Research Article

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Two-sided zero-divisor graphs of orientation-preserving and order-decreasing transformation semigroups

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Abstract: For $n \geq 4$, let $\mathcal{OPD}_n$ be the orientation-preserving and order-decreasing transformation semigroup on the finite chain $X_n = \{1 < \ldots < n\}$. First, we determine the set of two-sided zero-divisors of $\mathcal{OPD}_n$, and its cardinality. Then, we let $\Gamma(\mathcal{OPD}_n)$ be the graph whose vertices are the two-sided zero-divisors of $\mathcal{OPD}_n$ excluding the zero element θ and distinct two vertices α and β joined by an edge in case $\alpha\beta = \theta = \beta\alpha$. In this study, we prove that $\Gamma(\mathcal{OPD}_n)$ is a connected graph, and we find the diameter, girth, domination number, minimum degree, and maximum degree of $\Gamma(\mathcal{OPD}_n)$. Moreover, we give a lower bound for clique number of $\Gamma(\mathcal{OPD}_n)$ and we prove that $\Gamma(\mathcal{OPD}_n)$ is an imperfect graph.

Keywords: orientation-preserving and order-decreasing transformations, transformation semigroup, zero-divisor graph, clique number, domination number

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1 Introduction

In the literature, the zero-divisor graph of a commutative ring was defined by Beck [1]. In Beck's definition, the zero element is a vertex. Later, Anderson and Livingston redefined the zero-divisor graph without the zero element [2], which is now the standard definition of the zero-divisor graph of a commutative ring. Let R be a commutative ring, 0 be the zero element of R, and $Z(R)$ be the set of zero-divisors of R. The zero-divisor graph of R is an undirected graph $\Gamma(R)$ with vertex set $Z(R)^* = Z(R)\backslash\{0\}$ and distinct two vertices x and y in $Z(R)^*$ are adjacent vertices in $\Gamma(R)$ if and only if $xy = 0$. Similarly, the zero-divisor graph of a commutative semigroup was defined, and some properties of this graph were investigated [3,4]. Since then, zero-divisor graphs of some special commutative semigroups have been investigated (for example [5]). Redmond [6] defined four different zero-divisor graphs on a non-commutative ring. Those graphs can also be considered on a non-commutative semigroup with a zero element. Note that every zero-divisor graph is simple, meaning it has no loops or multiple edges. Let S be a non-commutative semigroup with 0. We assign the following subsets on S:

$$T(S) = \{x \in S : xy = 0 = zx \text{ for some } y, z \in S\backslash\{0\}\},$$
$$Z(S) = \{x \in S : xy = 0 \text{ or } yx = 0 \text{ for some } y \in S\backslash\{0\}\},$$
$$T(S)^* = T(S)\backslash\{0\} \quad \text{and} \quad Z(S)^* = Z(S)\backslash\{0\}.$$

We define four different zero-divisor graphs on S as follows:
- $\Gamma(S) = \Gamma_1(S)$ is the undirected graph with vertices $T(S)^*$ and distinct two vertices x and y are adjacent if and only if $xy = 0 = yx$;

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- $\Gamma_2(S)$ is the undirected graph with vertices $Z(S)^*$ and distinct two vertices x and y are adjacent if and only if $xy = 0 = yx$;
- $\Gamma_3(S)$ is the undirected graph with vertices $Z(S)^*$ and distinct two vertices x and y are adjacent if and only if $xy = 0$ or $yx = 0$; and
- $\Gamma_4(S)$ is the directed graph with vertices $Z(S)^*$ and for distinct two vertices x and y, $x \to y$ is a directed edge if and only if $xy = 0$.

In this study, we only consider the zero-divisor graph given in the first definition above.

For $n \in \mathbb{N}$, let $\mathcal{T}_n$ denote the full transformation semigroup on the chain $X_n = \{1, \dots, n\}$ under its natural order. An element $\alpha \in \mathcal{T}_n$ is called *order-preserving* if $x \le y$ implies $x\alpha \le y\alpha$ for all $x, y \in X_n$, and *order-decreasing* if $x\alpha \le x$ for all $x \in X_n$. Then, the subsemigroup consisting of all order-preserving transformations in $\mathcal{T}_n$ is denoted by $\mathcal{O}_n$, and the subsemigroup consisting of all order-decreasing transformations in $\mathcal{T}_n$ is denoted by $\mathcal{D}_n$, and the subsemigroup consisting of all order-preserving and order-decreasing transformations in $\mathcal{T}_n$ is denoted by C_n. Higgins [7] proved that the cardinality of C_n is the nth Catalan number, namely, $C_n = \frac{1}{n+1}\binom{2n}{n}$, that is why C_n is also known as the nth Catalan monoid. For a sequence $(x_1, x_2, \dots, x_r)$ on X_n, if there exists no more than one subscript i such that $x_i > x_{i+1}$, where $x_{r+1} = x_1$, then $(x_1, x_2, \dots, x_r)$ is called a cyclic. An element α in $\mathcal{T}_n$ is called *orientation-preserving* if $(1\alpha, 2\alpha, \dots, n\alpha)$ is a cyclic. Then, the subsemigroup consisting of all orientation-preserving transformations in $\mathcal{T}_n$ is denoted by $\mathcal{OP}_n$ and the subsemigroup consisting of all order-decreasing transformations in $\mathcal{OP}_n$ is denoted by $\mathcal{OPD}_n$. $\Gamma_1(C_n)$ was investigated [8]. Let $\mathcal{P}_n$ be the partial transformation semigroup on X_n, and let $\mathcal{SP}_n = \mathcal{P}_n \backslash \mathcal{T}_n$. The undirected graph $\Gamma(\mathcal{P}_n)$ was studied [9], the undirected graph $\Gamma_3(\mathcal{SP}_n)$ and the directed graph $\Gamma_4(\mathcal{SP}_n)$ were studied [10]. Recently, Korkmaz defined two undirected graphs on $\mathcal{T}_n$ and investigated some properties of these two graphs [11]. We refer to [12–14] for other terms in semigroup and graph theories, which are not explained here.

