) - \mathrm{e}^{f(s)}| < \varepsilon$$

were applied. Here, $||u||$ denotes the distance of u to the nearest integer.

In the case of transcendental α, the set $\{\log(m + \alpha) : m \in \mathbb{N}_0\}$ is linearly independent over $\mathbb{Q}$; therefore, methods of Diophantine analysis can be applied.

Bagchi in the thesis [11] proposed a new method for the proof of universality for zeta-functions based on probabilistic limit theorems for weakly convergent probability measures in the space $H(D)$ of analytic functions on D. This method is also convenient for the proof of Proposition 1.

In the cases $\alpha = 1$ and $\alpha = 1/2$, the assertion of Proposition 1 remains valid with a remark that the function $f(s)$ is nonvanishing on K.

The case of algebraic irrational α is the most complicated. The best result in this case is given in [12] with a certain restriction for the degree of α.

The second assertion of Proposition 1 was given in [13].

Proposition 1, as the other universality theorems for zeta-functions, is not effective. Though Proposition 1 implies that there are infinitely many shifts $\zeta(s + i\tau, \alpha)$ approximating a given function, any concrete shift is not known. A weaker problem is to indicate the interval containing τ with the approximating property. For the Riemann zeta-function, this was considered in [14–16] and in [12] for the function $\zeta(s,\alpha)$ with algebraic irrational α. Another way toward effectively realized universality theorems for zeta-functions is shortening of the length of the interval. The latter way for $\zeta(s)$ was proposed in [17], and improved in [18]. A universality theorem in short intervals for $\zeta(s,\alpha)$ with transcendental α was obtained in [2].

Proposition 2. [2] *Suppose that α is transcendental, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{H} \mathrm{meas}\left\{ \tau \in [T, T + H] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, "liminf" *can be replaced by* "lim" *for all but at most countably many $\varepsilon > 0$.*

In this article, we are interested in the joint universality of Hurwitz zeta-functions, i.e., simultaneous approximation of a tuple of analytic functions by shifts $(\zeta(s + i\tau, \alpha_1), \ldots, \zeta(s + i\tau, \alpha_r))$. The first joint universality theorem was obtained by Voronin [19], who proved joint universality for Dirichlet L-functions with pairwise nonequivalent Dirichlet characters. Clearly, in the joint case, the approximating functions must be independent in a certain sense. In the case of Dirichlet L-functions [19], this independence is realized by the pairwise nonequivalence of Dirichlet characters. For Hurwitz zeta-functions, a sufficient independence is achieved by using algebraically independent over $\mathbb{Q}$ parameters $\alpha_1. \ldots, \alpha_r$, i.e., that there is no polynomial $p(s_1, \ldots, s_r) \not\equiv 0$ with rational coefficients such that $p(\alpha_1, \ldots, \alpha_r) = 0$. The following joint universality theorem is known [20,21].

Proposition 3. *Suppose that the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}$ and $f_j \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas}\left\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

In [22], the algebraic independence over $\mathbb{Q}$ of the parameters $\alpha_1, \dots, \alpha_r$ was replaced by a weaker requirement that the set

$$L(\alpha_1, \dots, \alpha_r) \overset{\text{def}}{=} \{\log(m + \alpha_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r\}$$

is linearly independent over $\mathbb{Q}$.

Our aim is to give a joint generalization of Proposition 2. For the statement of the main results, some notation is needed. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of a topological space $\mathbb{X}$. Define the set

$$\Omega = \prod_{m \in \mathbb{N}_0} \{s \in \mathbb{C} : |s| = 1\}.$$

Elements of Ω are all functions defined on $\mathbb{N}_0$ with values from the unit circle on $\mathbb{C}$. With the product topology and pointwise multiplication, Ω is a compact topological Abelian group. Therefore, by the Tikhonov theorem [23], the Cartesian product

$$\Omega^r = \prod_{j=1}^{r} \Omega_j,$$

where $\Omega_j = \Omega$ for $j = 1, \dots, r$, again is a compact topological group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H exists. Notice that the measure m_H is the product of the Haar measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$, i.e., if

$$A = A_1 \times \dots \times A_r \in \mathcal{B}(\Omega^r)$$

with $A_j \in \mathcal{B}(\Omega_j)$, then

$$m_H(A) = m_{1H}(A_1) \cdot \dots \cdot m_{rH}(A_r).$$

Thus, we have the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. We equip the space $H(D)$ of analytic functions with the topology of uniform convergence on compact sets and set

$$H^r(D) = \underbrace{H(D) \times \dots \times H(D)}_{r}.$$

Now, on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$, define the $H^r(D)$-valued random element

$$\zeta(s, \underline{\alpha}, \omega) = (\zeta(s, \alpha_1, \omega_1), \dots, \zeta(s, \alpha_r, \omega_r)),$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\omega = (\omega_1, \dots, \omega_r) \in \Omega^r$, $\omega_j \in \Omega_j$, $\omega_j = (\omega_j(m) : m \in \mathbb{N}_0)$, $j = 1, \dots, r$, and

$$\zeta(s, \alpha_j, \omega_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

The main result of the article is the following theorem.

