Research Article

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The higher mapping cone axiom

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Abstract: The aim of the article is to better understand the higher octahedral axiom of n-angulated categories. We discuss several possible additional axioms for pre-n-angulated categories and prove that they are all equivalent to the higher octahedral axiom.

Keywords: n-angulated category, higher mapping cone

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1 Introduction

Triangulated categories were introduced independently in algebraic geometry by Verdier [1] and in algebraic topology by Puppe [2]. These constructions have since played a crucial role in representation theory, algebraic geometry, algebraic topology, commutative algebra and even theoretical physics. Let n be an integer greater than or equal to 3. Geiss et al. [3] introduced "higher dimensional" analogues of triangulated categories, called *n-angulated categories*. The classical triangulated categories are the special case $n = 3$. Other examples of n-angulated categories can be found in [3–5]. The fourth axiom for n-angulated categories is a generalization of the octahedral axiom for triangulated categories. Bergh and Thaule [6] discussed the axioms of n-angulated categories systematically and showed that the higher octahedral axiom is equivalent to the higher mapping cone axiom. They also proved that TR3 is redundant under the presence of TR4 and the other axioms also in the higher setting. Lin and Zheng [7] used homotopy cartesian diagrams to give several new equivalent statements of the higher mapping cone axiom.

The aim of this note is to introduce the higher base change, higher cobase change, higher mapping cone axiom and the higher octahedral axiom, and show all these axioms are equivalent in pre-n-angulated categories.

2 Some basic facts of pre-n-angulated categories

This section is devoted to recalling some notions and basic consequences of pre-n-angulated categories. For terminology, we shall follow [3,6].

Let $\mathscr{T}$ be an additive category with an automorphism $\Sigma : \mathscr{T} \to \mathscr{T}$ and n an integer greater than or equal to 3. A sequence of morphisms in $\mathscr{T}$

$$X_\bullet : X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1$$

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is an *n-Σ-sequence*. Its left and right *rotations* are the two n-Σ-sequences

$$X_2 \overset{\alpha_2}{\to} X_3 \overset{\alpha_3}{\to} \dots \overset{\alpha_n}{\to} \Sigma X_1 \overset{(-1)^n\Sigma\alpha_1}{\to} \Sigma X_2,$$
$$\Sigma^{-1}X_n \overset{(-1)^n\Sigma^{-1}\alpha_n}{\to} X_1 \overset{\alpha_1}{\to} X_2 \overset{\alpha_2}{\to} \dots \overset{\alpha_{n-2}}{\to} X_{n-1} \overset{\alpha_{n-1}}{\to} X_n,$$

respectively. A *trivial n-Σ-sequence* is a sequence of the form

$$(TX)_\bullet : X \overset{1}{\to} X \to 0 \to \dots \to 0 \to \Sigma X$$

or any of its rotations for $X \in \mathscr{T}$. An n-Σ-sequence $X_\bullet$ is *exact* if the induced sequence

$$\operatorname{Hom}_{\mathscr{T}}(-, X_\bullet) : \dots \to \operatorname{Hom}_{\mathscr{T}}(-, X_1) \to \dots \to \operatorname{Hom}_{\mathscr{T}}(-, X_n) \to \operatorname{Hom}_{\mathscr{T}}(-, \Sigma X_1) \to \dots$$

of representable functors $\mathscr{T}^{\mathrm{op}} \to \mathrm{Ab}$ is exact.

A *morphism* of n-Σ-sequences is given by a sequence of morphisms $\varphi_\bullet = (\varphi_1, \varphi_2, \dots, \varphi_n)$ such that the following diagram

$$\begin{array}{ccccccccccc}
X_\bullet & & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_\bullet} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_\bullet & & Y_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma Y_1
\end{array}$$

commutes. It is an *isomorphism* if $\varphi_1, \dots, \varphi_n$ are all isomorphisms in $\mathscr{T}$, and a *weak isomorphism* if φ_i and φ_{i+1} are isomorphisms for some $1 \le i \le n$ (with $\varphi_{n+1} = \Sigma\varphi_1$).

The category $\mathscr{T}$ is *pre-n-angulated* if there exists a collection $\mathscr{N}$ of n-Σ-sequences satisfying the following three axioms:

(N1) (a) $\mathscr{N}$ is closed under direct sums, direct summands and isomorphisms of n-Σ-sequences.
(b) For all $X \in \mathscr{T}$, the trivial n-Σ-sequence $(TX)_\bullet$ belongs to $\mathscr{N}$.
(c) For each morphism $\alpha_1 : X_1 \to X_2$ in $\mathscr{T}$, there exists an n-Σ-sequence in $\mathscr{N}$ whose first morphism is α_1.

(N2) An n-Σ-sequence $X_\bullet$ belongs to $\mathscr{N}$ if and only if its left rotation belongs to $\mathscr{N}$.

(N3) Each commutative diagram with rows in $\mathscr{N}$

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \vdots{\scriptstyle \varphi_3} & & & & \vdots{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & Y_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma Y_1
\end{array}$$

can be completed to a morphism of n-Σ-sequences.

In this case, the collection $\mathscr{N}$ is a pre-n-angulation of the category $\mathscr{T}$ (relative to the automorphism Σ), and the n-Σ-sequences in $\mathscr{N}$ are called *n-angles*.

Let $X_\bullet$ and $Y_\bullet$ be two n-Σ-sequences and φ, ψ be two morphisms from $X_\bullet$ to $Y_\bullet$. A *homotopy* Θ from φ to ψ is given by diagonal morphisms Θ_i

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
{\scriptstyle \varphi_1}\downdownarrows{\scriptstyle \psi_1} & \swarrow{\scriptstyle \Theta_1} & {\scriptstyle \varphi_2}\downdownarrows{\scriptstyle \psi_2} & \swarrow{\scriptstyle \Theta_2} & {\scriptstyle \varphi_3}\downdownarrows{\scriptstyle \psi_3} & & & & {\scriptstyle \varphi_n}\downdownarrows{\scriptstyle \psi_n} & \swarrow{\scriptstyle \Theta_n} & {\scriptstyle \Sigma\varphi_1}\downdownarrows{\scriptstyle \Sigma\psi_1} \\
Y_1 & \xrightarrow[\beta_1]{} & Y_2 & \xrightarrow[\beta_2]{} & Y_3 & \xrightarrow[\beta_3]{} & \cdots & \xrightarrow[\beta_{n-1}]{} & Y_n & \xrightarrow[\beta_n]{} & \Sigma Y_1
\end{array}$$

such that

$$\varphi_i - \psi_i = \Theta_i\alpha_i + \beta_{i-1}\Theta_{i-1} \quad \text{for } i = 2, 3, \dots, n,$$
$$\Sigma\varphi_1 - \Sigma\psi_1 = \Sigma\Theta_1\Sigma\alpha_1 + \beta_n\Theta_n.$$

In this case, we say that φ and ψ are *homotopic*. A morphism homotopic to the zero morphism is called *null-homotopic*.

Remark 2.1. Let

$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1$$

be an n-Σ-sequence in $\mathscr{N}$.

(1) One checks that if you change the sign on any $n-1$ of $\alpha_1, \dots, \alpha_n$, then the resulting n-Σ-sequence is still in $\mathscr{N}$.

(2) It follows from (N2) that the following n-Σ-sequence

$$\Sigma^\ell X_1 \xrightarrow{(-1)^{\ell n}\Sigma^\ell \alpha_1} \Sigma^\ell X_2 \xrightarrow{(-1)^{\ell n}\Sigma^\ell \alpha_2} \dots \xrightarrow{(-1)^{\ell n}\, \Sigma^\ell \alpha_{n-1}} \Sigma^\ell X_n \xrightarrow{(-1)^{\ell n}\Sigma^\ell \alpha_n} \Sigma^{\ell+1} X_1$$

is still in $\mathscr{N}$ for any $\ell \in \mathbb{Z}$.