In this study, we investigate some properties of $\Gamma(\mathcal{OPD}_n)$. Since $\mathcal{OPD}_n = C_n$ for $n = 1, 2$, we suppose that $n \ge 3$ and note that $\mathcal{OPD}_n$ is a non-commutative semigroup with the zero element $\theta = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 1 & \cdots & 1 \end{pmatrix}$. And the identity element of $\mathcal{OPD}_n$ will be denoted by 1_n. In this study, we prove that $\Gamma(\mathcal{OPD}_n)$ is a connected graph, and we find the diameter, girth, domination number, minimum degree, and maximum degree of $\Gamma(\mathcal{OPD}_n)$. Moreover, we give a lower bound for clique number of $\Gamma(\mathcal{OPD}_n)$ and prove that $\Gamma(\mathcal{OPD}_n)$ is an imperfect graph.

2 Zero-divisors of $\mathcal{OPD}_n$

For $n \ge 3$, let $\mathcal{OPD}_n^* = \mathcal{OPD}_n \backslash \{\theta\}$, and then we define the following sets:

$$\begin{aligned} L &= L(\mathcal{OPD}_n) = \{\alpha \in \mathcal{OPD}_n : \alpha\beta = \theta \text{ for some } \beta \in \mathcal{OPD}_n^*\}, \\ R &= R(\mathcal{OPD}_n) = \{\alpha \in \mathcal{OPD}_n : \gamma\alpha = \theta \text{ for some } \gamma \in \mathcal{OPD}_n^*\}, \text{ and} \\ T &= T(\mathcal{OPD}_n) = \{\alpha \in \mathcal{OPD}_n : \alpha\beta = \theta = \gamma\alpha \text{ for some } \beta, \gamma \in \mathcal{OPD}_n^*\} = L \cap R, \end{aligned}$$

which are called the set of left, right, and two-sided zero-divisors of $\mathcal{OPD}_n$, respectively. In this section, we determine the left, right, and two-sided zero-divisors of $\mathcal{OPD}_n$, and then, find their cardinalities. Let us remember known result from Korkmaz [11].

Proposition 2.1. [11, Proposition 1] *For any $\alpha, \beta \in \mathcal{T}_n$, $\alpha\beta = \theta$ if and only if* $\operatorname{im}(\alpha) \subseteq 1\beta^{-1}$. *In particular, $\alpha^2 = \theta$ if and only if* $\operatorname{im}(\alpha) \subseteq 1\alpha^{-1}$.

Let us now define some specific mappings that will be useful throughout this study. For each $2 \le k \le n$, let

$$\beta_k = \begin{pmatrix} 1 & \cdots & n-1 & n \\ 1 & \cdots & 1 & k \end{pmatrix} \quad \text{and} \tag{1}$$

$$\gamma_k = \begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & n \\ 1 & \cdots & 1 & 2 & 1 & \cdots & 1 \end{pmatrix}. \tag{2}$$

For each $1 \le r \le n-1$, let $\mathcal{OPD}(n,r) = \{\alpha \in \mathcal{OPD}_n : |\mathrm{im}(\alpha)| \le r\}$, and recall that Narayana number $N(n,r)$ is defined by $N(n,r) = \frac{1}{n}\binom{n}{r}\binom{n}{r-1}$. It is known that $\sum_{r=1}^{n} N(n,r) = C_n$ (for example [15]). It is shown [16, Lemma 1 (*i*)] that $|\mathcal{OPD}(n,r)| = -n+1+\sum_{m=1}^{r} C_m + \sum_{m=r+1}^{n}\sum_{k=1}^{r} N(m,k)$. Moreover, since $\mathcal{OPD}(n,n-1) = \mathcal{OPD}_n \backslash \{1_n\}$, we conclude that $|\mathcal{OPD}_n| = 1-n+\sum_{m=1}^{n} C_m$ for $n \ge 3$. Thus, we have the following result.

Lemma 2.2. *For $n \ge 3$, $L = \mathcal{OPD}_n \backslash \{1_n\}$, and so $|L| = -n + \sum_{m=1}^{n} C_m$.*

Proof. For any $\alpha \in \mathcal{OPD}_n \backslash \{1_n\}$, let $A = X_n \backslash \mathrm{im}(\alpha)$. Since $A \ne \varnothing$ and $1 \in \mathrm{im}(\alpha)$, we have $\max(A) = k$ for some $2 \le k \le n$. If we consider γ_k as defined in (2), then it is clear that $\gamma_k \in \mathcal{OPD}_n^*$ and $\alpha\gamma_k = \theta$, and so α is a left-zero divisor element of $\mathcal{OPD}_n$. Since 1_n is the identity, we have $L = \mathcal{OPD}_n \backslash \{1_n\}$, and so $|L| = -n + \sum_{m=1}^{n} C_m$. □

For any $\alpha \in \mathcal{T}_n$ and $\varnothing \ne Y \subseteq X_n$, we denote the restriction map of α to Y by $\alpha_{|_Y}$. For any $\alpha \in \mathcal{OP}_n$, *the order-preserving degree* of α is defined by

$$\mathrm{opd}(\alpha) = \max\{m : \alpha_{|_Y} \in \mathcal{O}_m\}.$$

Thus, $\mathrm{opd}(\alpha) = n$ for all $\alpha \in \mathcal{C}_n$ and $2 \le \mathrm{opd}(\alpha) \le n-1$ for all $\alpha \in \mathcal{OPD}_n \backslash \mathcal{C}_n$ since $1\alpha = 1$ for all $\alpha \in \mathcal{D}_n$. For any $\alpha \in \mathcal{OPD}_n \backslash \mathcal{C}_n$, if $\mathrm{opd}(\alpha) = m$, then it is clear that α has the following tabular form

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & m & m+1 & \ldots & n \\ 1 & 2\alpha & \cdots & m\alpha & 1 & \ldots & 1 \end{pmatrix}$$

with the property that $1 \le 2\alpha \le \ldots \le m\alpha \le m$.