Theorem 4. *Suppose that the parameters $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{H} \operatorname{meas}\left\{\tau \in [T, T + H] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Moreover, the limit

$$\lim_{T \to \infty} \frac{1}{H} \operatorname{meas}\left\{\tau \in [T, T + H] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s + i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\}$$
$$= m_H\left\{\omega \in \Omega^r : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |\zeta(s, \alpha_j, \omega_j) - f_j(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

The form of Theorem 4 suggests that, for its proof, probabilistic arguments connected to the space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$ will be applied.

2 Limit theorem

The main ingredient of the proof of Theorem 4 is a joint limit theorem for Hurwitz zeta-functions in the space $H^r(D)$ in short intervals. For $A \in \mathcal{B}(H^r(D))$, define

$$P_{T,H,\underline{a}}(A) = \frac{1}{H}\text{meas}\{\tau \in [T, T+H] : \zeta(s+i\tau, \underline{\alpha}) \in A\},$$

where

$$\zeta(s, \underline{\alpha}) = (\zeta(s, \alpha_1), \dots, \zeta(s, \alpha_r)).$$

Denote by $P_{\zeta,\underline{\alpha}}$ the distribution of the $H^r(D)$-valued random element $\zeta(s, \underline{\alpha}, \omega)$, i.e.,

$$P_{\zeta,\underline{\alpha}}(A) = m_H\{\omega \in \Omega^r : \zeta(s, \underline{\alpha}, \omega) \in A\}, \quad A \in \mathcal{B}(H^r(D)).$$

Theorem 5. *Suppose that the parameters $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Then $P_{T,H,\underline{\alpha}}$ converges weakly to $P_{\zeta,\underline{\alpha}}$ as $T \to \infty$. Moreover, the support of the measure $P_{\zeta,\underline{\alpha}}$ is the whole space $H^r(D)$.*

Since joint limit theorems in short intervals for zeta-functions are not known, we will give a full proof of Theorem 5. For this, we will use separate lemmas.

For $A \in \mathcal{B}(\Omega^r)$, define

$$P^{\Omega^r}_{T,H,\underline{\alpha}}(A) = \frac{1}{H}\text{meas}\{\tau \in [T, T+H] : (((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0)) \in A\}.$$

Lemma 6. *Suppose that the parameters $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Then $P^{\Omega^r}_{T,H,\underline{\alpha}}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. A classical way for investigation of the weak convergence of probability measures in compact groups is the Fourier transform method, see, for example, [24]. The characters of the group Ω^r are of the form

$$\prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0}^{*} \omega_j^{k_{jm}}(m),$$

where the sign * indicates that only a finite number of $k_{jm} \in \mathbb{Z}$ are not zero. Hence, the Fourier transform $F_{T,H,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jm} : k_{jm} \in \mathbb{Z},\ m \in \mathbb{N}_0)$, $j = 1, \dots, r$, of $P^{\Omega^r}_{T,H,\underline{\alpha}}$ is

$$F_{T,H,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = \int_{\Omega^r} \left(\prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0}^{*} \omega_j^{k_{jm}}(m) \right) \mathrm{d}P^{\Omega^r}_{T,H,\underline{\alpha}},$$

and, by the definition of $P^{\Omega^r}_{T,H,\underline{\alpha}}$, we have

$$\begin{aligned} F_{T,H,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) &= \frac{1}{H} \int_T^{T+H} \prod_{j=1}^{r} \prod_{m \in \mathbb{N}_0}^{*} (m+\alpha_j)^{-ik_{jm}\tau}\, \mathrm{d}\tau \\ &= \frac{1}{H} \int_T^{T+H} \exp\left\{ -i\tau \sum_{j=1}^{r} \sum_{m \in \mathbb{N}_0}^{*} k_{jm} \log(m+\alpha_j) \right\} \mathrm{d}\tau. \end{aligned} \tag{1}$$

We need to show that $F_{T,H,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r)$, as $T \to \infty$, converges to the Fourier transform of the Haar measure m_H, i.e.,

$$\lim_{T\to\infty} F_{T,H,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = \begin{cases} 1 & \text{if } (\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0}), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Obviously, equality (1) gives

$$F_{T,H,\underline{\alpha}}(\underline{0}, \dots, \underline{0}) = 1. \tag{3}$$

Now, suppose that $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$, We will prove that, in this case,

$$A(\underline{k}_1, \dots, \underline{k}_r) \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} \sum_{m\in\mathbb{N}_0}^{*} k_{jm} \log(m + \alpha_j) \neq 0. \tag{4}$$

