Research Article

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A family of commuting contraction semigroups on $l^1(\mathbb{N})$ and $l^\infty(\mathbb{N})$

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Abstract: A family of commuting contraction semigroups $(P_n(t))_{n\in\mathbb{N}}$, defined on $l^1(\mathbb{N})$, is presented. For this family, the product semigroup $\prod_{n=1}^{\infty}P_n(t)$ exists and has bounded generator. The infinite product of the corresponding family of adjoint semigroups $(P_n^*(t))_{n\in\mathbb{N}}$, defined on $l^\infty(\mathbb{N})$, also exists and its generator is bounded. Explicit formulae for these generators are also given.

Keywords: infinite product of semigroups of operators, contraction semigroup, bounded generator

MSC 2020: 47D03, 60J27, 60J35

1 Introduction

It is well known that if B is a bounded linear operator in a Banach space $(X, ||\cdot||)$, then the family of operators $(P(t))_{t\geq 0}$ given by

$$P(t) = e^{tB} = \sum_{k=0}^{\infty}\frac{(tB)^k}{k!}, \quad t \geq 0,$$

forms a strongly continuous semigroup of bounded operators in X [1, pp. 67, 251]. This means that it satisfies $P(t+s) = P(t)P(s)$, for all $t, s \geq 0$, and $\lim_{t\to 0+}P(t)x = x$, for every $x \in X$. $P(0)$ denotes I_X, the identity operator in X. Moreover, the semigroup so defined is in fact uniformly continuous, that is $\lim_{t\to 0+}||P(t) - I_X|| = 0$, and B is its infinitesimal generator, see p. 251, ibid.

In this article, we provide a family of bounded linear operators $(B_n)_{n\in\mathbb{N}}$, defined on $l^1(\mathbb{N})$, such that the corresponding family of semigroups $(P_n(t))_{n\in\mathbb{N}}$, where $P_n(t) = e^{tB_n}$, satisfies the following conditions:

$$||P_n(t)|| \leq 1, \quad P_m(t)P_n(s) = P_n(s)P_m(t), \tag{1.1}$$

for any $m, n \in \mathbb{N}$ and $t, s \geq 0$. Throughout we use the convention that $\mathbb{N} = \{1, 2, 3, \ldots\}$.

In other words, $(P_n(t))_{n\in\mathbb{N}}$ is a sequence of commuting contraction semigroups on $l^1(\mathbb{N})$. The reason for considering such semigroups is that then the product $\prod_{n=1}^{N}P_n(t)$, for any $N \geq 2$, is also a semigroup of contractions and its generator equals $\sum_{n=1}^{N}B_n$ [2, p. 24]. We also give conditions under which the infinite product of those semigroups exists and its generator $A = \sum_{n=1}^{\infty}B_n$ is bounded, i.e.,

$$\prod_{n=1}^{\infty}P_n(t) \coloneqq \lim_{N\to\infty}\prod_{n=1}^{N}P_n(t) = e^{tA}, \quad t \geq 0. \tag{1.2}$$

The limit in (1.2) is in the strong topology. An explicit formula for A is given in Theorem 3.3, which follows from a general convergence theorem for semigroups satisfying (1.1) proved in [3]. Some results, based on this theorem, have been obtained recently, see [4,5] and [6, p. 85].

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This article is organized as follows. In Section 2, we summarize the previous results and define operators $(B_n)_{n\in\mathbb{N}}$. Two specific examples are also provided. Main results of this article are contained in Section 3. Section 3.1 is devoted to the proof of (1.1). The Lumer-Phillips theorem is used to show that the $B_n's$ generate semigroups of contractions, see Lemma 3.1. In Section 3.2, we prove (1.2), and in Section 3.3, an analogous result is obtained for $(P_n^*(t))_{n\in\mathbb{N}}$, where $P_n^*(t)$ denotes the adjoint semigroup to $P_n(t)$. An auxiliary lemma is proved in Section A.