Lemma 2.3. *For $n \ge 3$, $R = \{\alpha \in \mathcal{OPD}_n : |1\alpha^{-1}| \ge 2\}$, or equivalently, $R = (\mathcal{OPD}_n \backslash \mathcal{C}_n) \cup \{\alpha \in \mathcal{C}_n : 2\alpha = 1\}$, and so,*

$$|R| = 1 - n + C_n + \sum_{m=1}^{n-2} C_m.$$

Proof. Let $A = \{\alpha \in \mathcal{OPD}_n : |1\alpha^{-1}| \ge 2\}$, and for any $\alpha \in A$, let $\mathrm{opd}(\alpha) = m$ and suppose that $m \ne n$. If we consider β_n as defined in (1), then since $n\alpha = 1$, we have $\beta_n\alpha = \theta$, and so α is a right-zero divisor element of $\mathcal{OPD}_n$. Suppose that $m = n$. Since $\alpha \in \mathcal{C}_n$ and $|1\alpha^{-1}| \ge 2$, we must have $1\alpha = 2\alpha = 1$. If we consider β_2 as defined in (1), then we have $\beta_2\alpha = \theta$, and so α is a right-zero divisor element of $\mathcal{OPD}_n$.

For any $\alpha \in \mathcal{OPD}_n$, let $|1\alpha^{-1}| = 1$, that is $1\alpha^{-1} = \{1\}$. If $\beta\alpha = \theta$ for some $\beta \in \mathcal{OPD}_n$, then it follows from Proposition 2.1 that $\mathrm{im}(\beta) \subseteq 1\alpha^{-1} = \{1\}$, and so $\beta = \theta$. Thus, α cannot be a right-zero divisor. Therefore, we have

$$R = \{\alpha \in \mathcal{OPD}_n : |1\alpha^{-1}| \ge 2\} = (\mathcal{OPD}_n \backslash \mathcal{C}_n) \cup \{\alpha \in \mathcal{C}_n : 2\alpha = 1\}.$$

Moreover, if we define the mapping $\psi : \{\alpha \in \mathcal{C}_n : 2\alpha = 2\} \to \mathcal{C}_{n-1}$ by

$$\alpha\psi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 \\ 1 & 3\alpha - 1 & 4\alpha - 1 & \cdots & n\alpha - 1 \end{pmatrix}$$

for all $\alpha \in \{\alpha \in \mathcal{C}_n : 2\alpha = 2\}$, then it is clear that ψ is a well-defined mapping. For any $\alpha_1, \alpha_2 \in \{\alpha \in \mathcal{C}_n : 2\alpha = 2\}$, suppose that $\alpha_1\psi = \alpha_2\psi$. Then, since $1\alpha_1 = 1 = 1\alpha_2$, $2\alpha_1 = 2 = 2\alpha_2$ and $k\alpha_1 = k\alpha_2$ for $3 \le k \le n$, it follows that $\alpha_1 = \alpha_2$, and so ψ is one to one. For any $\beta \in \mathcal{C}_{n-1}$, define

$$\hat{\beta} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 2\beta + 1 & \cdots & (n-1)\beta + 1 \end{pmatrix}.$$

Then, it is clear that $\hat{\beta} \in \{\alpha \in C_n : 2\alpha = 2\}$, and that $\hat{\beta}\psi = \beta$. Thus, ψ is onto, and so ψ is a bijection. Moreover, since $\{\alpha \in C_n : 2\alpha = 1\} = C_n \backslash \{\alpha \in C_n : 2\alpha = 2\}$, it follows that

$$|\{\alpha \in C_n : 2\alpha = 1\}| = |C_n| - |\{\alpha \in C_n : 2\alpha = 2\}| = C_n - C_{n-1}.$$

Therefore, from [16, Lemma 1 (*i*)], we have

$$\begin{aligned} |R| &= |\mathcal{OPD}_n \backslash C_n| + |\{\alpha \in C_n : 2\alpha = 1\}| \\ &= \left(1 - n + \sum_{m=1}^{n-1} C_m\right) + (C_n - C_{n-1}) = 1 - n + C_n + \sum_{m=1}^{n-2} C_m, \end{aligned}$$

as required. □

Since $R \subseteq L$, we have the following immediate corollary.

Corollary 2.4. *For $n \geq 3$, $T = R$.*

3 Zero-divisor graph of $\mathcal{OPD}_n$

We denote the vertex set and the edge set of a simple graph G by $V(G)$ and $E(G)$, respectively. For any $n+1$ different vertices $u = v_0, v_1, \ldots, v_n = v$ in $V(G)$, if there exists an edge $v_i - v_{i+1}$ in $E(G)$ for each $0 \leq i \leq n-1$, then $u = v_0 - v_1 - \ldots - v_{n-1} - v_n = v$ is called a *path* between u and v, and n is called the *length of the path*. The length of a shortest path between u and v in G is denoted by $d_G(u, v)$. If there exist a path for all two distinct vertices in G, then G is called a *connected graph*. The *eccentricity* of a vertex v in a connected simple graph G, denoted by $\mathrm{ecc}(v)$, is defined by

$$\mathrm{ecc}(v) = \max\{d_G(u, v) : u \in V(G)\}.$$

The *diameter* of a connected simple graph G, denoted by $\mathrm{diam}(G)$, is defined by

$$\mathrm{diam}(G) = \max\{\mathrm{ecc}(v) : v \in V(G)\}.$$

Observe that $V(\Gamma(\mathcal{OPD}_3)) = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}\right\}$ and $E(\Gamma(\mathcal{OPD}_3))$ contains only one edge, namely, $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$. Next we let

$$\delta_k = \begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & n \\ 1 & \cdots & 1 & k & 1 & \cdots & 1 \end{pmatrix} \quad \text{for } 2 \leq k \leq n, \tag{3}$$

$$\lambda_k = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & n \\ 1 & \cdots & 1 & k+1 & \cdots & k+1 \end{pmatrix} \quad \text{for } 2 \leq k \leq n-1, \tag{4}$$

$$\rho = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 1 & 3 & \ldots & n \end{pmatrix}, \quad \text{and} \tag{5}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 1 & 2 & \ldots & n-1 \end{pmatrix}. \tag{6}$$