Actually, if, for some nonzero $k_{l_1m_1}, \dots, k_{l_vm_v} \in \mathbb{Z}$,

$$k_{l_1m_1} \log(m_1 + \alpha_{l_1}) + \dots + k_{l_vm_v} \log(m_v + \alpha_{l_v}) = 0,$$

then

$$(m_1 + \alpha_{l_1})^{k_{m_1l_1}} \cdot \dots \cdot (m_v + \alpha_{l_v})^{k_{m_vl_v}} = 1,$$

and this contradicts the algebraic independence of the numbers $\alpha_1, \dots, \alpha_r$. Thus, (4) holds, and, after integration in (1), we find

$$F_{T,H,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = \frac{\exp\{-iTA(\underline{k}_1, \dots, \underline{k}_r)\} - \exp\{-i(T+H)A(\underline{k}_1, \dots, \underline{k}_r)\}}{iHA(\underline{k}_1, \dots, \underline{k}_r)}.$$

Since $H \to \infty$ as $T \to \infty$, this shows that, for $(\underline{k}_1, \dots, \underline{k}_r) \neq (\underline{0}, \dots, \underline{0})$,

$$\lim_{T\to\infty} F_{T,H,\underline{\alpha}}(\underline{k}_1, \dots, \underline{k}_r) = 0,$$

and this together with (3) proves (2). □

We notice that $H \to \infty$ as $T \to \infty$ is the sufficient requirement for H in Lemma 6. Moreover, in place of algebraic independence of $\alpha_1, \dots, \alpha_r$, we may use the linear independence over $\mathbb{Q}$ for the set $L(\alpha_1, \dots, \alpha_r)$.

Next, we will apply Lemma 6 for the proof of a limit lemma in the space $H^r(D)$. Let $\beta > 1/2$ be a fixed number, and, for $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $j = 1, \dots, r$,

$$v_n(m, \alpha_j) = \exp\left\{-\left(\frac{m + \alpha_j}{n}\right)^{\beta}\right\}.$$

Define

$$\zeta_n(s, \underline{\alpha}) = (\zeta_n(s, \alpha_1), \dots, \zeta_n(s, \alpha_r)),$$

where

$$\zeta_n(s, \alpha_j) = \sum_{m=0}^{\infty} \frac{v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

It is obvious, that the latter series are absolutely convergent in any half-plane $\sigma > \sigma_0$ with finite σ_0.

For $A \in \mathcal{B}(H^r(D))$, set

$$P_{T,H,n,\underline{\alpha}}(A) = \frac{1}{H} \text{meas}\{\tau \in [T, T+H] : \zeta_n(s + i\tau, \underline{\alpha}) \in A\}.$$

For the definition of the limit measure of $P_{T,H,n,\underline{\alpha}}$ as $T \to \infty$, we introduce the function $u_{n,\underline{\alpha}} : \Omega^r \to H^r(D)$ given by

$$u_{n,\underline{\alpha}}(\omega) = \zeta_n(s, \underline{\alpha}, \omega),$$

where

$$\zeta_n(s, \underline{\alpha}, \omega) = (\zeta_n(s, \alpha_1, \omega_1), \dots, \zeta_n(s, \alpha_r, \omega_r))$$

and

$$\zeta_n(s, \alpha_j, \omega_j) = \sum_{m=0}^{\infty} \frac{\omega_j(m) v_n(m, \alpha_j)}{(m + \alpha_j)^s}, \quad j = 1, \dots, r.$$

In addition, the latter series converges absolutely for $\sigma > \sigma_0$, thus, uniformly with respect to ω_j. Hence, the function $u_{n,\underline{\alpha}}$ is continuous; therefore, it is $(\mathcal{B}(\Omega^r(D)), \mathcal{B}(H^r(D)))$ measurable. This shows that the Haar measure m_H on $(\Omega^r, \mathcal{B}(\Omega^r))$ induces the unique probability measure $m_H u_{n,\underline{\alpha}}^{-1}$ on $(H^r(D), \mathcal{B}(H^r(D)))$ defined by

$$m_H u_{n,\underline{\alpha}}^{-1}(A) = m_H(u_{n,\underline{\alpha}}^{-1} A), \quad A \in \mathcal{B}(H^r(D)).$$

For brevity, let $Q_{n,\underline{\alpha}} = m_H u_{n,\underline{\alpha}}^{-1}$.