2 A family of operators $(B_n)_{n\in\mathbb{N}}$

2.1 Motivation from the theory of Markov chains

Recall that $l^1(\mathbb{N})$ is the Banach space of all absolutely summable sequences. This means that $x = (\xi_i)_{i\in\mathbb{N}}$ is an element of $l^1(\mathbb{N})$ if and only if $\sum_{i\in\mathbb{N}}|\xi_i| < \infty$ and then $||x||_{l^1(\mathbb{N})} = \sum_{i\in\mathbb{N}}|\xi_i|$. It is a separable space and any $x \in l^1(\mathbb{N})$ can be written as $\sum_{i\in\mathbb{N}}\xi_i e_i$, where $\{e_i\}_{i\in\mathbb{N}}$ is the standard Schauder basis in this space, i.e., $e_i = (\ldots, 0, 1, 0, \ldots)$ with 1 in the ith coordinate. For more details about $l^1(\mathbb{N})$, see [7, Chapter 7].

In [4,5], the sequence of operators $(B_n)_{n\in\mathbb{N}}$, defined in $l^1(\mathbb{N})$, was considered, where

$$B_n x = (\eta_i)_{i\in\mathbb{N}} = \begin{cases} -\beta_n\xi_i + \alpha_n\xi_{i+2^{n-1}}, & \text{if } i \bmod 2^n \in S_n^1, \\ -\alpha_n\xi_i + \beta_n\xi_{i-2^{n-1}}, & \text{if } i \bmod 2^n \in S_n^2. \end{cases} \tag{2.1}$$

The sets S_n^1, S_n^2 in (2.1) partition the set $\{0, 1, 2, \ldots, 2^n - 1\}$ and are given by

$$S_n^1 = \{1, 2, 3, \ldots, 2^{n-1}\}, \quad S_n^2 = \{0, 2^{n-1} + 1, 2^{n-1} + 2, \ldots, 2^n - 1\}. \tag{2.2}$$

The sequences $(\beta_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$ are assumed to be sequences of real positive numbers. The operators $(B_n)_{n\in\mathbb{N}}$ are in fact isomorphic images of the operators from [6, p. 83] and are generators of the transition semigroups associated with two-state Markov chains. These were used by Blackwell [8] to construct a Markov chain whose all states are instantaneous. To be more precise, he considered a Markov process $(X(t))_{t\geq 0}$ defined as follows:

$$X(t) = (X_1(t), X_2(t), X_3(t), \ldots), \quad t \geq 0, \tag{2.3}$$

where $X_1(t), X_2(t), X_3(t), \ldots$ is an infinite sequence of mutually independent, two-state Markov chains. The set of states of every component of $X(t)$ is $\{0, 1\}$ and the transition semigroup e^{tB_n}, associated with $X_n(t)$, is thus characterized by β_n and α_n. An explicit formula for this semigroup exists [6, p. 83].

Let $\mathbb{S}$ denote the state space of $X(t)$. Then $\mathbb{S} \subset \{0, 1\}^\infty$, and this last set is uncountable by the known Cantor's diagonal argument. So in general $\mathbb{S}$ may be uncountable. However, if we assume that the sequences $(\beta_n)_{n\in\mathbb{N}}, (\alpha_n)_{n\in\mathbb{N}}$ satisfy (the first condition by Blackwell)

$$\sum_{n=1}^{\infty} \frac{\beta_n}{\alpha_n + \beta_n} < +\infty, \tag{2.4}$$

then, with probability one, elements of $\mathbb{S}$ are sequences with only a finite number of 1's. Therefore, $\mathbb{S}$ is countable and a bijection between $\mathbb{S}$ and $\mathbb{N}$ exists, and it is given in [5]. Moreover, the condition (2.4) also guarantees that the transition semigroup $P(t)$, associated with $X(t)$, is well defined. This means that it is a strongly continuous semigroup of contractions, and it is given by

$$P(t)x = \lim_{N\to\infty} \prod_{n=1}^{N} e^{tB_n}x, \quad x \in l^1(\mathbb{N}), t \geq 0. \tag{2.5}$$