(3) All n-angles are exact by [3, Proposition 2.5].

Lemma 2.2. *Let* $X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1 \in \mathscr{N}$. *Then,*

$$\dots \to \operatorname{Hom}_{\mathscr{T}}(\Sigma X_1, Y) \to \operatorname{Hom}_{\mathscr{T}}(X_n, Y) \to \dots \to \operatorname{Hom}_{\mathscr{T}}(X_1, Y) \to \dots$$

is exact in Ab *for any* $Y \in \mathscr{T}$.

Proof. This is almost verbatim the dual argument as the corresponding statement for pre-triangulated categories, see for example [8, Section 1.1.10]. □

Let $\mathscr{C}$ be an additive category and $f : A \to B$ a morphism in $\mathscr{C}$. A *weak cokernel* of f is a morphism $g : B \to C$ such that for all $C' \in \mathscr{C}$ the sequence

$$\operatorname{Hom}_{\mathscr{C}}(C, C') \xrightarrow{g^*} \operatorname{Hom}_{\mathscr{C}}(B, C') \xrightarrow{f^*} \operatorname{Hom}_{\mathscr{C}}(A, C')$$

is exact in Ab. Equivalently, g is a weak cokernel of f if $gf = 0$ and for each morphism $h : B \to C'$ with $hf = 0$ there exists a morphism $p : C \to C'$ such that $h = pg$. The concept of *weak kernel* is defined dually.

Lemma 2.3. *Let* $X_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{i-1}} X_i$ *be a sequence in* $\mathscr{T}$. *If* α_{j-1} *is a weak kernel of* α_j *for* $2 \le j \le i-1$, *then there exists an* n*-angle*

$$X_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{i-1}} X_i \xrightarrow{\alpha_i} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1.$$

Proof. This follows directly from (N1) for $i = 2$. Assume that $3 \le i \le n-2$. Applying (N1) and (N2) for α_2, there exists an n-angle

$$X_1' \xrightarrow{\alpha_1'} X_2 \xrightarrow{\alpha_2} X_3 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n'} \Sigma X_1'.$$

Since $\alpha_2 \alpha_1' = 0$, there is $\varphi_1 : X_1' \to X_1$ such that the following diagram

$$\begin{array}{ccccccccccc}
X_1' & \xrightarrow{\alpha_1'} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n'} & \Sigma X_1' \\
\downarrow{\scriptstyle \varphi_1} & & \| & & \| & & & & \| & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{(\Sigma\varphi_1)\alpha_n'} & \Sigma X_1
\end{array}$$

commutes. By the higher analogue of the 5-Lemma, φ_1 is an isomorphism. We thus obtain the desired n-angle. Assume that $i = n-1$. There exists an n-angle as follows by induction:

$$X_1' \xrightarrow{\alpha_1'} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n'} \Sigma X_1'.$$

By analogy with the preceding proof, we obtain the desired n-angle. □

Lemma 2.4. *Each commutative diagram with rows in $\mathcal{N}$*

$$\begin{array}{ccccccccccccccc} X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{i-1}} & X_i & \xrightarrow{\alpha_i} & X_{i+1} & \xrightarrow{\alpha_{i+1}} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\ \downarrow{\varphi_1} & & \downarrow{\varphi_2} & & & & \downarrow{\varphi_i} & & \vdots{\varphi_{i+1}} & & & & \vdots{\varphi_n} & & \downarrow{\Sigma\varphi_1} \\ Y_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{i-1}} & Y_i & \xrightarrow{\beta_i} & Y_{i+1} & \xrightarrow{\beta_{i+1}} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma Y_1 \end{array}$$

can be completed to a morphism of n-Σ-sequences, where $2 \le i \le n-1$. In particular, the sequence constructed is homotopic to the one obtained by (N3) *from* (φ_1, φ_2).

Proof. If $i = 2$, then we are done. Assume that $i > 2$. By (N3), we may choose $\varphi_3', \dots, \varphi_n'$ such that $(\varphi_1, \varphi_2, \varphi_3', \dots, \varphi_n')$ is a morphism of n-Σ-sequences. Since $\mathrm{Hom}_{\mathcal{T}}(X_{j+1}, Y_j) \xrightarrow{\alpha_j^*} \mathrm{Hom}_{\mathcal{T}}(X_j, Y_j) \xrightarrow{\alpha_{j-1}^*} \mathrm{Hom}_{\mathcal{T}}(X_{j-1}, Y_j)$ is exact by Lemma 2.2, there is $\Theta_j : X_{j+1} \to Y_j$ such that $(\varphi_j - \varphi_j') - \beta_{j-1}\Theta_{j-1} = \Theta_j\alpha_j$ for $3 \le j \le i$, where $\Theta_2 = 0$. Set $\varphi_{i+1} = \beta_i\Theta_i + \varphi_{i+1}'$, $\varphi_j = \varphi_j'$ for $i+2 \le j \le n$ and $\Theta_j = 0$ for $i+1 \le j \le n$. Then, $(\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_{n-1}, \varphi_n)$ is a morphism of n-Σ-sequences. □

Next we show that if an n-Σ-sequence looks like it might admit one of these n-angles as a direct summand, then it actually does (loosely speaking).

Lemma 2.5. *Suppose we have an n-angle of the form*

$$A \oplus X_1 \xrightarrow{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}} A \oplus X_2 \xrightarrow{[a \ \ \alpha_2]} X_3 \xrightarrow{\alpha_3} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\begin{bmatrix} b \\ \alpha_n \end{bmatrix}} \Sigma(A \oplus X_1).$$

If $\alpha \in \mathrm{Aut}(A)$, then it is isomorphic to a direct sum of n-angles

$$A \xrightarrow{1} A \to 0 \to \dots \to 0 \to \Sigma A,$$

$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1,$$

where $\alpha_1 = \delta - \gamma\alpha^{-1}\beta$.

Proof. Since the composition of maps in an n-angle is zero, we have that $a = -\alpha_2\gamma\alpha^{-1}$ and $(\Sigma\alpha)b + (\Sigma\beta)\alpha_n = 0$. This yields an isomorphism of n-Σ-sequences

$$\begin{array}{ccccccccccccc} A \oplus X_1 & \xrightarrow{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}} & A \oplus X_2 & \xrightarrow{[a \ \ \alpha_2]} & X_3 & \xrightarrow{\alpha_3} & X_4 & \xrightarrow{\alpha_4} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\begin{bmatrix} b \\ \alpha_n \end{bmatrix}} & \Sigma(A \oplus X_1) \\ \downarrow{\begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}} & & \downarrow{\begin{bmatrix} 1 & 0 \\ -\gamma\alpha^{-1} & 1 \end{bmatrix}} & & \| & & \| & & & & \| & & \downarrow{\begin{bmatrix} \Sigma\alpha & \Sigma\beta \\ 0 & 1 \end{bmatrix}} \\ A \oplus X_1 & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & \alpha_1 \end{bmatrix}} & A \oplus X_2 & \xrightarrow{[0 \ \ \alpha_2]} & X_3 & \xrightarrow{\alpha_3} & X_4 & \xrightarrow{\alpha_4} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\begin{bmatrix} 0 \\ \alpha_n \end{bmatrix}} & \Sigma(A \oplus X_1), \end{array}$$

where $\alpha_1 = \delta - \gamma\alpha^{-1}\beta$. The proof is complete. □

Use the same arguments as above, we have the following lemmas.

Lemma 2.6. *Suppose we have an n-angle of the form*

$$X_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{i-2}} X_{i-1} \xrightarrow{\begin{bmatrix} a \\ \alpha_{i-1} \end{bmatrix}} A \oplus X_i \xrightarrow{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}} A \oplus X_{i+1} \xrightarrow{[b \ \ \alpha_{i+1}]} X_{i+2} \xrightarrow{\alpha_{i+2}} \dots \to X_n \xrightarrow{\alpha_n} \Sigma X_1$$

for $i = 2, \dots, n-1$. If $\alpha \in \mathrm{Aut}(A)$, then it is isomorphic to a direct sum of n-angles

$$0 \to \dots \to 0 \to A \xrightarrow{1} A \to 0 \to \dots \to 0,$$

$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1,$$

where $\alpha_i = \delta - \gamma\alpha^{-1}\beta$ for $i = 2, \dots, n-1$.