Lemma 7. *Suppose that the parameters $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Then $P_{T,H,n,\underline{\alpha}}$ converges weakly to $Q_{n,\underline{\alpha}}$ as $T \to \infty$.*

Proof. From the definitions of $P^{\Omega^r}_{T,H,\underline{\alpha}}$, $P_{T,H,n,\underline{\alpha}}$ and $u_{n,\underline{\alpha}}$, it follows that, for $A \in \mathcal{B}(H^r(D))$,

$$\begin{aligned} P_{T,H,n,\underline{\alpha}}(A) &= \frac{1}{T}\text{meas}\{\tau \in [T, T+H] : (((m+\alpha_1)^{-i\tau} : m \in \mathbb{N}_0), \dots, ((m+\alpha_r)^{-i\tau} : m \in \mathbb{N}_0)) \in u_{N,\underline{\alpha}}^{-1} A\} \\ &= P^{\Omega^r}_{T,H,\underline{\alpha}}(u_{n,\underline{\alpha}}^{-1} A) = P^{\Omega^r}_{T,H,\underline{\alpha}} u_{n,\underline{\alpha}}^{-1}(A). \end{aligned}$$

Since $A \in \mathcal{B}(H^r(D))$ is arbitrary, we have $P_{T,H,n,\underline{\alpha}} = P^{\Omega^r}_{T,H,\underline{\alpha}} u_{n,\underline{\alpha}}^{-1}$. Now, this equality, continuity of the function $u_{n,\underline{\alpha}}$ and preservation of weak convergence under continuous mappings, see, for example, Section 5 of [25], together with Lemma 6 show that $P_{T,H,n,\underline{\alpha}}$ converges weakly to the measure $Q_{n,\underline{\alpha}}$ as $T \to \infty$. □

In addition, define two measures

$$P_{T,\underline{\alpha}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0, T] : \zeta(s + i\tau, \underline{\alpha}) \in A\}, \quad A \in \mathcal{B}(H^r(D)),$$

and

$$P_{T,n,\underline{\alpha}}(A) = \frac{1}{T}\text{meas}\{\tau \in [0, T] : \zeta_n(s + i\tau, \underline{\alpha}) \in A\}, \quad A \in \mathcal{B}(H^r(D)).$$

Then in [22], using the linear independence of the set $L(\alpha_1, \dots, \alpha_r)$, it was obtained that $P_{T,n,\underline{\alpha}}$ converges weakly to $Q_{n,\underline{\alpha}}$, and $P_{T,\underline{\alpha}}$ converges weakly to $P_{\zeta,\underline{\alpha}}$ as $T \to \infty$. Moreover, $Q_{n,\underline{\alpha}}$, as $n \to \infty$, and $P_{T,\underline{\alpha}}$, as $T \to \infty$, converges weakly to the same limit measure, i.e., to $P_{\zeta,\underline{\alpha}}$. Since the algebraic independence over $\mathbb{Q}$ of numbers $\alpha_1, \dots, \alpha_r$ implies the linear independence of $L(\alpha_1, \dots, \alpha_r)$, we have the following statement.

Lemma 8. *Suppose that the parameters $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then $Q_{n,\underline{\alpha}}$ converges weakly to $P_{\zeta,\underline{\alpha}}$ as $n \to \infty$.*

From Lemma 8, it follows that, for the proof of Theorem 5, it suffices to show that the measures $P_{T,H,\underline{\alpha}}$, as $T \to \infty$, and $Q_{n,\underline{\alpha}}$, as $n \to \infty$, have the same limit measure. For this aim, we need some mean value estimate. Before the statement of a mean value lemma, we recall a metric in $H^r(D)$. For $g_1, g_2 \in H(D)$, set

$$d(g_1, g_2) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup_{s \in K_m} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_m} |g_1(s) - g_2(s)|},$$

where $\{K_m : m \in \mathbb{N}\} \subset D$ is the sequence of compact embedded sets such that the strip D is the union of all K_m, and each compact set $K \subset D$ lies in some K_m. Then d is a metric in $H(D)$, which induces the topology of uniform convergence on compacta. Setting, for $\underline{g}_1 = (g_{11}, \dots, g_{1r})$, $\underline{g}_2 = (g_{21}, \dots g_{2r}) \in H^r(D)$,

$$d_r(\underline{g}_1, \underline{g}_2) = \max_{1 \leqslant j \leqslant r} d(g_{1j}, g_{2j})$$

gives the metric in $H^r(D)$ which induces the product topology.