The fact that $\prod_{n=1}^{\infty} e^{tB_n}$ is the transition semigroup of $X(t)$ follows from the independence of its components. The second condition introduced by Blackwell requires that

$$\sum_{n=1}^{\infty} \beta_n = +\infty, \tag{2.6}$$

and it implies that all states of $X(t)$ are instantaneous, i.e., the Q-matrix of the process has $-\infty$ in all the diagonal entries. Incidentally, the condition (2.6), together with (2.4), implies also that $\sum_{n=1}^{\infty} \alpha_n = +\infty$.

From these two conditions, it follows the generator of $P(t)$, denoted as A_{gen}, is densely defined and unbounded. In [5], it was found, by the application of [3, Proposition 2.7], that $A_{\text{gen}} = \overline{A}$, where

$$Ax := \lim_{N\to\infty} \sum_{n=1}^{N} B_n x, \quad x \in D(A) \subset l^1(\mathbb{N}). \tag{2.7}$$

Here, $D(A)$ denotes the domain of A. Because A is not bounded, $D(A) \neq l^1(\mathbb{N})$. For example, any x with a finite number of non-zero components does not belong to $D(A)$. It should be added that in this case, the operator A^* defined as the limit of $\sum_{n=1}^{N} B_n^*$, when $N \to \infty$, is not densely defined in $l^\infty(\mathbb{N})$. Thus, we cannot apply the proposition from [3] and conclude the continuity of the adjoint semigroup $P^*(t)$.

Blackwell was interested in the case in which all states of (2.3) are instantaneous, and his example is one of the few such known, see [6, p. 82] and the references contained therein. This chain is so interesting because some explicit calculations, just described earlier, can be carried out that are related to it.

If instead of (2.6) we assume that $(\beta_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$ satisfy (in addition to (2.4))

$$\sum_{n=1}^{\infty} (\alpha_n + \beta_n) < +\infty, \tag{2.8}$$

then all states of $X(t)$ are stable, which means the Q-matrix of the process has finite values in all its diagonal entries. Moreover, the operator A determined by (2.7) is then bounded, $D(A) = l^1(\mathbb{N})$, and it generates the transition semigroup of $X(t)$ [4]. This also follows from [3, Proposition 2.7], and the semigroup is defined by (2.5). Let us also emphasize that the condition (2.8) (or (2.6)) does not contradict (2.4).

The application of [3, Proposition 2.7] in the stable case also shows that (2.8) overrides (2.4). In other words, the condition (2.8) is sufficient for A to be bounded. Recall that (2.4) ensures that the set of states of $X(t)$ is countable, i.e., it is a Markov chain.

Therefore, when the condition (2.8) holds true, we can omit (2.4). The operator A and the semigroup $P(t)$ are still well defined and $P(t) = e^{tA}$. However, because we no longer assume (2.4), the connection between $P(t)$ and Markov chains is lost. The state space of (2.3) could be uncountable, and $X(t)$ would be a Markov process. It is not clear what $P(t)$ then describes. Currently, I am not able to provide any interpretation. Similarly, the meaning of B_n from definition 1, as well as A in Theorem 3.3, is unclear. Nevertheless, from the point of view of semigroups, these objects are well defined.

2.2 Definition of the B_n's

This special form of the operators given by (2.1) suggests how to generalize them so that they still commute and generate semigroups of contractive operators. The goal of this article is to prove that this generalization works and the construction is based on a finite collection of sequences of numbers, so we begin with them.