Lemma 2.7. *Suppose we have an n-angle of the form*

$$A \oplus X_1 \xrightarrow{[a\ \alpha_1]} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-2}} X_{n-1} \xrightarrow{\begin{bmatrix} b \\ \alpha_{n-1} \end{bmatrix}} \Sigma A \oplus X_n \xrightarrow{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}} \Sigma(A \oplus X_1).$$

If $\alpha \in \mathrm{Aut}(A)$, *then it is isomorphic to a direct sum of n-angles*

$$A \to 0 \to \dots \to 0 \to \Sigma A \xrightarrow{1} \Sigma A,$$
$$X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1,$$

where $\alpha_n = \delta - \gamma\alpha^{-1}\beta$.

3 Additional axioms

The aim of this section is to introduce some possible additional axioms for a pre-n-angulated category, which are inspired by works of Hubery [9]. A pre-n-angulated category $(\mathscr{T}, \Sigma, \mathscr{N})$ is called *n-angulated* if it satisfies any of these extra axioms. We briefly describe the axioms before giving the precise formulation.

(N4) The following commutative diagram with rows in $\mathscr{N}$

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \vdots{\scriptstyle \varphi_3} & & & & \vdots{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & Y_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma Y_1
\end{array}$$

can be completed to a morphism of n-angles such that the mapping cone

$$X_2 \oplus Y_1 \xrightarrow{\begin{bmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} \dots \xrightarrow{\begin{bmatrix} -\alpha_n & 0 \\ \varphi_n & \beta_{n-1} \end{bmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{bmatrix} -\Sigma\alpha_1 & 0 \\ \Sigma\varphi_1 & \beta_n \end{bmatrix}} \Sigma X_2 \oplus \Sigma Y_1$$

is in $\mathscr{N}$.

Axiom A. The following commutative diagram with rows in $\mathscr{N}$

$$\begin{array}{ccccccccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{i-1}} & X_i & \xrightarrow{\alpha_i} & X_{i+1} & \xrightarrow{\alpha_{i+1}} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_i} & & \vdots{\scriptstyle \varphi_{i+1}} & & & & \vdots{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{i-1}} & Y_i & \xrightarrow{\beta_i} & Y_{i+1} & \xrightarrow{\beta_{i+1}} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma Y_1
\end{array}$$

for $2 \leq i \leq n-1$ can be completed to a morphism of n-angles such that the mapping cone

$$X_2 \oplus Y_1 \xrightarrow{\begin{bmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} \dots \xrightarrow{\begin{bmatrix} -\alpha_n & 0 \\ \varphi_n & \beta_{n-1} \end{bmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{bmatrix} -\Sigma\alpha_1 & 0 \\ \Sigma\varphi_1 & \beta_n \end{bmatrix}} \Sigma X_2 \oplus \Sigma Y_1$$

is in $\mathscr{N}$.

Axiom B_0. The following commutative diagram with rows in $\mathscr{N}$

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \vdots{\scriptstyle \varphi_3} & & & & \vdots{\scriptstyle \varphi_n} & & \| \\
X_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & Y_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma X_1
\end{array}$$

can be completed to a morphism of n-angles such that the n-Σ-sequence

$$X_2 \overset{\left[\begin{smallmatrix}-\alpha_2\\ \varphi_2\end{smallmatrix}\right]}{\to} X_3 \oplus Y_2 \overset{\left[\begin{smallmatrix}-\alpha_3 & 0\\ \varphi_3 & \beta_2\end{smallmatrix}\right]}{\to} \cdots \overset{\left[\begin{smallmatrix}-\alpha_{n-1} & 0\\ \varphi_{n-1} & \beta_{n-2}\end{smallmatrix}\right]}{\to} X_n \oplus Y_{n-1} \overset{[\varphi_n\ \beta_{n-1}]}{\to} Y_n \overset{(\Sigma\alpha_1)\beta_n}{\to} \Sigma X_2$$

is in $\mathscr{N}$.

Axioms B_0 is a special case of Axiom (N4) when one of the maps is known to be an isomorphism. Axiom B_0 can be thought of as analogous to the existence of $(n-2)$-pushout diagrams and $(n-2)$-pullback diagrams in an $(n-2)$-abelian category [10].

Axiom B_1. The following commutative diagram with rows in $\mathscr{N}$

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} & & \| \\
X_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-2}} & Y_{n-1} & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma X_1
\end{array}$$

can be completed to a morphism of n-angles such that the n-Σ-sequence

$$X_2 \overset{\left[\begin{smallmatrix}-\alpha_2\\ \varphi_2\end{smallmatrix}\right]}{\to} X_3 \oplus Y_2 \overset{\left[\begin{smallmatrix}-\alpha_3 & 0\\ \varphi_3 & \beta_2\end{smallmatrix}\right]}{\to} \cdots \overset{\left[\begin{smallmatrix}-\alpha_{n-1} & 0\\ \varphi_{n-1} & \beta_{n-2}\end{smallmatrix}\right]}{\to} X_n \oplus Y_{n-1} \overset{[\varphi_n\ \beta_{n-1}]}{\to} Y_n \overset{(\Sigma\alpha_1)\beta_n}{\to} \Sigma X_2$$

is in $\mathscr{N}$.

Axiom C. Given an n-Σ-sequence in $\mathscr{N}$

$$X_2 \overset{\left[\begin{smallmatrix}-\alpha_2\\ \varphi_2\end{smallmatrix}\right]}{\to} X_3 \oplus Y_2 \overset{\left[\begin{smallmatrix}-\alpha_3 & 0\\ \varphi_3 & \beta_2\end{smallmatrix}\right]}{\to} \cdots \overset{\left[\begin{smallmatrix}-\alpha_{n-1} & 0\\ \varphi_{n-1} & \beta_{n-2}\end{smallmatrix}\right]}{\to} X_n \oplus Y_{n-1} \overset{[\varphi_n\ \beta_{n-1}]}{\to} Y_n \overset{\delta}{\to} \Sigma X_2.$$