Lemma 9. *Suppose that the parameters $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and $T^{27/82} \leqslant H \leqslant T^{1/2}$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} d_r(\underline{\zeta}(s + i\tau, \underline{\alpha}), \underline{\zeta}_n(s + i\tau, \underline{\alpha}))\mathrm{d}\tau = 0.$$

Proof. Since the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, they are transcendental. Therefore, in view of Lemma 5 of [2],

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} d(\zeta(s + i\tau, \alpha_j), \zeta_n(s + i\tau, \alpha_j))\mathrm{d}\tau = 0, \quad j = 1, \dots, r. \tag{5}$$

This and the definition of the metric d_r prove the lemma.

For the convenience of the reader, we will present a proof of (5). By the definition of the metric d, the latter equalities follow from

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \sup_{s\in K} |\zeta(s + i\tau, \alpha_j) - \zeta_n(s + i\tau, \alpha_j)|\mathrm{d}\tau, \quad j = 1, \dots r, \tag{6}$$

for every compact set $K \subset D$.

Let

$$l_n(s) = \frac{1}{\beta}\Gamma\left(\frac{s}{\beta}\right)n^s.$$

Then the classical Mellin formula implies, for $\sigma > 1$, the representation ($\alpha_j = \alpha$)

$$\zeta_n(s, \alpha) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \zeta(s + z, \alpha) l_n(z)\mathrm{d}z. \tag{7}$$

Suppose $K \subset D$ is a compact set. Then there exists $\delta > 0$ such that $1/2 + 2\delta \leqslant \sigma \leqslant 1 - \delta$ for all $\sigma + it \in K$. Let $\beta = 1/2 + \delta$ and $\beta_1 = 1/2 + \delta - \sigma$. Then the residue theorem, in view of (7), gives

$$\zeta_n(s, \alpha) - \zeta(s, \alpha) = \frac{1}{2\pi} \int_{\beta_1 - i\infty}^{\beta_1 + i\infty} \zeta(s + z, \alpha) l_n(z)\mathrm{d}z + l_n(1 - s).$$

Therefore,

$$\sup_{s\in K} |\zeta_n(s + i\tau, \alpha) - \zeta(s + i\tau, \alpha)| \ll \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + \delta + i\tau + iv, \alpha\right)\right| \sup_{s\in K} \left|l_n\left(\frac{1}{2} + \delta - s + iv\right)\right| \mathrm{d}v + \sup_{s\in K} |l_n(1 - s - i\tau)|.$$

Hence, we have

$$\begin{aligned} &\frac{1}{H} \int_T^{T+H} \sup_{s\in K} |\zeta_n(s + i\tau, \alpha) - \zeta(s + i\tau, \alpha)|\mathrm{d}\tau \\ &\ll_{\delta,\alpha,K} \int_{-\log^2 T}^{\log^2 T} \left(\frac{1}{H} \int_T^{T+H} \left|\zeta\left(\frac{1}{2} + \delta + i\tau + iv, \alpha\right)\right| \mathrm{d}\tau\right) \sup_{s\in K} \left|l_n\left(\frac{1}{2} + \delta - s + iv\right)\right| \mathrm{d}v \\ &\quad + \frac{1}{H} \int_T^{T+H} \sup_{s\in K} |l_n(1 - s - i\tau)|\mathrm{d}\tau + \frac{n^{-\delta} \exp\{-c_1 \log^2 T\}}{H} \int_T^{T+H} (|\tau| + 1)^{1/2}\mathrm{d}\tau \\ &\stackrel{\text{def}}{=} I_1 + I_2 + I_3, \quad c_1 > 0. \end{aligned} \tag{8}$$

For estimation of I_1, the mean square estimate in short intervals obtained in [2] is applied. Namely, if $\alpha \neq 1/2$ and 1, and $1/2 < \sigma \leqslant 7/12$, then, for $T^{27/82} \leqslant H \leqslant T^{\sigma}$, uniformly in H,

$$\int\limits_{T}^{T+H} |\zeta(\sigma + it, \alpha)|^2 \mathrm{d}\tau \ll_{\sigma,\alpha} H.$$

Application of the latter result gives, for $|v| \leqslant \log^2 T$,

$$\frac{1}{H}\int\limits_{T}^{T+H} \left|\zeta\left(\frac{1}{2} + \delta + i\tau + iv, \alpha\right)\right| \mathrm{d}\tau \ll_{\delta,\alpha} \left(\frac{1}{H}(H + |v|)\right)^{1/2} \ll_{\delta,\alpha} (1 + |v|),$$

and

$$I_1 \ll_{\delta,\alpha} n^{-\delta}. \tag{9}$$

Properties of the function $\Gamma(s)$ lead to

$$I_2 \ll_K \frac{n^{1/2-2\delta}}{H}. \tag{10}$$

Since $I_3 = o(1)$ as $T \to \infty$, this (8) and (9) prove (6) □

For convenience, we additionally state one lemma on the convergence of random elements in distribution ($\xrightarrow{\mathcal{D}}$).