Let an integer $d \geq 2$ be fixed and suppose that there are given $d \cdot (d-1)$ sequences of positive numbers, denoted by $(\beta_{i,j}^n)_{n\in\mathbb{N}}$, i.e.,

$$\beta_{i,j}^n > 0, \quad n \geq 1, i, j = 1, 2, \ldots, d \text{ and } i \neq j. \tag{2.9}$$

These numbers, for fixed n, can be thought of as off-diagonal entries in a $d \times d$ matrix, see Examples 1, 2. Based on (2.9), define $(\beta_{i,i}^n)_{n\in\mathbb{N}}$ as follows:

$$\beta_{i,i}^n := -\sum_{j=1, j\neq i}^{d} \beta_{i,j}^n, \quad i = 1, 2, \ldots, d. \tag{2.10}$$

The numbers $\beta^n_{i,i}$ are diagonal entries, in the matrix analogy. It should be added that the matrix analogy is only perfect for $n = 1$. From (2.10), it follows that if n is fixed, then $\sum_{j=1}^{d}\beta^n_{i,j} = 0$, for every $i = 1, 2, \ldots, d$.

The sets S^1_n, S^2_n, determined by (2.2), partition the set $\mathbb{Z}_{2^n-1} = \{0, 1, 2, \ldots, 2^n - 1\}$ into two parts of equal size. In a similar way, we introduce a partition of $\mathbb{Z}_{d^n-1} = \{0, 1, 2, \ldots, d^n - 1\}$, denoted as $S^1_n, S^2_n, \ldots, S^d_n$, into d parts of equal size. Namely, define

$$\begin{cases} S^k_n = \{(k-1)d^{n-1} + 1, (k-1)d^{n-1} + 2, \ldots, kd^{n-1}\}, & k = 1, 2, \ldots, d-1, \\ S^d_n = \{0, (d-1)d^{n-1} + 1, (d-1)d^{n-1} + 2, \ldots, d^n - 1\}. \end{cases} \tag{2.11}$$

In other words, $|S^k_n| = d^{n-1}$ for $k = 1, 2, \ldots, d$ and

$$\mathbb{Z}_{d^n-1} = \bigcup_{k=1}^{d} S^k_n \quad \text{and} \quad S^{k_1}_n \cap S^{k_2}_n = \emptyset, \quad k_1 \neq k_2.$$

The sets $S^1_n, S^2_n, \ldots, S^d_n$ can also be defined recursively and in order to do this let us first introduce some notation. If S is a subset of $\mathbb{Z}$, where $\mathbb{Z}$ stands for the set of integers, and $a \in \mathbb{Z}$, then $S + a$ denotes the set $\{s + a : s \in S\}$. So $\{1, 2\} + 3 = \{4, 5\}$. With a little abuse of notation, we also introduce

$$[a, b] \coloneqq \{a, a+1, a+2, \ldots, b-1, b\}, \quad a \le b, a, b \in \mathbb{Z}.$$

This should not lead to misunderstandings, since all indices in a vector (ξ_i) or in a sum $\sum_{j=j_1}^{j_2}\xi_j$ are assumed to be integers. With this notation, we write

$$(\xi_i)_{i\in[1,d^n]} = (\xi_1, \xi_2, \ldots, \xi_{d^n}), \quad S^k_n = [(k-1)d^{n-1} + 1, kd^{n-1}].$$

The recursive definition of (2.11) would be to assume $S^1_n = \{1, 2, 3, \ldots, d^{n-1}\}$ and

$$S^k_n = (S^1_n + (k-1)d^{n-1}) \bmod d^n, \quad k = 2, 3, \ldots, d,$$

where the sum taken modulus d^n ensures that $0 \in S^d_n$. We can now give the following definition.

Definition 1. For $n \ge 1$ define $B_n : l^1(\mathbb{N}) \to l^1(\mathbb{N})$ as follows: for $x = (\xi_i)_{i\in\mathbb{N}}$ let $B_n x \coloneqq (\eta_i)_{i\in\mathbb{N}}$, where

$$\eta_i = \sum_{j=1}^{k-1}\beta^n_{k-j,k}\xi_{i-jd^{n-1}} + \beta^n_{k,k}\xi_i + \sum_{j=1}^{d-k}\beta^n_{k+j,k}\xi_{i+jd^{n-1}}, \quad \text{if } i \bmod d^n \in S^k_n, \tag{2.12}$$

where the set S^k_n is determined by (2.11). □

In this article, we use the convention that if in a sum $\sum_{j=j_1}^{j_2}\xi_j$ we have $j_2 < j_1$, then the sum equals zero. For example, taking $i = 1$ in (2.12), since $S^1_n = \{1, 2, \ldots, d^{n-1}\}$, we obtain

$$\eta_1 = \beta^n_{1,1}\xi_1 + \sum_{j=1}^{d-1}\beta^n_{1+j,1}\xi_{1+jd^{n-1}}.$$