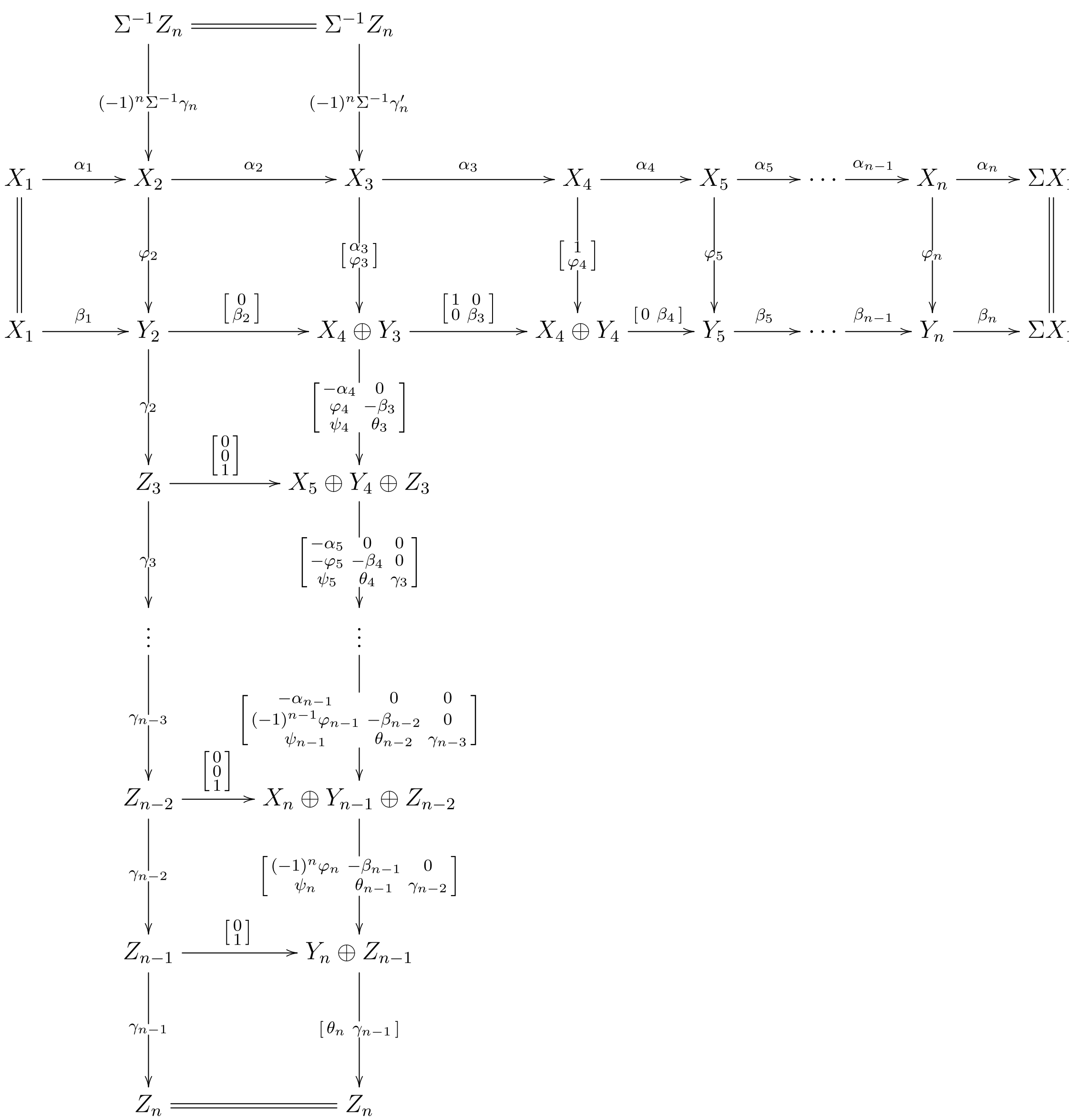
If $X_1 \overset{\alpha_1}{\to} X_2 \overset{\alpha_2}{\to} \cdots \overset{\alpha_{n-1}}{\to} X_n \overset{\alpha_n}{\to} \Sigma X_1 \in \mathscr{N}$ or $X_1 \overset{\beta_1}{\to} Y_2 \overset{\beta_2}{\to} \cdots \overset{\beta_{n-1}}{\to} Y_n \overset{\beta_n}{\to} \Sigma X_1 \in \mathscr{N}$, then there exists a commutative diagram with rows in $\mathscr{N}$

$$\begin{array}{ccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_n} & & \| \\
X_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma X_1
\end{array}$$

such that $\delta = (\Sigma\alpha_1)\beta_n$.

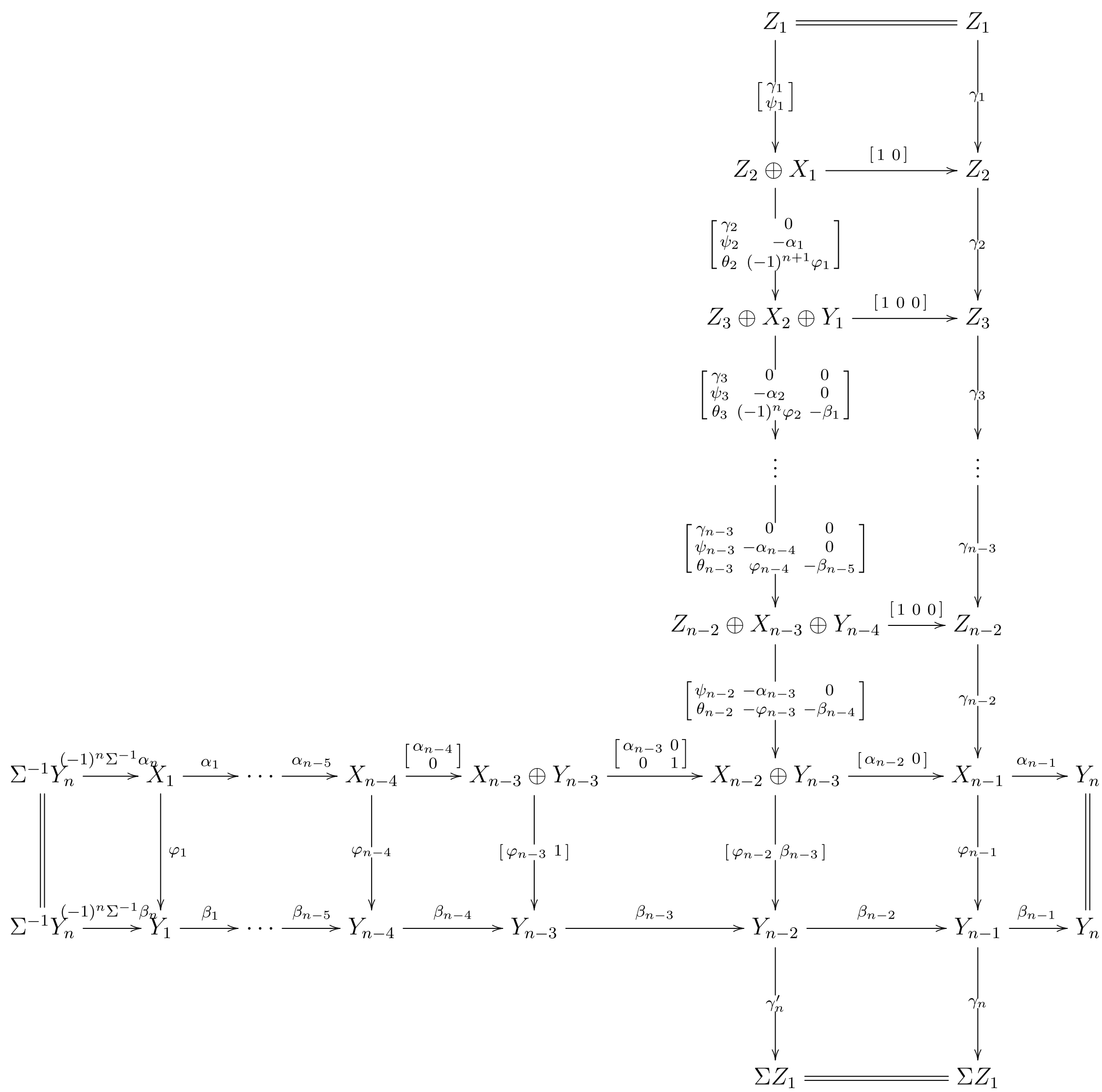
Axiom C is a kind of converse to Axiom B_1 and Axiom B_1', and can be thought of as analogous to the fact that parallel maps in an $(n-2)$-pullback/$(n-2)$-pushout diagram, α_2, β_2 have isomorphic kernels and $\alpha_{n-1}, \beta_{n-1}$ have isomorphic cokernels.

Axiom D. For any n-Σ-sequence $X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1 \in \mathcal{N}$ and any morphism $\varphi_2 : X_2 \to Y_2$, there exists a commutative diagram of n-angles



such that $\gamma_n\theta_n = (\Sigma\alpha_1)\beta_n$.

Axiom D'. For any n-Σ-sequence $Y_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-1}} Y_n \xrightarrow{\beta_n} \Sigma Y_1 \in \mathcal{N}$ and any morphism $\varphi_{n-1} : X_{n-1} \to Y_{n-1}$, there exists a commutative diagram of n-angles



such that $(\Sigma\psi_1)\gamma_n = \alpha_n\beta_{n-1}$.