Lemma 10. *Suppose that the metric space $(\mathbb{X}, \rho)$ is separable, the $\mathbb{X}$-valued random elements X_{nk} and Y_n, $k, n \in \mathbb{N}$, are defined on the same probability space $(\widehat{\Omega}, \mathcal{A}, v)$, for every $k \in \mathbb{N}$,*

$$X_{nk} \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X_k$$

and

$$X_k \underset{k\to\infty}{\xrightarrow{\mathcal{D}}} X.$$

If, for every $\delta > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} v\{\rho(X_{nk}, Y_n) \geqslant \delta\} = 0,$$

then $Y_n \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X$.

Proof of the lemma can be found, for example, in [25], Theorem 4.2.

Proof of Theorem 5. On a certain probability space $(\widehat{\Omega}, \mathcal{A}, v)$, define a random variable $\theta_{T,H}$, and suppose that it is uniformly distributed on $[T, T + H]$. Introduce the $H^r(D)$-valued random elements

$$\zeta_{T,H,\underline{\alpha}} = \zeta_{T,H,\underline{\alpha}}(s) = \zeta(s + i\theta_{T,H}, \underline{\alpha})$$

and

$$\zeta_{T,H,n,\underline{\alpha}} = \zeta_{T,H,n,\underline{\alpha}}(s) = \zeta_n(s + i\theta_{T,H}, \underline{\alpha}),$$

and let $\zeta_{n,\underline{\alpha}} = \zeta_{n,\underline{\alpha}}(s)$ be the $H^r(D)$-valued random element having the distribution $Q_{n,\underline{\alpha}}$, where the measure $Q_{n,\underline{\alpha}}$ is the same as in Lemma 8. Lemma 8 implies the relation

$$\zeta_{n,\underline{\alpha}} \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} P_{\zeta,\underline{\alpha}}, \tag{11}$$

while Lemma 7 gives

$$\zeta_{T,H,n,\underline{\alpha}} \underset{T\to\infty}{\xrightarrow{\mathcal{D}}} \zeta_{n,\underline{\alpha}}. \tag{12}$$

Moreover, using Lemma 9, we obtain that, for every $\delta > 0$,

$$\begin{aligned}
&\lim_{n\to\infty}\limsup_{T\to\infty}\nu\{d_r(\zeta_{T,H,\underline{\alpha}}, \zeta_{T,H,n,\underline{\alpha}}) \geqslant \delta\} \\
&= \lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{H}\mathrm{meas}\{\tau \in [T, T+H] : d_r(\zeta(s+i\tau, \underline{\alpha}), \zeta_n(s+i\tau, \underline{\alpha})) \geqslant \delta\} \\
&\leqslant \lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{\delta H}\int_T^{T+H} d_r(\zeta(s+i\tau, \underline{\alpha}), \zeta_n(s+i\tau, \underline{\alpha}))\mathrm{d}\tau = 0.
\end{aligned}$$

This together with relations (11) and (12) shows that all conditions of Lemma 10 satisfied for the random elements $\zeta_{T,H,\underline{\alpha}}$, $\zeta_{T,H,n,\underline{\alpha}}$ and $\zeta_{n,\underline{\alpha}}$. Therefore, we have

$$\zeta_{T,H,\underline{\alpha}} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\zeta,\underline{\alpha}},$$

and this gives the first assertion of the theorem.

In [22], under a hypothesis that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$, it was proved that the support of the measure $P_{\zeta,\underline{\alpha}}$ is the whole space $H^r(D)$. Since the algebraic independence of $\alpha_1, \dots, \alpha_r$ implies the linear independence of $L(\alpha_1, \dots, \alpha_r)$, this proves the second assertion of the theorem. □

3 Proof of universality

We recall the Mergelyan theorem on approximation of analytic functions by polynomials [26,27].

Lemma 11. [26] *Let K be a compact set with a connected complement on the complex plane, and $g(s)$ a continuous function on K analytic in the interior of K. Then, for every $\delta > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K}|g(s) - p(s)| < \delta.$$

Proof of Theorem 4. Consider the set

$$G_\varepsilon = \left\{(g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|g_j(s) - p_j(s)| < \frac{\varepsilon}{2}\right\},$$

where $p_1(s), \dots, p_r(s)$ are polynomials such that

$$\sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|f_j(s) - p_j(s)| < \frac{\varepsilon}{2}. \tag{13}$$