For convenience, we use the notation Γ instead of $\Gamma(\mathcal{OPD}_n)$. For any $v \in V(G)$, the *neighborhood* of v is denoted by $N(v)$ and defined by

$$N(v) = \{u \in V(G) : u - v \in E(G)\}.$$

Moreover, for $\alpha \in V(\Gamma)$, we let

$$N_1(\alpha) = N(\alpha) \cap C_n \quad \text{and} \quad N_2(\alpha) = N(\alpha) \cap (\mathcal{OPD}_n \backslash C_n).$$

Then, we state and prove one of the main results in this study.

Theorem 3.1. *For* $n \geq 4$, Γ *is connected and* $\mathrm{diam}(\Gamma) = 4$.

Proof. For any distinct $\alpha, \beta \in V(\Gamma)$, we consider three cases; both of them in C_n, just one of them in C_n and none of them in C_n.
Case 1: Suppose that $\alpha, \beta \in C_n$. First note that $2\alpha = 1 = 2\beta$ since $\alpha, \beta \in V(\Gamma) = (\mathcal{OPD}_n \backslash C_n) \cup \{\alpha \in C_n : 2\alpha = 1\}$. Then, we consider three subcases; none of the image sets contains n, just one of the image sets contains n, and both the image sets contain n.
Subcase (i). Suppose that $n\alpha,\ n\beta < n$. If one of them is β_2 as defined in (1), then we have the path $\alpha - \beta$; and if none of them is β_2, then we have the path $\alpha - \beta_2 - \beta$.
Subcase (ii). Without loss of generality, suppose that $n\alpha = n$ and $n\beta < n$. If $k = \min(X_n \backslash \mathrm{im}(\alpha))$, then since $2 \leq k \leq n-1$, we consider γ_k, β_2, and β_3 as defined in (1), and (2). Suppose that $\beta \notin \{\beta_2, \beta_3\}$. If $k \neq 2$, then $\alpha - \gamma_k - \beta_2 - \beta$ is a path in Γ, and if $k = 2$, then $\alpha - \gamma_2 - \beta_3 - \beta_2 - \beta$ is a path in Γ, and so, $d_\Gamma(\alpha, \beta) \leq 4$. If $\beta \in \{\beta_2, \beta_3\}$, then it is similarly shown that α and β are connected, and $d_\Gamma(\alpha, \beta) \leq 3$.
Subcase (iii). Suppose that $n\alpha = n$ and $n\beta = n$. If $k = \min(X_n \backslash \mathrm{im}(\alpha))$ and $l = \min(X_n \backslash \mathrm{im}(\beta))$, then since $2 \leq k, l \leq n-1$, we consider δ_2, γ_k, γ_l, and β_n as defined in (1), (2), and (3). If $k = l = 2$, then $\alpha - \delta_2 - \beta$ is a path in Γ since $1\alpha = 2\alpha = 1 = 1\beta = 2\beta$. Without loss of generality, suppose that $k \neq 2$ and $l = 2$. Similarly, if $\alpha, \beta \notin \{\beta_n\}$, then $\alpha - \gamma_k - \beta_n - \delta_2 - \beta$ is a path in Γ and if $\alpha = \beta_n$ or $\beta = \beta_n$, then it is clear that $d_\Gamma(\alpha, \beta) \leq 2$. Now, suppose $k, l \geq 3$. If $k = l$, then $\alpha - \gamma_k - \beta$ is a path in Γ, and if $k \neq l$, then $\alpha - \gamma_k - \gamma_l - \beta$ is a path in Γ.
Case 2: Suppose that $\alpha \in C_n$ and $\beta \in \mathcal{OPD}_n \backslash C_n$. Let $k = \min(X_n \backslash \mathrm{im}(\alpha))$. If $n\alpha = n$, $\alpha \neq \beta_n$ and $\beta \neq \gamma_k$, then $\alpha - \gamma_k - \beta_n - \beta$ is a path in Γ. If $n\alpha = n$ and $\alpha = \beta_n$ or $n\alpha = n$ and $\beta = \gamma_k$, then it is clear that α and β are adjacent vertices in Γ. If $n\alpha < n, \alpha \neq \beta_2$ and $\beta \neq \delta_3$, then $\alpha - \beta_2 - \delta_3 - \beta_n - \beta$ is a path in Γ. If $n\alpha < n$ and $\alpha = \beta_2$ or $n\alpha < n$ and $\beta = \delta_3$, then it is also clear that $d_\Gamma(\alpha, \beta) \leq 3$.
Case 3: Suppose that both α and β in $\mathcal{OPD}_n \backslash C_n$. Since $n\alpha = 1 = n\beta$, it follows that $\alpha - \beta_n - \beta$ is a path in Γ.

Therefore, Γ is a connected graph and $d_\Gamma(\alpha, \beta) \leq 4$ for all $\alpha, \beta \in V(\Gamma)$. If we consider ρ and τ as defined in (5) and (6), then we note that ρ and τ are non-adjacent vertices in Γ. Moreover, if $\rho\eta = \eta\rho = \theta$ and $\tau\mu = \mu\tau = \theta$, then we have $\eta = \delta_2$ and $\mu = \beta_2$. Finally, since $\beta_2\delta_2 \neq \theta$, it follows that $d_\Gamma(\rho, \tau) \geq 4$, and so $\mathrm{diam}(\Gamma) = 4$. □

For any simple graph G and $v \in V(G)$, the degree of v, denoted by $\deg_G(v)$, is defined as the number of adjacent vertices to v in G. Moreover, the minimum degree of G, denoted by $\delta(G)$, is defined by

$$\delta(G) = \min\{\deg_G(v) : v \in V(G)\},$$

and the maximum degree of G, denoted by $\Delta(G)$, is defined by

$$\Delta(G) = \max\{\deg_G(v) : v \in V(G)\}.$$

Thus, if G is connected, then $\delta(G) \geq 1$. Moreover, since $\deg_\Gamma(\rho) = \deg_\Gamma(\tau) = 1$, we have the following immediate corollary.