Axiom D and Axiom D′ can be thought to be “higher dimensional” analogues of cobase change and base change for triangulated categories [11].

4 Equivalence of the additional axioms

In this section, we prove that all possible axioms in the above section are equivalent, which can be applied to explain the higher mapping axiom (N4).

(N4) implies A. Suppose that (N4) holds and that we are given a diagram as in A. We may choose $\varphi_3', \dots, \varphi_n'$ such that the mapping cone

$$X_2 \oplus Y_1 \xrightarrow{\begin{bmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3' & \beta_2 \end{bmatrix}} \dots \xrightarrow{\begin{bmatrix} -\alpha_n & 0 \\ \varphi_n' & \beta_{n-1} \end{bmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{bmatrix} -\Sigma\alpha_1 & 0 \\ \Sigma\varphi_1 & \beta_n \end{bmatrix}} \Sigma X_2 \oplus \Sigma Y_1 \tag{†}$$

is in $\mathcal{N}$. It follows from Lemma 2.4 that $(\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n)$ and $(\varphi_1, \varphi_2, \varphi_3', \ldots, \varphi_n')$ are homotopic. Hence, [4, Lemma 2.1] implies that the mapping cone

$$X_2 \oplus Y_1 \xrightarrow{\begin{bmatrix}-\alpha_2 & 0\\ \varphi_2 & \beta_1\end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix}-\alpha_3 & 0\\ \varphi_3 & \beta_2\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}-\alpha_n & 0\\ \varphi_n & \beta_{n-1}\end{bmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{bmatrix}-\Sigma\alpha_1 & 0\\ \Sigma\varphi_1 & \beta_n\end{bmatrix}} \Sigma X_2 \oplus \Sigma Y_1 \tag{‡}$$

is isomorphic to (†), and so (‡) belongs to $\mathcal{N}$.

A implies (N4). Suppose that A holds and that we are given a diagram as in (N4). Then, there is a morphism $(\varphi_1, \varphi_2, \ldots \varphi_i, \varphi_{i+1}', \ldots, \varphi_n')$ of n-angles by (N3), and so we have a diagram as in A. Hence, A yields $\varphi_{i+1}, \ldots, \varphi_n$ such that the mapping cone

$$X_2 \oplus Y_1 \xrightarrow{\begin{bmatrix}-\alpha_2 & 0\\ \varphi_2 & \beta_1\end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix}-\alpha_3 & 0\\ \varphi_3 & \beta_2\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}-\alpha_n & 0\\ \varphi_n & \beta_{n-1}\end{bmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{bmatrix}-\Sigma\alpha_1 & 0\\ \Sigma\varphi_1 & \beta_n\end{bmatrix}} \Sigma X_2 \oplus \Sigma Y_1$$

is in $\mathcal{N}$.

(N4) implies B_0. Suppose that (N4) holds and that we are given a diagram as in B_0. We may choose $\varphi_3, \ldots, \varphi_n$ such that the following n-Σ-sequence

$$X_1 \oplus X_2 \xrightarrow{\begin{bmatrix}0 & -\alpha_2\\ \beta_1 & \varphi_2\end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix}-\alpha_3 & 0\\ \varphi_3 & \beta_2\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}-\alpha_n & 0\\ \varphi_n & \beta_{n-1}\end{bmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{bmatrix}1 & \beta_n\\ -\Sigma\alpha_1 & 0\end{bmatrix}} \Sigma X_1 \oplus \Sigma X_2$$

is in $\mathcal{N}$. Thus, by Lemma 2.7, this has the following n-Σ-sequence

$$X_2 \xrightarrow{\begin{bmatrix}-\alpha_2\\ \varphi_2\end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix}-\alpha_3 & 0\\ \varphi_3 & \beta_2\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}-\alpha_{n-1} & 0\\ \varphi_{n-1} & \beta_{n-2}\end{bmatrix}} X_n \oplus Y_{n-1} \xrightarrow{[\varphi_n\ \beta_{n-1}]} Y_n \xrightarrow{(\Sigma\alpha_1)\beta_n} \Sigma X_2$$

as a direct summand. Hence, B_0 holds.

B_0 implies (N4). Suppose that B_0 holds and that we are given a diagram as in (N4). We begin by considering an isomorphism of n-Σ-sequences

$$\begin{array}{ccccccccccc}
X_1 \oplus Y_1 & \xrightarrow{[0\ \beta_1]} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-2}} & Y_{n-1} & \xrightarrow{\begin{bmatrix}0\\ \beta_{n-1}\end{bmatrix}} & \Sigma X_1 \oplus Y_n & \xrightarrow{\begin{bmatrix}-1 & 0\\ 0 & \beta_n\end{bmatrix}} & \Sigma X_1 \oplus \Sigma Y_1\\
\downarrow{\scriptstyle\begin{bmatrix}1 & 0\\ -\varphi_1 & 1\end{bmatrix}} & & \| & & & & \| & & \| & & \downarrow{\scriptstyle\begin{bmatrix}1 & 0\\ -\Sigma\varphi_1 & 1\end{bmatrix}}\\
X_1 \oplus Y_1 & \xrightarrow{[\varphi_2\alpha_1\ \beta_1]} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-2}} & Y_{n-1} & \xrightarrow{\begin{bmatrix}0\\ \beta_{n-1}\end{bmatrix}} & \Sigma X_1 \oplus Y_n & \xrightarrow{\begin{bmatrix}-1 & 0\\ \Sigma\varphi_1 & \beta_n\end{bmatrix}} & \Sigma X_1 \oplus \Sigma Y_1.
\end{array}$$

Since the top row is in $\mathcal{N}$, so is the bottom row. Applying B_0, the following commutative diagram with rows in $\mathcal{N}$

$$\begin{array}{ccccccccccccc}
X_1 \oplus Y_1 & \xrightarrow{\begin{bmatrix}\alpha_1 & 0\\ 0 & 1\end{bmatrix}} & X_2 \oplus Y_1 & \xrightarrow{[\alpha_2\ 0]} & X_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\begin{bmatrix}\alpha_n\\ 0\end{bmatrix}} & \Sigma X_1 \oplus \Sigma Y_1\\
\| & & \downarrow{\scriptstyle[\varphi_2\ \beta_1]} & & \downarrow{\scriptstyle\varphi_3} & & & & \downarrow{\scriptstyle\varphi_{n-1}} & & \downarrow{\scriptstyle\begin{bmatrix}-a\\ \varphi_n\end{bmatrix}} & & \|\\
X_1 \oplus Y_1 & \xrightarrow{[\varphi_2\alpha_1\ \beta_1]} & Y_2 & \xrightarrow{\beta_2} & Y_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-2}} & Y_{n-1} & \xrightarrow{\begin{bmatrix}0\\ \beta_{n-1}\end{bmatrix}} & \Sigma X_1 \oplus Y_n & \xrightarrow{\begin{bmatrix}-1 & 0\\ \Sigma\varphi_1 & \beta_n\end{bmatrix}} & \Sigma X_1 \oplus \Sigma Y_1
\end{array}$$

can be completed to a morphism of n-angles and such that the n-Σ-sequence

$$X_2 \oplus Y_1 \xrightarrow{\begin{bmatrix}-\alpha_2 & 0\\ \varphi_2 & \beta_1\end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix}-\alpha_3 & 0\\ \varphi_3 & \beta_2\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}-a & 0\\ \varphi_n & \beta_{n-1}\end{bmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{bmatrix}-\Sigma\alpha_1 & 0\\ \Sigma\varphi_1 & \beta_n\end{bmatrix}} \Sigma X_2 \oplus \Sigma Y_1$$

is in $\mathcal{N}$. Note that $\begin{bmatrix}\alpha_n\\ 0\end{bmatrix} = \begin{bmatrix}-1 & 0\\ \Sigma\varphi_1 & \beta_n\end{bmatrix}\begin{bmatrix}-a\\ \varphi_n\end{bmatrix}$, and so $a = \alpha_n$. Hence, (N4) holds.