Furthermore, it is worth noting that if $i \in [1, d^n]$ with $i \bmod d^n \in S^k_n$, then both $i - jd^{n-1} \in [1, d^n]$, for $j = 1, 2, \ldots, k-1$, as well as $i + jd^{n-1} \in [1, d^n]$, for $j = 1, 2, \ldots, d-k$. This follows from the fact that i lies between $(k-1)d^{n-1} + 1$ and kd^{n-1}. Therefore, in the first case, we have

$$i - jd^{n-1} \ge (k-1)d^{n-1} + 1 - (k-1)d^{n-1} = 1$$

and, in the second case,

$$i + jd^{n-1} \le kd^{n-1} + (d-k)d^{n-1} = d^n.$$

In a similar way, if $i \in S_l$, where $S_l = [1, d^n] + ld^n$, for some integer $l \ge 1$, and $i \bmod d^n \in S^k_n$, then $i - jd^{n-1} \in S_l$, for $j = 1, 2, \ldots, k-1$ and $i + jd^{n-1} \in S_l$, for $j = 1, 2, \ldots, d-k$.

We draw an important conclusion from these observations. Namely, the one that the (infinite) matrix of B_n, denoted by $M(B_n)$, can be written in the following way:

$$M(B_n) = \begin{bmatrix} M(\widetilde{B}_n) & 0 & 0 & \dots \\ 0 & M(\widetilde{B}_n) & 0 & \dots \\ 0 & 0 & M(\widetilde{B}_n) & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \tag{2.13}$$

where 0 denotes the $d^n \times d^n$ matrix with all entries zero and $M(\widetilde{B}_n)$ is the $d^n \times d^n$ matrix of B_n truncated to the first d^n coordinates. In other words, it is the matrix of the map $\widetilde{B}_n : \mathbb{R}^{d^n} \to \mathbb{R}^{d^n}$ defined as follows:

$$\widetilde{B}_n x \coloneqq (\widetilde{\eta}_i)_{i\in[1,d^n]} \quad \text{with } \widetilde{\eta}_i = \eta_i,\ i \in [1, d^n],$$

where η_i is given by (2.12). In this case, $x = (\xi_i)_{i\in[1,d^n]}$. The fact that $M(B_n)$ is the matrix of B_n simply means

$$B_n x = x \cdot M(B_n), \quad x \in l^1(\mathbb{N}).$$

So it is clear that B_n is a bounded linear operator on $l^1(\mathbb{N})$, and using the triangle inequality, we obtain an estimate

$$\|B_n x\|_{l^1(\mathbb{N})} \le c_n \|x\|_{l^1(\mathbb{N})}, \tag{2.14}$$

where c_n is given by

$$c_n = d \cdot \sum_{i,j=1;\ i\ne j}^{d} \beta^n_{i,j}.$$

Example 1. For $d = 2$, the B_n's are given by (2.1), and two independent sequences $(\beta^n_{1,2})_{n\in\mathbb{N}}$, $(\beta^n_{2,1})_{n\in\mathbb{N}}$, introduced in (2.9), are denoted by $(\beta_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$, respectively. This notation is used in [4, 5] and in [6, p. 83]. Rewriting (2.1) in terms of $(\beta^n_{1,2})_{n\in\mathbb{N}}$ and $(\beta^n_{2,1})_{n\in\mathbb{N}}$ gives