Corollary 3.2. *For* $n \geq 4$, $\delta(\Gamma) = 1$.

To find $\Delta(\Gamma)$, we need some preliminary results. Let $U_1 = \{\alpha \in C_n : 2\alpha = 2\}$ and $U_2 = \{\alpha \in C_n : n\alpha = n\}$. As shown in the proof of Lemma 2.3 that $|U_1| = C_{n-1}$, one can easily show that $|U_2| = C_{n-1}$ and $|U_1 \cap U_2| = |\{\alpha \in C_n : 2\alpha = 2 \text{ and } n\alpha = n\}| = C_{n-2}$. Since $\{\alpha \in C_n : 2\alpha = 1 \text{ and } n\alpha < n\} = C_n \backslash (U_1 \cup U_2)$, we conclude that the cardinality of the set $\{\alpha \in C_n : 2\alpha = 1 \text{ and } n\alpha < n\}$ is $C_n - 2C_{n-1} + C_{n-2}$ (see, also [8, proof of Lemma 3.3]).

For $2 \leq m \leq n$, let $Q_m = \{\alpha \in \mathcal{OPD}_n : \mathrm{opd}(\alpha) = m\}$. Then, it is shown [16, Lemma 1] that

(i) $\mathcal{OPD}_n = \bigcup_{m=2}^{n} Q_m$,
(ii) $Q_m \cap Q_{m'} = \varnothing$ for all $2 \leq m \neq m' \leq n$, and
(iii) $Q_m \cup \{\theta\}$ and C_m are isomorphic for each $2 \leq m \leq n$.

By using these results, we prove the following lemma.

Lemma 3.3. *For* $n \geq 4$, $\deg_\Gamma(\beta_2) = C_n - C_{n-1} + C_{n-2} - n - 1$.

Proof. For $n \geq 4$, let

$$\mathcal{A} = \{\alpha \in V(\Gamma) \cap C_n : n\alpha < n\}, \quad \mathcal{B} = \{\alpha \in \mathcal{OPD}_n \backslash C_n : 2\alpha = 1\}, \quad \text{and}$$
$$C = \{\alpha \in \mathcal{OPD}_n \backslash C_n : 2\alpha = 2\}.$$

Then, it is clear that $\beta_2 \in \mathcal{A}$, $\mathcal{B} \cup C = \mathcal{OPD}_n \backslash C_n$, and $\mathcal{B} \cap C = \emptyset$. Since $\mathcal{A} \cup \{\theta\} = \{\alpha \in C_n : 2\alpha = 1 \text{ and } n\alpha < n\}$, it follows that $|\mathcal{A}| = C_n - 2C_{n-1} + C_{n-2} - 1$. For $2 \leq m \leq n-1$, if $Q'_m = \{\alpha \in \mathcal{OPD}_n : \text{opd}(\alpha) = m \text{ and } 2\alpha = 2\}$, then C is a disjoint union of $Q'_2, \ldots, Q'_{n-1}$, and that for each $2 \leq m \leq n-1$, Q'_m and $\{\alpha \in C_m : 2\alpha = 2\}$ are isomorphic semigroups, and $|\{\alpha \in C_m : 2\alpha = 2\}| = C_{m-1}$, it follows that $|C| = \sum_{m=2}^{n-1} C_{m-1} = \sum_{m=1}^{n-2} C_m$. Moreover, from the proof of Lemma 2.3, we have $|\mathcal{B}| = |\mathcal{OPD}_n \backslash C_n| - |C| = C_{n-1} - n + 1$. Therefore, since any α in $V(\Gamma)$ is an adjacent vertex to β_2 if and only if $\alpha \in (\mathcal{A} \cup \mathcal{B}) \backslash \{\beta_2\}$, it follows that $\deg_\Gamma(\beta_2) = |\mathcal{A}| - 1 + |\mathcal{B}| = C_n - C_{n-1} + C_{n-2} - n - 1$. □

It is known that the Catalan numbers satisfy the recurrence relation $C_n = \sum_{m=0}^{n-1} C_m C_{n-1-m}$, where $C_0 = 1$ [17]. Therefore, we have the following proposition.

Proposition 3.4. *For* $n \geq 4$, $\deg_\Gamma(\beta_2) > \sum_{i=2}^{n-1}(C_i - 1)$.

Proof. For $n \geq 4$, since $C_{n-2} \geq 2$, it follows from Lemma 3.3 that

$$\begin{aligned}\deg_\Gamma(\beta_2) &= \left(\sum_{m=0}^{n-1} C_m C_{n-1-m}\right) - C_{n-1} + C_{n-2} - n - 1 \\ &= \left(\sum_{m=1}^{n-1} C_m C_{n-1-m}\right) + C_{n-2} - n - 1 \geq \left(\sum_{m=1}^{n-1} C_m C_{n-1-m}\right) - n + 1 \\ &> \left(\sum_{m=1}^{n-1} C_m\right) - n + 1 = \sum_{m=1}^{n-1}(C_m - 1) = \sum_{m=2}^{n-1}(C_m - 1),\end{aligned}$$

as required. □

Proposition 3.5. *For* $n \geq 4$ *and* $3 \leq k \leq n$, $\deg_\Gamma(\beta_k) < \deg_\Gamma(\beta_2)$.

Proof. First, we show that $\deg_\Gamma(\beta_n) < \deg_\Gamma(\beta_2)$. It is clear that $N_1(\beta_n) = \emptyset$ and $N_2(\beta_n) = \mathcal{OPD}_n \backslash C_n$, and so from Proposition 3.4, $\deg_\Gamma(\beta_n) = \sum_{i=2}^{n-1}(C_i - 1) < \deg_\Gamma(\beta_2)$.