The proof of A ⇔ B_1 is similar to that of (N4) ⇔ B_0.

B_1 implies C. Suppose that B_1 holds and that we are given an n-angle as in C. Suppose further that we fix the n-angle $X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} \Sigma X_1$. It is easy to verify that $X_2 \xrightarrow{-\alpha_2} X_3 \xrightarrow{-\alpha_3} \cdots \xrightarrow{-\alpha_{n-1}} X_n \xrightarrow{-\alpha_n} \Sigma X_1 \xrightarrow{-\Sigma\alpha_1} \Sigma X_2 \in \mathcal{N}$.

Applying B_1, the following commutative diagram

$$
\begin{array}{ccccccccccc}
X_2 & \xrightarrow{\begin{bmatrix}-\alpha_2\\ \varphi_2\end{bmatrix}} & X_3\oplus Y_2 & \xrightarrow{\begin{bmatrix}-\alpha_3 & 0\\ \varphi_3 & \beta_2\end{bmatrix}} & \cdots & \xrightarrow{\begin{bmatrix}-\alpha_{n-1} & 0\\ \varphi_{n-1} & \beta_{n-2}\end{bmatrix}} & X_n\oplus Y_{n-1} & \xrightarrow{[\varphi_n\ \beta_{n-1}]} & Y_n & \xrightarrow{\delta} & \Sigma X_2\\
\| & & \downarrow{\scriptstyle [1\ 0]} & & & & \downarrow{\scriptstyle [1\ 0]} & & \vdots{\scriptstyle -\beta_n} & & \|\\
X_2 & \xrightarrow{-\alpha_2} & X_3 & \xrightarrow{-\alpha_3} & \cdots & \xrightarrow{-\alpha_{n-1}} & X_n & \xrightarrow{-\alpha_n} & \Sigma X_1 & \xrightarrow{-\Sigma\alpha_1} & \Sigma X_2
\end{array}
$$

can be completed to a morphism of n-angles and such that the n-Σ-sequence

$$
X_3\oplus Y_2 \xrightarrow{\begin{bmatrix}1 & 0\\ \alpha_3 & 0\\ -\varphi_3 & -\beta_2\end{bmatrix}} X_3\oplus X_4\oplus Y_3 \xrightarrow{\begin{bmatrix}-\alpha_3 & 1 & 0\\ 0 & \alpha_4 & 0\\ 0 & -\varphi_4 & -\beta_3\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}-\alpha_{n-2} & 1 & 0\\ 0 & \alpha_{n-1} & 0\\ 0 & -\varphi_{n-1} & -\beta_{n-2}\end{bmatrix}} X_{n-1}\oplus X_n\oplus Y_{n-1} \xrightarrow{\begin{bmatrix}-\alpha_{n-1} & 1 & 0\\ 0 & -\varphi_n & -\beta_{n-1}\end{bmatrix}} X_n\oplus Y_n
$$
$$
\xrightarrow{[-\alpha_n\ -\beta_n]} \Sigma X_1 \xrightarrow{\begin{bmatrix}0\\ -\Sigma(\varphi_2\alpha_1)\end{bmatrix}} \Sigma(X_3\oplus Y_2)
$$

is in $\mathscr{N}$. Note that $\delta = (\Sigma\alpha_1)\beta_n$ and $\alpha_n = \beta_n\varphi_n$. By Lemma 2.5, this has the n-Σ-sequence

$$
Y_2 \xrightarrow{\begin{bmatrix}0\\ -\beta_2\end{bmatrix}} X_4\oplus Y_3 \xrightarrow{\begin{bmatrix}1 & 0\\ \alpha_4 & 0\\ -\varphi_4 & -\beta_3\end{bmatrix}} X_4\oplus X_5\oplus Y_4 \xrightarrow{\begin{bmatrix}-\alpha_4 & 1 & 0\\ 0 & \alpha_5 & 0\\ 0 & -\varphi_5 & -\beta_4\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}-\alpha_{n-2} & 1 & 0\\ 0 & \alpha_{n-1} & 0\\ 0 & -\varphi_{n-1} & -\beta_{n-2}\end{bmatrix}} X_{n-1}\oplus X_n\oplus Y_{n-1} \xrightarrow{\begin{bmatrix}-\alpha_{n-1} & 1 & 0\\ 0 & -\varphi_n & -\beta_{n-1}\end{bmatrix}} X_n\oplus Y_n
$$
$$
\xrightarrow{[-\alpha_n\ -\beta_n]} \Sigma X_1 \xrightarrow{-\Sigma(\varphi_2\alpha_1)} \Sigma Y_2
$$

as a direct summand. Rotating the above n-angle and using Lemma 2.5 again, we obtain the following n-angle:

$$
Y_3 \xrightarrow{\begin{bmatrix}0\\ -\beta_3\end{bmatrix}} X_5\oplus Y_4 \xrightarrow{\begin{bmatrix}1 & 0\\ \alpha_5 & 0\\ -\varphi_5 & -\beta_4\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}-\alpha_{n-2} & 1 & 0\\ 0 & \alpha_{n-1} & 0\\ 0 & -\varphi_{n-1} & -\beta_{n-2}\end{bmatrix}} X_{n-1}\oplus X_n\oplus Y_{n-1} \xrightarrow{\begin{bmatrix}-\alpha_{n-1} & 1 & 0\\ 0 & -\varphi_n & -\beta_{n-1}\end{bmatrix}} X_n\oplus Y_n \xrightarrow{[-\alpha_n\ -\beta_n]} \Sigma X_1 \xrightarrow{-\Sigma(\varphi_2\alpha_1)} \Sigma Y_2
$$
$$
\xrightarrow{(-1)^{n+1}\Sigma\beta_2} \Sigma Y_3.
$$

Continuing this process, rotating the n-Σ-sequence and setting $\beta_1 = \varphi_2\alpha_1$. We obtain the required commutative diagram with rows in $\mathscr{N}$.

Suppose instead we had fixed an n-angle $X_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-1}} Y_n \xrightarrow{\beta_n} \Sigma X_1$. We apply the above construction to obtain n-angles $X_1 \xrightarrow{\overline{\alpha}_1} X_2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\overline{\alpha}_n} \Sigma X_1$ and $X_1 \xrightarrow{\overline{\beta}_1} Y_2 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_{n-1}} Y_n \xrightarrow{\overline{\beta}_n} \Sigma X_1$ satisfying Axiom C. By Lemma 2.4, we have the following commutative diagram of n-angles

$$
\begin{array}{ccccccccc}
X_1 & \xrightarrow{\overline{\beta}_1} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\overline{\beta}_n} & \Sigma X_1\\
\downarrow{\scriptstyle a} & & \| & & & & \| & & \downarrow{\scriptstyle \Sigma a}\\
X_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma X_1.
\end{array}
$$

By the higher analogue of the 5-Lemma, $a \in \mathrm{Aut}(X_1)$. Set $\alpha_1 = \overline{\alpha}_1 a^{-1}$ and $\alpha_n = (\Sigma a)\overline{\alpha}_n$. Then, we have a commutative diagram with rows in $\mathscr{N}$:

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
\downarrow{\scriptstyle a^{-1}} & & \| & & \| & & & & \| & & \downarrow{\scriptstyle \Sigma a^{-1}} \\
X_1 & \xrightarrow{\overline{\alpha}_1} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\overline{\alpha}_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \| \\
X_1 & \xrightarrow{\overline{\beta}_1} & Y_2 & \xrightarrow{\beta_2} & Y_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\overline{\beta}_n} & \Sigma X_1 \\
\downarrow{\scriptstyle a} & & \| & & \| & & & & \| & & \downarrow{\scriptstyle \Sigma a} \\
X_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & Y_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma X_1.
\end{array}$$

We obtain the required commutative diagram of n-angles.