$$B_n x = (\eta_i)_{i\in\mathbb{N}} = \begin{cases} -\beta^n_{1,2}\xi_i + \beta^n_{2,1}\xi_{i+2^{n-1}}, & \text{if } i \bmod 2^n \in S^1_n, \\ -\beta^n_{2,1}\xi_i + \beta^n_{1,2}\xi_{i-2^{n-1}}, & \text{if } i \bmod 2^n \in S^2_n, \end{cases}$$

where the sets S^1_n, S^2_n are given by (2.2). In this case, $\beta^n_{1,1} = -\beta^n_{1,2}$, $\beta^n_{2,2} = -\beta^n_{2,1}$, for $n \ge 1$. In particular, $S^1_1 = \{1\}$, $S^2_1 = \{0\}$, $S^1_2 = \{1, 2\}$, $S^2_2 = \{0, 3\}$ and

$$M(\widetilde{B}_1) = \begin{bmatrix} -\beta^1_{1,2} & \beta^1_{1,2} \\ \beta^1_{2,1} & -\beta^1_{2,1} \end{bmatrix} \quad M(\widetilde{B}_2) = \begin{bmatrix} -\beta^2_{1,2} & 0 & \beta^2_{1,2} & 0 \\ 0 & -\beta^2_{1,2} & 0 & \beta^2_{1,2} \\ \beta^2_{2,1} & 0 & -\beta^2_{2,1} & 0 \\ 0 & \beta^2_{2,1} & 0 & -\beta^2_{2,1} \end{bmatrix}.$$

For $n = 3$, we have $S^1_3 = \{1, 2, 3, 4\}$, $S^2_3 = \{0, 5, 6, 7\}$, and

$$M(\widetilde{B}_3) = \begin{bmatrix} -\beta^3_{1,2} & 0 & 0 & 0 & \beta^3_{1,2} & 0 & 0 & 0 \\ 0 & -\beta^3_{1,2} & 0 & 0 & 0 & \beta^3_{1,2} & 0 & 0 \\ 0 & 0 & -\beta^3_{1,2} & 0 & 0 & 0 & \beta^3_{1,2} & 0 \\ 0 & 0 & 0 & -\beta^3_{1,2} & 0 & 0 & 0 & \beta^3_{1,2} \\ \beta^3_{2,1} & 0 & 0 & 0 & -\beta^3_{2,1} & 0 & 0 & 0 \\ 0 & \beta^3_{2,1} & 0 & 0 & 0 & -\beta^3_{2,1} & 0 & 0 \\ 0 & 0 & \beta^3_{2,1} & 0 & 0 & 0 & -\beta^3_{2,1} & 0 \\ 0 & 0 & 0 & \beta^3_{2,1} & 0 & 0 & 0 & -\beta^3_{2,1} \end{bmatrix}.$$

Example 2. Consider $d = 3$. In this case, there are six independent sequences: $(\beta^n_{1,2})_{n\in\mathbb{N}}$, $(\beta^n_{1,3})_{n\in\mathbb{N}}$, $(\beta^n_{2,1})_{n\in\mathbb{N}}$, $(\beta^n_{2,3})_{n\in\mathbb{N}}$, $(\beta^n_{3,1})_{n\in\mathbb{N}}$, $(\beta^n_{3,2})_{n\in\mathbb{N}}$. Then

$$\beta^n_{1,1} = -(\beta^n_{1,2} + \beta^n_{1,3}), \quad \beta^n_{2,2} = -(\beta^n_{2,1} + \beta^n_{2,3}), \quad \beta^n_{3,3} = -(\beta^n_{3,1} + \beta^n_{3,2}), \quad n \geq 1,$$