For all $\alpha \in V(\Gamma) \cap C_n$, we recall $2\alpha = 1$, and we let $3 \leq k \leq n-1$. Since $\lambda_{k-1} \in N_1(\beta_2) \backslash N_1(\beta_k)$, it follows that $|N_1(\beta_k)| < |N_1(\beta_2)|$. Moreover, if $\alpha \in N_2(\beta_k)$, then α has the following tabular form

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n-1 & n \\ 1 & 2\alpha & \cdots & (k-1)\alpha & 1 & (k+1)\alpha & \cdots & (n-1)\alpha & 1 \end{pmatrix}.$$

If we let

$$\tilde{\alpha} = \begin{pmatrix} 1 & 2 & 3 & \cdots & k & k+1 & \cdots & n-1 & n \\ 1 & 1 & 2\alpha & \cdots & (k-1)\alpha & (k+1)\alpha & \cdots & (n-1)\alpha & 1 \end{pmatrix}, \tag{7}$$

then we have $\tilde{\alpha} \in N_2(\beta_2)$. If we define the mapping $\varphi_k : N_2(\beta_k) \to N_2(\beta_2)$ by $\alpha\varphi_k = \tilde{\alpha}$ for all $\alpha \in N_2(\beta_k)$, then it is clear that φ_k is a well-defined one to one mapping, and so $|N_2(\beta_k)| \leq |N_2(\beta_2)|$. Therefore, we conclude that $\deg_\Gamma(\beta_k) < \deg_\Gamma(\beta_2)$ for each $3 \leq k \leq n$. □

Proposition 3.6. *For* $n \geq 4$ *and* $2 \leq m \leq n-1$, *we have* $\deg_\Gamma(\gamma_m) < \deg_\Gamma(\beta_2)$.

Proof. Let $2 \le m \le n-1$. It is clear that $N_2(\gamma_m) \subseteq N_2(\beta_2)$ for all $2 \le m \le n-1$. Moreover, since $\gamma_m \in N_2(\beta_2)$ if $m > 2$, and since $\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 2 & \cdots & 2 & 1 \end{pmatrix} \in N_2(\beta_2)\backslash N_2(\gamma_2)$, it follows that $|N_2(\gamma_m)| + 1 \le |N_2(\beta_2)|$ for all $2 \le m \le n-1$.

For $2 \le m \le n$, let $J(m) = \{\alpha \in C_n : 2\alpha = 1 \text{ and} m \notin \text{im}(\alpha)\}$. Then, we show that $|J(m)| < |J(n)|$ for all $2 \le m \le n-1$. For $3 \le m \le n$ and $\alpha \in J(m-1)$, let $\hat{\alpha} : X_n \to X_n$ be defined by

$$x\hat{\alpha} = \begin{cases} x\alpha & x\alpha \ne m \\ m-1 & x\alpha = m \end{cases}$$

for all $x \in X_n$. Similarly, the mapping $\psi_m : J(m-1) \to J(m)$ defined by $\alpha\psi_m = \hat{\alpha}$ for all $\alpha \in J(m-1)$ is one to one, and so $|J(m-1)| \le |J(m)|$ for all $3 \le m \le n$. Moreover, if we consider the transformation

$$\beta = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 1 & 3 & \cdots & n-1 & n-1 \end{pmatrix},$$

then it is clear that $\beta \in J(n)\backslash J(n-1)$ and $\beta \ne \hat{\alpha}$ for all $\alpha \in J(n-1)$, and so ψ_n is not onto, i.e., $|J(n-1)| < |J(n)|$. Therefore, we have $|J(m)| < |J(n)|$ for all $2 \le m \le n-1$. Since

$$N_1(\gamma_m) = J(m)\backslash\{\theta\} \text{ and } N_1(\beta_2) \cup \{\beta_2\} = J(n)\backslash\{\theta\},$$

it follows that $|N_1(\gamma_m)| - 1 < |N_1(\beta_2)|$ for all $2 \le m \le n-1$. Therefore, $\deg_\Gamma(\gamma_m) < \deg_\Gamma(\beta_2)$ for all $2 \le m \le n-1$. □

Theorem 3.7. *$\Delta(\Gamma) = C_n - C_{n-1} + C_{n-2} - n - 1$ for $n \ge 4$.*

Proof. Let $\alpha \in V(\Gamma)$. For $\alpha \in V(\Gamma) \cap C_n$, let $k = \min(\text{im}(\alpha)\backslash\{1\})$, consider β_k and suppose $\alpha \ne \beta_k$. For any $\xi \in N(\alpha)$, it follows from Proposition 2.1 that $\text{im}(\xi) \subseteq 1\alpha^{-1} \subseteq X_{n-1} = 1\beta_k^{-1}$ and $\text{im}(\beta_k) \subseteq \text{im}(\alpha) \subseteq 1\xi^{-1}$, and so $\xi\beta_k = \theta = \beta_k\xi$. Thus, if $\xi \ne \beta_k$, then $\xi \in N(\beta_k)$, and if $\xi = \beta_k$, then $\alpha \in N(\beta_k)$. Therefore, $\deg_\Gamma(\alpha) \le \deg_\Gamma(\beta_k)$, and so from Proposition 3.5, $\deg_\Gamma(\alpha) \le \deg_\Gamma(\beta_2)$.