C implies B_1. Suppose that C holds and that we are given a diagram as in B_1. Let $A \in \mathscr{T}$. Applying the functor $\mathrm{Hom}_{\mathscr{T}}(A, -)$ to the commutative diagram in B_1, we obtain the following exact sequence of abelian groups:

$$\mathrm{Hom}_{\mathscr{T}}(A, X_2) \xrightarrow{\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}_*} \mathrm{Hom}_{\mathscr{T}}(A, X_3 \oplus Y_2) \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}_*} \dots \xrightarrow{\begin{bmatrix} -\alpha_{n-1} & 0 \\ \varphi_{n-1} & \beta_{n-2} \end{bmatrix}_*} \mathrm{Hom}_{\mathscr{T}}(A, X_n \oplus Y_{n-1}).$$

Hence, Lemma 2.3 yields an n-angle:

$$X_2 \xrightarrow{\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} \dots \xrightarrow{\begin{bmatrix} -\alpha_{n-1} & 0 \\ \varphi_{n-1} & \beta_{n-2} \end{bmatrix}} X_n \oplus Y_{n-1} \xrightarrow{[a \ b]} W \xrightarrow{c} \Sigma X_2.$$

We can apply C to obtain a commutative diagram of n-angles

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{n-2}} & X_{n-1} & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle a} & & \| \\
X_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-2}} & Y_{n-1} & \xrightarrow{b} & W & \xrightarrow{d} & \Sigma X_1
\end{array}$$

such that $c = (\Sigma\alpha_1)d$. By Lemma 2.4, we have a commutative diagram of n-angles

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-2}} & Y_{n-1} & \xrightarrow{b} & W & \xrightarrow{d} & \Sigma X_1 \\
\| & & \| & & & & \| & & \downarrow{\scriptstyle \theta} & & \| \\
X_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{n-2}} & Y_{n-1} & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma X_1.
\end{array}$$

Clearly, θ is an isomorphism. Set $\varphi_n = \theta a$. Then, $(1, \varphi_2, \dots, \varphi_{n-1}, \varphi_n)$ is a morphism of n-angles. Thus, the following diagram is an isomorphism of n-Σ-sequences

$$\begin{array}{ccccccccccc}
X_2 & \xrightarrow{\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}} & X_3 \oplus Y_2 & \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} & \cdots & \xrightarrow{\begin{bmatrix} -\alpha_{n-1} & 0 \\ \varphi_{n-1} & \beta_{n-2} \end{bmatrix}} & X_n \oplus Y_{n-1} & \xrightarrow{[a \ b]} & W & \xrightarrow{c} & \Sigma X_2 \\
\| & & \| & & & & \| & & \downarrow{\scriptstyle \theta} & & \| \\
X_2 & \xrightarrow{\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}} & X_3 \oplus Y_2 & \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} & \cdots & \xrightarrow{\begin{bmatrix} -\alpha_{n-1} & 0 \\ \varphi_{n-1} & \beta_{n-2} \end{bmatrix}} & X_n \oplus Y_{n-1} & \xrightarrow{[\varphi_n \ \beta_{n-1}]} & Y_n & \xrightarrow{(\Sigma\alpha_1)\beta_n} & \Sigma X_2.
\end{array}$$

Since the top row is in $\mathscr{N}$, so is the bottom row. Hence, B_1 holds.

***B_0* implies D.** Suppose that B_0 holds and that we are given an n-angle and a morphism as in D. By (N1), there are two n-angles

$$X_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{n-1}} Y_n \xrightarrow{\beta_n} \Sigma X_1,$$
$$X_2 \xrightarrow{\varphi_2} Y_2 \xrightarrow{\gamma_2} Z_3 \xrightarrow{\gamma_3} \cdots \xrightarrow{\gamma_{n-1}} Z_n \xrightarrow{\gamma_n} \Sigma X_2,$$

where $\beta_1 = \varphi_2\alpha_1$. One can check that $X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3 \xrightarrow{-\alpha_3} \cdots \xrightarrow{-\alpha_{n-1}} X_n \xrightarrow{(-1)^{n+1}\alpha_n} \Sigma X_1$ belongs to $\mathcal{N}$. Applying B_0, the following commutative diagram

$$\begin{array}{ccccccccccccc}
X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & X_3 & \xrightarrow{-\alpha_3} & \cdots & \xrightarrow{-\alpha_{n-2}} & X_{n-1} & \xrightarrow{-\alpha_{n-1}} & X_n & \xrightarrow{(-1)^{n+1}\alpha_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle (-1)^n\varphi_{n-1}} & & \downarrow{\scriptstyle (-1)^{n+1}\varphi_n} & & \| \\
X_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & Y_3 & \xrightarrow{\beta_3} & \cdots & \xrightarrow{\beta_{n-2}} & Y_{n-1} & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma X_1
\end{array}$$

can be completed to a morphism of n-angles and such that the n-Σ-sequence

$$X_2 \xrightarrow{\begin{bmatrix}-\alpha_2\\ \varphi_2\end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix}\alpha_3 & 0\\ \varphi_3 & \beta_2\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}\alpha_{n-1} & 0\\ (-1)^n\varphi_{n-1} & \beta_{n-2}\end{bmatrix}} X_n \oplus Y_{n-1} \xrightarrow{[(-1)^{n+1}\varphi_n \;\; \beta_{n-1}]} Y_n \xrightarrow{(\Sigma\alpha_1)\beta_n} \Sigma X_2$$

is an n-angle. Again applying B_0, the following commutative diagram

$$\begin{array}{ccccccccccccc}
X_2 & \xrightarrow{\begin{bmatrix}-\alpha_2\\ \varphi_2\end{bmatrix}} & X_3 \oplus Y_2 & \xrightarrow{\begin{bmatrix}\alpha_3 & 0\\ \varphi_3 & \beta_2\end{bmatrix}} & X_4 \oplus Y_3 & \xrightarrow{\begin{bmatrix}\alpha_4 & 0\\ -\varphi_4 & \beta_3\end{bmatrix}} & \cdots & \xrightarrow{\begin{bmatrix}\alpha_{n-1} & 0\\ (-1)^n\varphi_{n-1} & \beta_{n-2}\end{bmatrix}} & X_n \oplus Y_{n-1} & \xrightarrow{[(-1)^{n+1}\varphi_n \;\; \beta_{n-1}]} & Y_n & \xrightarrow{(\Sigma\alpha_1)\beta_n} & \Sigma X_2 \\
\| & & \downarrow{\scriptstyle [0\;1]} & & \downarrow{\scriptstyle [\psi_4\;\theta_3]} & & & & \downarrow{\scriptstyle [\psi_n\;\theta_{n-1}]} & & \downarrow{\scriptstyle \theta_n} & & \| \\
X_2 & \xrightarrow{\varphi_2} & Y_2 & \xrightarrow{\gamma_2} & Z_3 & \xrightarrow{\gamma_3} & \cdots & \xrightarrow{\gamma_{n-2}} & Z_{n-1} & \xrightarrow{\gamma_{n-1}} & Z_n & \xrightarrow{\gamma_n} & \Sigma X_2
\end{array}$$

can be completed to a morphism of n-angles and such that the mapping cone is in $\mathcal{N}$. Hence, Lemma 2.5 implies that the following n-Σ-sequence

$$X_3 \xrightarrow{\begin{bmatrix}\alpha_3\\ \varphi_3\end{bmatrix}} X_4 \oplus Y_3 \xrightarrow{\begin{bmatrix}-\alpha_4 & 0\\ \varphi_4 & -\beta_3\\ \psi_4 & \theta_3\end{bmatrix}} X_5 \oplus Y_4 \oplus Z_3 \xrightarrow{\begin{bmatrix}-\alpha_5 & 0 & 0\\ -\varphi_5 & -\beta_4 & 0\\ \psi_5 & \theta_4 & \gamma_3\end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix}-\alpha_{n-1} & 0 & 0\\ (-1)^{n-1}\varphi_{n-1} & -\beta_{n-2} & 0\\ \psi_{n-1} & \theta_{n-2} & \gamma_{n-3}\end{bmatrix}} X_n \oplus Y_{n-1} \oplus Z_{n-2} \xrightarrow{\begin{bmatrix}(-1)^n\varphi_n & -\beta_{n-1} & 0\\ \psi_n & \theta_{n-1} & \gamma_{n-2}\end{bmatrix}} Y_n \oplus Z_{n-1}$$
$$\xrightarrow{[\theta_n \;\; \gamma_{n-1}]} Z_n \xrightarrow{\gamma_n'} \Sigma X_3$$

is in $\mathcal{N}$, where $\gamma_n' = (\Sigma\alpha_2)\gamma_n$. Note that $\gamma_n\theta_n = (\Sigma\alpha_1)\beta_n$. Hence, D holds.