and (2.12) takes the form

$$B_n x = (\eta_i)_{i\in\mathbb{N}} = \begin{cases} \beta^n_{1,1}\xi_i + \beta^n_{2,1}\xi_{i+3^{n-1}} + \beta^n_{3,1}\xi_{i+2\cdot 3^{n-1}}, & \text{if } i \bmod 3^n \in S^1_n, \\ \beta^n_{1,2}\xi_{i-3^{n-1}} + \beta^n_{2,2}\xi_i + \beta^n_{3,2}\xi_{i+3^{n-1}}, & \text{if } i \bmod 3^n \in S^2_n, \\ \beta^n_{1,3}\xi_{i-2\cdot 3^{n-1}} + \beta^n_{2,3}\xi_{i-3^{n-1}} + \beta^n_{3,3}\xi_i, & \text{if } i \bmod 3^n \in S^3_n, \end{cases}$$

where the sets S^1_n, S^2_n, S^3_n, for $n \geq 1$, are given by

$$S^1_n = \{1, 2, 3, \ldots, 3^{n-1}\}$$
$$S^2_n = \{3^{n-1} + 1, 3^{n-1} + 2, \ldots, 2 \cdot 3^{n-1}\}$$
$$S^3_n = \{0, 2 \cdot 3^{n-1} + 1, 2 \cdot 3^{n-1} + 2, \ldots, 3^n - 1\}.$$

In particular, for $n = 1$, we have $S^1_1 = \{1\}$, $S^2_1 = \{2\}$, $S^3_1 = \{0\}$ and $M(\widetilde{B}_1)$ is as follows:

$$M(\widetilde{B}_1) = \begin{bmatrix} \beta^1_{1,1} & \beta^1_{1,2} & \beta^1_{1,3} \\ \beta^1_{2,1} & \beta^1_{2,2} & \beta^1_{2,3} \\ \beta^1_{3,1} & \beta^1_{3,2} & \beta^1_{3,3} \end{bmatrix} = \begin{bmatrix} -(\beta^1_{1,2} + \beta^1_{1,3}) & \beta^1_{1,2} & \beta^1_{1,3} \\ \beta^1_{2,1} & -(\beta^1_{2,1} + \beta^1_{2,3}) & \beta^1_{2,3} \\ \beta^1_{3,1} & \beta^1_{3,2} & -(\beta^1_{3,1} + \beta^1_{3,2}) \end{bmatrix}.$$

For $n = 2$, we have $S^1_2 = \{1, 2, 3\}$, $S^2_2 = \{4, 5, 6\}$, $S^3_2 = \{0, 7, 8\}$ and $M(\widetilde{B}_2)$ can be written as follows:

$$M(\widetilde{B}_2) = \begin{bmatrix} \beta^2_{1,1} & 0 & 0 & \beta^2_{1,2} & 0 & 0 & \beta^2_{1,3} & 0 & 0 \\ 0 & \beta^2_{1,1} & 0 & 0 & \beta^2_{1,2} & 0 & 0 & \beta^2_{1,3} & 0 \\ 0 & 0 & \beta^2_{1,1} & 0 & 0 & \beta^2_{1,2} & 0 & 0 & \beta^2_{1,3} \\ \beta^2_{2,1} & 0 & 0 & \beta^2_{2,2} & 0 & 0 & \beta^2_{2,3} & 0 & 0 \\ 0 & \beta^2_{2,1} & 0 & 0 & \beta^2_{2,2} & 0 & 0 & \beta^2_{2,3} & 0 \\ 0 & 0 & \beta^2_{2,1} & 0 & 0 & \beta^2_{2,2} & 0 & 0 & \beta^2_{2,3} \\ \beta^2_{3,1} & 0 & 0 & \beta^2_{3,2} & 0 & 0 & \beta^2_{3,3} & 0 & 0 \\ 0 & \beta^2_{3,1} & 0 & 0 & \beta^2_{3,2} & 0 & 0 & \beta^2_{3,3} & 0 \\ 0 & 0 & \beta^2_{3,1} & 0 & 0 & \beta^2_{3,2} & 0 & 0 & \beta^2_{3,3} \end{bmatrix},$$

where $\beta^2_{1,1} = -(\beta^2_{12} + \beta^2_{13})$, $\beta^2_{2,2} = -(\beta^2_{21} + \beta^2_{23})$, and $\beta^2_{3,3} = -(\beta^2_{31} + \beta^2_{32})$.