If $\alpha \in V(\Gamma) \cap (\mathcal{OPD}_n\backslash C_n)$ and $\text{opd}(\alpha) = m$, then α has the following tabular form

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & m-1 & m & m+1 & \cdots & n \\ 1 & 2\alpha & \cdots & (m-1)\alpha & k & 1 & \cdots & 1 \end{pmatrix}$$

with the property $1 \le 2\alpha \le \ldots \le (m-1)\alpha \le k \le m \le n-1$ and $k \ge 2$. If we consider $\bar{\alpha} = \begin{pmatrix} 1 & \cdots & m-1 & m & m+1 & \cdots & n \\ 1 & \cdots & 1 & k & 1 & \cdots & 1 \end{pmatrix}$, then we similarly have $\deg_\Gamma(\alpha) \le \deg_\Gamma(\bar{\alpha})$. Now, consider γ_m. If $k = 2$, then $\bar{\alpha} = \gamma_m$, and so suppose $k \ge 3$. If $\zeta \in N_1(\bar{\alpha})$, then it is clear that $\zeta \in N_1(\gamma_m)$, and so, $N_1(\bar{\alpha}) \subseteq N_1(\gamma_m)$. Moreover, we have $\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 2 & \cdots & 2 \end{pmatrix} \in N_1(\gamma_m)\backslash N_1(\bar{\alpha})$ since $m \ge k \ge 3$. Thus, $|N_1(\bar{\alpha})| \le |N_1(\gamma_m)| - 1$.

If $\zeta \in N_2(\bar{\alpha})$, then $k\zeta = 1$, and so, ζ has the following tabular form:

$$\zeta = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k & k+1 & \cdots & n-1 & n \\ 1 & 2\zeta & \cdots & (k-1)\zeta & 1 & (k+1)\zeta & \cdots & (n-1)\zeta & 1 \end{pmatrix}.$$

Thus, we have $i\zeta \ne m$ for $i \in \{2, 3, \ldots, k-1, k+1, \ldots, n-1\}$ since ζ and $\bar{\alpha}$ are adjacent vertices in Γ. If we define $\tilde{\zeta} = \begin{pmatrix} 1 & 2 & 3 & \cdots & k & k+1 & \cdots & n-1 & n \\ 1 & 1 & 2\zeta & \cdots & (k-1)\zeta & (k+1)\zeta & \cdots & (n-1)\zeta & 1 \end{pmatrix}$ as in (7), then we have $\text{im}(\tilde{\zeta}) \subseteq 1\gamma_m^{-1}$ and $\text{im}(\gamma_m) \subseteq 1\tilde{\zeta}^{-1}$, and so, $\tilde{\zeta} \in N_2(\gamma_m)$ if $\tilde{\zeta} \ne \gamma_m$. If we define the mapping $f : N_2(\bar{\alpha}) \to N_2(\gamma_m) \cup \{\gamma_m\}$ by $\zeta f = \tilde{\zeta}$ for all $\zeta \in N_2(\bar{\alpha})$, then it is clear that f is a well-defined one to one mapping, and so, $|N_2(\bar{\alpha})| \le |N_2(\gamma_m)| + 1$. Thus, from Proposition 3.6, $\deg_\Gamma(\alpha) \le \deg_\Gamma(\bar{\alpha}) \le \deg_\Gamma(\gamma_m) < \deg_\Gamma(\beta_2)$. Therefore, from Lemma 3.3, we have $\Delta(\Gamma) = \deg_\Gamma(\beta_2) = C_n - C_{n-1} + C_{n-2} - n - 1$. □

The length of a shortest cycle contained in a graph G is called the *girth* of G and it is denoted by $\mathrm{gr}(G)$. Moreover, if G does not contain any cycles, then its girth is defined as infinity. Thus, if G is a simple connected graph, then the length of a shortest cycle must be at least 3, and so $\mathrm{gr}(G) \geq 3$. Then, we have the following corollary.

Corollary 3.8. *For* $n \geq 4$, $\mathrm{gr}(\Gamma) = 3$.

Proof. If we consider $\alpha = \begin{pmatrix} 1 & 2 & \dots & n-2 & n-1 & n \\ 1 & 1 & \dots & 1 & 2 & 2 \end{pmatrix}$, β_2 and γ_3 as defined in (1) and (2), then $\alpha - \beta_2 - \gamma_3 - \alpha$ is a cycle in Γ, and so $\mathrm{gr}(\Gamma) = 3$. □

Let D be a non-empty subset of the vertex set $V(G)$ of a graph G. For each $v \in V(G)$, if $v \in D$, or if there exists $u \in D$ such that $u - v$ is an edge in $E(G)$, then D is called a *dominating set* for G. The *domination number* of G, denoted by $\Upsilon(G)$, is defined by

$$\Upsilon(G) = \min\{|D| : D \text{ is a dominating set for } G\}.$$

To find $\Upsilon(\Gamma)$, we also consider the following vertices in $V(\Gamma)$:

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & 3 & \dots & n \end{pmatrix},$$

$$\mu_k = \begin{pmatrix} 1 & 2 & 3 & \dots & k+1 & k+2 & \dots & n \\ 1 & 1 & 2 & \dots & k & k+2 & \dots & n \end{pmatrix} \quad (2 \leq k \leq n-2),$$

$$\mu_{n-1} = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 1 & 2 & \dots & n-1 \end{pmatrix}, \quad \text{and}$$

$$\mu_n = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 1 & 2 & \dots & n-1 & 1 \end{pmatrix}.$$

With these notations, we have the following theorem.

Theorem 3.9. *For* $n \geq 4$, $\Upsilon(\Gamma) = n$.

Proof. For each $1 \leq k \leq n-1$, it is easy to check that $\mu_k\alpha = \alpha\mu_k = \theta$ if and only if $\alpha = \gamma_{k+1}$, and that $\mu_n\alpha = \alpha\mu_n = \theta$ if and only if $\alpha = \beta_n$. Thus, $\deg_\Gamma(\mu_k) = 1$ for each $1 \leq k \leq n$. Now, let $\mathcal{U} = \{\gamma_2, \gamma_3, \dots, \gamma_n, \beta_n\}$. If D is a dominating set for Γ, then either μ_k or its unique adjacent vertex in D, and so $\Upsilon(\Gamma) \geq n$.