D implies B_0. Suppose that D holds and that we are given a diagram as in B_0. Then, we have a commutative diagram of n-angles

$$
\begin{array}{ccccccccccccccc}
 & & X_2 & = & X_2 & & & & & & & & & & \\
 & & \big\downarrow{\scriptstyle \begin{bmatrix} -1 \\ \varphi_2 \end{bmatrix}} & & \big\downarrow{\scriptstyle \begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}} & & & & & & & & & & \\
X_1 & \xrightarrow{\begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}} & X_2 \oplus Y_2 & \xrightarrow{\begin{bmatrix} \alpha_2 & 0 \\ 0 & 1 \end{bmatrix}} & X_3 \oplus Y_2 & \xrightarrow{\begin{bmatrix} \alpha_3 & 0 \end{bmatrix}} & X_4 & \xrightarrow{\alpha_4} & X_5 & \xrightarrow{\alpha_5} & \cdots & \xrightarrow{\alpha_{n-1}} & X_n & \xrightarrow{\alpha_n} & \Sigma X_1 \\
\Vert & & \big\downarrow{\scriptstyle \begin{bmatrix} \varphi_2 & 1 \end{bmatrix}} & & \big\downarrow{\scriptstyle \begin{bmatrix} \alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} & & \big\downarrow{\scriptstyle \begin{bmatrix} 1 \\ \varphi_4 \end{bmatrix}} & & \big\downarrow{\scriptstyle \varphi_5} & & & & \big\downarrow{\scriptstyle \varphi_n} & & \Vert \\
X_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\begin{bmatrix} 0 \\ \beta_2 \end{bmatrix}} & X_4 \oplus Y_3 & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & \beta_3 \end{bmatrix}} & X_4 \oplus Y_4 & \xrightarrow{\begin{bmatrix} 0 & \beta_4 \end{bmatrix}} & Y_5 & \xrightarrow{\beta_5} & \cdots & \xrightarrow{\beta_{n-1}} & Y_n & \xrightarrow{\beta_n} & \Sigma X_1 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \begin{bmatrix} -\alpha_4 & 0 \\ \varphi_4 & -\beta_3 \end{bmatrix}} & & & & & & & & & & \\
 & & \vdots & & \vdots & & & & & & & & & & \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \begin{bmatrix} -\alpha_{n-1} & 0 & \\ (-1)^{n-1}\varphi_{n-1} & -\beta_{n-2} & 0 \end{bmatrix}} & & & & & & & & & & \\
 & & 0 & \longrightarrow & X_n \oplus Y_{n-1} & & & & & & & & & & \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \begin{bmatrix} (-1)^n\varphi_n & -\beta_{n-1} \end{bmatrix}} & & & & & & & & & & \\
 & & 0 & \longrightarrow & Y_n & & & & & & & & & & \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle (-1)^{n+1}(\Sigma\alpha_1)\beta_n} & & & & & & & & & & \\
 & & \Sigma X_2 & = & \Sigma X_2. & & & & & & & & & &
\end{array}
$$

One can check that the following n-Σ-sequence

$$X_2 \xrightarrow{\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix} \alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix} \alpha_{n-1} & 0 \\ (-1)^n\varphi_{n-1} & \beta_{n-2} \end{bmatrix}} X_n \oplus Y_{n-1} \xrightarrow{\begin{bmatrix} (-1)^{n+1}\varphi_n & \beta_{n-1} \end{bmatrix}} Y_n \xrightarrow{(\Sigma\alpha_1)\beta_n} \Sigma X_2$$

is an n-angle. We also have an isomorphism of n-Σ-sequences

$$
\begin{array}{ccccccccccc}
X_2 \oplus X_1 & \xrightarrow{\begin{bmatrix} -\alpha_2 & 0 \\ \varphi_2 & \beta_1 \end{bmatrix}} & X_3 \oplus Y_2 & \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} & \cdots & \xrightarrow{\begin{bmatrix} -\alpha_{n-1} & 0 \\ \varphi_{n-1} & \beta_{n-2} \end{bmatrix}} & X_n \oplus Y_{n-1} & \xrightarrow{\begin{bmatrix} -\alpha_n & 0 \\ \varphi_n & \beta_{n-1} \end{bmatrix}} & \Sigma X_1 \oplus Y_n & \xrightarrow{\begin{bmatrix} -\Sigma\alpha_1 & 0 \\ 1 & \beta_n \end{bmatrix}} & \Sigma(X_2 \oplus X_1) \\
\big\downarrow{\scriptstyle \begin{bmatrix} 1 & \alpha_1 \\ 0 & 1 \end{bmatrix}} & & \big\downarrow{\scriptstyle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} & & & & \big\downarrow{\scriptstyle \begin{bmatrix} (-1)^{n+1} & 0 \\ 0 & 1 \end{bmatrix}} & & \big\downarrow{\scriptstyle \begin{bmatrix} 1 & \beta_n \\ 0 & 1 \end{bmatrix}} & & \big\downarrow{\scriptstyle \begin{bmatrix} 1 & \Sigma\alpha_1 \\ 0 & 1 \end{bmatrix}} \\
X_2 \oplus X_1 & \xrightarrow{\begin{bmatrix} -\alpha_2 & 0 \\ \varphi_2 & 0 \end{bmatrix}} & X_3 \oplus Y_2 & \xrightarrow{\begin{bmatrix} \alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} & \cdots & \xrightarrow{\begin{bmatrix} \alpha_{n-1} & 0 \\ (-1)^n\varphi_{n-1} & \beta_{n-2} \end{bmatrix}} & X_n \oplus Y_{n-1} & \xrightarrow{\begin{bmatrix} 0 & 0 \\ (-1)^{n+1}\varphi_n & \beta_{n-1} \end{bmatrix}} & \Sigma X_1 \oplus Y_n & \xrightarrow{\begin{bmatrix} 0 & (\Sigma\alpha_1)\beta_n \\ 1 & 0 \end{bmatrix}} & \Sigma(X_2 \oplus X_1).
\end{array}
$$

Note that the lower row is an n-angle, it follows from Lemma 2.7 that

$$X_2 \xrightarrow{\begin{bmatrix} -\alpha_2 \\ \varphi_2 \end{bmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{bmatrix} -\alpha_3 & 0 \\ \varphi_3 & \beta_2 \end{bmatrix}} \cdots \xrightarrow{\begin{bmatrix} -\alpha_{n-1} & 0 \\ \varphi_{n-1} & \beta_{n-2} \end{bmatrix}} X_n \oplus Y_{n-1} \xrightarrow{\begin{bmatrix} \varphi_n & \beta_{n-1} \end{bmatrix}} Y_n \xrightarrow{(\Sigma\alpha_1)\beta_n} \Sigma X_2$$

is an n-angle. Hence, B_0 holds.

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