This is possible by Lemma 11. Since, by Theorem 5, the support $S_{\zeta,\underline{\alpha}}$ of $P_{\zeta,\underline{\alpha}}$ is the set $H^r(D)$, $(p_1(s), \dots, p_r(s))$ is an element of $S_{\zeta,\underline{\alpha}}$. Therefore, the set G_ε is an open neighborhood of an element of the support $S_{\zeta,\underline{\alpha}}$. Hence, by the property of supports,

$$P_{\zeta,\underline{\alpha}}(G_\varepsilon) > 0. \tag{14}$$

Let

$$\mathcal{G}_\varepsilon = \left\{(g_1, \dots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r}\sup_{s\in K_j}|g_j(s) - f_j(s)| < \varepsilon\right\}.$$

Then (13), and the definitions of G_ε and $\mathcal{G}_\varepsilon$ show that the set G_ε lies in $\mathcal{G}_\varepsilon$. Thus, taking into account (14), we have

$$P_{\zeta,\underline{\alpha}}(\mathcal{G}_\varepsilon) > 0. \tag{15}$$

The latter inequality, Theorem 5 and the equivalence of weak convergence of probability measures in terms of open sets (see Theorem 2.1 of [25]) lead to

$$\liminf_{T\to\infty} P_{T,H,\underline{\alpha}}(\mathcal{G}_\varepsilon) \geqslant P_{\zeta,\underline{\alpha}}(\mathcal{G}_\varepsilon) > 0,$$

and the definitions of $P_{T,H,\underline{\alpha}}$ and $\mathcal{G}_\varepsilon$ prove the first assertion of the theorem.

To prove the second assertion of the theorem, we will apply the equivalent of weak convergence in terms of continuity sets. We recall that $A \in \mathcal{B}(H^r(D))$ is a continuity set of $P_{\zeta,\underline{\alpha}}$ if $P_{\zeta,\underline{\alpha}}(\partial A) = 0$, where ∂A denotes the boundary of A.

Observing that the boundary of $\partial\mathcal{G}_\varepsilon$ of the set $\mathcal{G}_\varepsilon$ lies in the set

$$\left\{(g_1, \ldots, g_r) \in H^r(D) : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |g_j(s) - f_j(s)| = \varepsilon\right\},$$

we obtain that the boundaries $\partial\mathcal{G}_{\varepsilon_1}$ and $\partial\mathcal{G}_{\varepsilon_2}$ have no common elements for $\varepsilon_1 \neq \varepsilon_2$. This remark implies that $P_{\zeta,\underline{\alpha}}(\partial\mathcal{G}_\varepsilon) > 0$ for at most countably many $\varepsilon > 0$. Therefore, the set $\mathcal{G}_\varepsilon$ is a continuity set of the measure $P_{\zeta,\underline{\alpha}}$ for all but at most countably many $\varepsilon > 0$. Hence, Theorem 5 and the equivalence of weak convergence of probability measures in terms of continuity sets (see Theorem 2.1 of [25]) give, by (15), that

$$\lim_{T\to\infty} P_{T,H,\underline{\alpha}}(\mathcal{G}_\varepsilon) = P_{\zeta,\underline{\alpha}}(\mathcal{G}_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. The definitions of $P_{T,H,\underline{\alpha}}$, $\mathcal{G}_\varepsilon$ and $P_{\zeta,\underline{\alpha}}$ show that the limit

$$\begin{aligned}&\lim_{T\to\infty} \frac{1}{H}\text{meas}\left\{\tau \in [T, T+H] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j}|\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon\right\}\\ &= m_H\left\{\omega \in \Omega^r : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j}|\zeta(s, \alpha_j, \omega_j) - f_j(s)| < \varepsilon\right\}\end{aligned}$$

exists and is positive for all but at most countably many $\varepsilon > 0$. The theorem is proved. □

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References

[1] A. Hurwitz, *Einige Eigenschaften der Dirichletschen Funktionen* $F(s) = \sum\left(\frac{D}{n}\right)\frac{1}{n^s}$, *die bei der Bestimmung der Klassenanzahlen binärer quadratischer Formen auftreten*. J. Appl. Math. Physics **27** (1882), 86–101.

[2] A. Laurinčikas, *Universality of the Hurwitz zeta-function in short intervals*, Bol. Soc. Mat. Mex. **31** (2025), no.1, 17, DOI: https://doi.org/10.1007/s40590-024-00694-5.

[3] S.M. Voronin, *Theorem on the "universality" of the Riemann zeta-function*, Math. USSR Izv. **9** (1975), no. 3, 443–453, DOI: https://doi.org/10.1070/IM1975v009n03ABEH001485.

[4] S.M. Voronin, *Analytic Properties of Arithmetic Objects*, Doctoral Thesis, Steklov Mathematical Institute, Moscow, 1977.