Next we show that $\mathcal{U}$ is a dominating set for Γ. For any $\alpha \in V(\Gamma)\backslash\mathcal{U}$, we consider two cases: either $\alpha \in \mathcal{OPD}_n\backslash C_n$ or $\alpha \in C_n$. In the first case, since $n\alpha = 1$, it is clear that α and β_n are adjacent vertices in Γ. In the second case, we have two subcases, either $n\alpha < n$ or $n\alpha = n$. If $\alpha \in C_n$ and $n\alpha < n$, then α and γ_n are adjacent vertices in Γ. If $\alpha \in C_n$ and $n\alpha = n$, then there exists at least one $k \in X_n\backslash\mathrm{im}(\alpha)$ such that $2 \leq k \leq n-1$. Then, it is easy to see that α and γ_k are adjacent vertices in Γ. Therefore, $\mathcal{U}$ is a dominating set of Γ, and so $\Upsilon(\Gamma) = n$. □

Let G be a simple graph and C be a non-empty subset of $V(G)$. If every two distinct vertices in C are adjacent, then C is called a *clique* in G, and moreover, the subgraph whose vertex set is C and edge set contains all edges of G, which have endpoints in C is called the subgraph of G induced by C. Thus, C is a clique if and only if the subgraph of G induced by C is a complete graph. Moreover, the number of vertices of any maximal clique in G is called the *clique number* of G, and it is denoted by $\omega(G)$.

For any real number x, we denote the smallest integer greater than or equal to x by $\lceil x \rceil$. For any $p, q, r \in \mathbb{N}$ with $p < q$, let

$$Y_{q,p} = \{p+1, p+2, \dots, q\} \text{ and } X_r = \{1, 2, \dots, r\}$$

under their natural order. Moreover, let O^* denote the set of all order-preserving mappings from $Y_{q,p}$ to X_r. For any $\alpha \in O^*$, it is clear that $(|1\alpha^{-1}|, |2\alpha^{-1}|, \dots, |r\alpha^{-1}|)$ is a solution of the equation

$$x_1 + x_2 + \dots + x_r = q - p, \quad \text{where all } x_i \in \mathbb{N} \cup \{0\}.$$

Conversely, for each solution of this equation $(c_1, c_2, \ldots, c_r)$, there exists unique $\alpha \in O^*$ such that $|i\alpha^{-1}| = c_i$ for each $1 \le i \le r$. Therefore, since the number of solutions of the above equation is the same with the cardinality of O^*, we have

$$|O^*| = \binom{q-p+r-1}{r-1}, \tag{8}$$

[17]. Now, we give a lower bound for clique number of Γ in the following theorem.

Theorem 3.10. *For $n \ge 4$, if $\lceil \frac{n}{2} \rceil = s$, then $\omega(\Gamma) \ge \binom{n}{s} - (n-s+1)$.*

Proof. For $2 \le r \le n-2$, if we let

$$V_r = \{\alpha \in \mathcal{OPD}_n : \{1\} \ne \operatorname{im}(\alpha) \subseteq X_r \subseteq 1\alpha^{-1}\},$$

then it is clear that $\emptyset \ne V_r \subseteq V(\Gamma)$. Moreover, let Λ be the subgraph of Γ induced by V_r. Then, for any distinct two vertices α and β in V_r, since $\operatorname{im}(\alpha) \subseteq 1\beta^{-1}$ and $\operatorname{im}(\beta) \subseteq 1\alpha^{-1}$, it follows from Proposition 2.1 that $\alpha\beta = \theta = \beta\alpha$, and so Λ is a complete graph.

Moreover, for any $\alpha \in V_r$, if $\operatorname{opd}(\alpha) = m$, then we observe that $r+1 \le m \le n$, $x\alpha = 1$ for all $x \in X_r \cup Y_{n,m}$, and that the restriction $\alpha_{|Y_{m,r}}$ is an order-preserving transformation from $Y_{m,r}$ to X_r. From (8), there exist $\binom{m-1}{r-1}$ many order-preserving mappings from $Y_{m,r}$ to X_r, and one of them is $\theta_{|Y_{m,r}}$, and so

$$|V_r| = \sum_{m=r+1}^{n}\left(\binom{m-1}{r-1} - 1\right) = \sum_{k=0}^{n-r}\binom{r-1+k}{r-1} - (n-r+1) = \binom{n}{r} - (n-r+1).$$

If $\lceil \frac{n}{2} \rceil = s$, then it is clear that $\binom{n}{r} - (n-r+1) \le \binom{n}{s} - (n-s+1)$ for all $2 \le r \le n-2$, and so $\omega(\Gamma) \ge \binom{n}{s} - (n-s+1)$. □

For $n = 4$, it is easy to see that $\omega(\Gamma) = \binom{4}{2} - (4-2+1) = 3$. In particular, the subgraph of Γ induced by $\left\{\begin{pmatrix}1&2&3&4\\1&1&1&2\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&2&1\end{pmatrix}, \begin{pmatrix}1&2&3&4\\1&1&2&2\end{pmatrix}\right\}$ is a maximal complete subgraph of Γ. However, we have not proved $\omega(\Gamma) = \binom{n}{s} - (n-s+1)$ in general.

If we color all the vertices in G with the rule of no two adjacent vertices have the same color, then the minimum number of colors needed to color of G is called the *chromatic number* of G, and it is denoted by $\chi(G)$. If $\omega(H) = \chi(H)$ for every induced subgraph H of G, then G is called a perfect graph, otherwise it is called imperfect graph.

Theorem 3.11. *Γ is an imperfect graph for $n \ge 4$.*

Proof. For $n \ge 4$, let $\xi = \begin{pmatrix}1&2&\ldots&n-2&n-1&n\\1&1&\ldots&1&n-2&1\end{pmatrix} \in V(\Gamma)$ and $H = \{\xi, \gamma_{n-2}, \lambda_{n-2}, \beta_{n-2}, \beta_n\}$. If Π is the subgraph of Γ induced by H, then it is a routine matter to check that Π is a cycle graph with the cycle

$$\lambda_{n-2} - \gamma_{n-2} - \beta_n - \xi - \beta_{n-2} - \lambda_{n-2}.$$

Therefore, since $\omega(\Pi) = 2$ and $\chi(\Pi) = 3$, Γ is an imperfect graph. □

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