[5] A.A. Karatsuba and S.M. Voronin, *The Riemann Zeta-Function*, Walter de Gruiter, Berlin, New York, 1992.
[6] S.M. Voronin, *Selected works: Mathematics*, A. A. Karatsuba, ed., Publishing House MGTU Im. NE Baumana, Moscow, 2006 (in Russian).
[7] J. Steuding, *Value-Distribution of L-Functions*, Lecture Notes Mathematics, vol. 1877, Springer, Berlin, Heidelberg, New York, 2007, DOI: https://doi.org/10.1007/978-3-540-44822-8.
[8] K. Matsumoto, *A survey on the theory of universality for zeta and L-functions*, in: M. Kaneko, Sh. Kanemitsu, J. Liu (Eds), Number Theory: Plowing and Starring Through High Wave Forms, Proceedings 7th China-Japan Seminar (Fukuoka 2013), Series on Number Theory and Its Applications, World Scientific Publishing Co., New Jersey, London, Singapore, Bejing, Shanghai, Hong Kong, Taipei, Chennai, 2015, pp. 95–144.
[9] H. Bohr and R. Courant, *Neue anwendungen der theorie der diophantischen approximationen auf die Riemannsche zetafunktion*, J. Reine Angew. Math. **144** (1914), 249–274, DOI: https://doi.org/10.1515/crll.1914.144.249.
[10] S. M. Gonek, *Analytic Properties of Zeta and L-Functions*, Ph.D. Thesis, University of Michigan, Ann Arbor, 1975.
[11] B. Bagchi, *The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series*, Ph. D. Thesis, Indian Statistical Institute, Calcutta, 1981.
[12] A. Sourmelidis and J. Steuding, *On the value distribution of Hurwitz zeta-function with algebraic irrational parameter*, Constr. Approx. **55** (2022), no. 3, 829–860, DOI: https://doi.org/10.1007/s00365-021-09561-2.
[13] L. Meška, *Modified Universality Theorems for the Riemann and Hurwitz Zeta-Functions*, Ph.D. Thesis, Vilnius University, Vilnius, 2017.
[14] A. Good, *On the distribution of the values of Riemannas zeta-function*, Acta Arith. **38** (1981), no. 4, 347–388, DOI: https://doi.org/10.4064/aa-38-4-347-388.
[15] R. Garunkštis, *The effective universality theorem for the Riemann zeta-function*, in: Proceedings of the Session on Analytic Number Theory and Diophantine Equations, Bonner Math. Schriften, vol. 360, University Bonn, Bonn, 2003.
[16] R. Garunkštis, A. Laurinčikas, K. Matsumoto, J. Steuding and R. Steuding, *Effective uniform approximation by the Riemann zeta-function*, Publ. Mat. **54** (2010), no. 1, 209–219.
[17] A. Laurinčikas, *Universality of the Riemann zeta-function in short intervals*, J. Number Theory **204** (2019), 279–295, DOI: https://doi.org/10.1016/j.jnt.2019.04.006.
[18] A. Andersson, R. Garunkštis, R. Kačinskaitė, K. Nakai, L. Pańkowski, A. Sourmelidis, et al., *Notes on universality in short intervals and exponential shifts*, Lith. Math. J. **64** (2024), no. 2, 125–137, DOI: https://doi.org/10.1007/s10986-024-09631-5.
[19] S. M. Voronin, *On the functional independence of Dirichlet L-functions*, Acta Arith. **27** (1975), 443–453 (in Russian), DOI: https://doi.org/10.4064/aa-27-1-493-503.
[20] A. Laurinčikas, *The joint universality for periodic Hurwitz zeta-functions*, Analysis (Munich) **26** (2006), no. 3, 419–428, DOI: https://doi.org/10.1524/anly.2006.26.99.419.
[21] T. Nakamura, *The existence and the non-existence of joint t-universality for Lerch zeta-functions*, J. Number Theory **125** (2007), no. 2, 424–441, DOI: https://doi.org/10.1016/j.jnt.2006.12.008.
[22] A. Laurinčikas, *The joint universality of Hurwitz zeta-functions*, Šiauliai Math. Semin. **3** (2008), no. 11, 169–187.
[23] A. Tychonoff, *Über einen Funktionenraum*, Math. Ann. **111** (1935), 762–766, DOI: https://doi.org/10.1007/BF01472255.
[24] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer Verlag, Berlin, Heidelberg, New York, 1977, DOI: https://doi.org/10.1007/978-3-642-66706-0.
[25] P. Billingsley, *Convergence of Probability Measures*, Second Edition, John Wiley & Sons, New York, 1999, DOI: https://doi.org/10.1002/9780470316962.
[26] S. N. Mergelyan, *Uniform approximations to functions of a complex variable*, American Mathematical Society Translations, Vol. 101, American Mathematical Society, Providence, RI, 1954.
[27] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society: Colloquium Publications, vol. 20, American Mathematical Society, Providence, RI, 1960.