3 Main results

3.1 Proofs of (1.1)

Lemma 3.1. *The operator B_n, given by* (2.12)*, generates a semigroup of contractions, i.e.,*

$$\|P_n(t)x\|_{l^1(\mathbb{N})} = \|e^{tB_n}x\|_{l^1(\mathbb{N})} \leq \|x\|_{l^1(\mathbb{N})} \tag{3.1}$$

for every $t \geq 0$ and $x \in l^1(\mathbb{N})$.

Proof. We prove (3.1) using the Lumer-Phillips theorem, see [9, Theorem 2.1]. This theorem states that a bounded linear operator B, defined in a Banach space $(X, ||\cdot||)$, generates a semigroup of contractive operators if and only if it is dissipative, which means that it satisfies the condition $||(\lambda - B)x|| \geq \lambda||x||$, for all $\lambda > 0$ [10, p. 75].

Dissipativity can be checked by somewhat simpler condition. Namely, if X is a real Banach space, then B is dissipative if and only if for every $x \in X$ there exists $j(x) \in J(x)$ such that

$$\langle Bx, j(x)\rangle \leq 0, \tag{3.2}$$

where

$$J(x) \coloneqq \{x^* \in X^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2\},$$

see [10, Proposition 3.23]. It is worth to mention that $J(x)$ is always nonempty by the Hahn-Banach theorem, see [7, p. 181]. Recall that X^* stands for the dual space of X consisting of the bounded linear functionals on X and $\langle x, x^*\rangle$ denotes $x^*(x)$. It is also a Banach space, see [7, p. 180].

We show (3.2) for $B = B_n$. In our case, $X = l^1(\mathbb{N})$. It is well known, see [7, p. 207], that X^* can be identified with the Banach space of all bounded sequences, denoted by $l^\infty(\mathbb{N})$. If $x^* = (\xi_i)_{i\in\mathbb{N}}$ is an element of $l^\infty(\mathbb{N})$, then $||x^*||_{l^\infty(\mathbb{N})} = \sup_{i\in\mathbb{N}}|\xi_i|$.

Let $x = (\xi_i)_{i\in\mathbb{N}}$ be any non-zero element of $l^1(\mathbb{N})$ and define $x^* \in l^\infty(\mathbb{N})$ as follows:

$$x^* \coloneqq ||x||_{l^1(\mathbb{N})}(\operatorname{sgn}(\xi_i))_{i\in\mathbb{N}}, \tag{3.3}$$

where $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(a) = |a|/a$ for a non-zero $a \in \mathbb{R}$. Then $x^* \in J(x)$, due to

$$\langle x, x^*\rangle = ||x||_{l^1(\mathbb{N})}\sum_{i=1}^{\infty}\xi_i \operatorname{sgn}(\xi_i) = ||x||^2_{l^1(\mathbb{N})}.$$

The main tool used in proving $\langle B_n x, x^*\rangle \leq 0$ is the following elementary inequality

$$a \cdot \operatorname{sgn}(b) \leq a \cdot \operatorname{sgn}(a) = |a|, \quad a, b \in \mathbb{R}. \tag{3.4}$$

By (2.12) and (3.3), we need to prove

$$\langle B_n x, x^*\rangle = ||x||_{l^1(\mathbb{N})}\sum_{j\in J}\left(\sum_{i=j+1}^{j+d^n}\eta_i \operatorname{sgn}(\xi_i)\right) \leq 0, \tag{3.5}$$

where $J = \{md^n : m \geq 0, m \in \mathbb{Z}\}$. The outer sum in (3.5) is simply from $j = 0$ to $+\infty$, where j increases by a multiple of d^n. From (2.13), it is clear that to prove (3.5), it is enough to show

$$\Omega \coloneqq \sum_{i=1}^{d^n}\eta_i \operatorname{sgn}(\xi_i) \leq 0. \tag{3.6}$$

By substituting for η_i in (3.6), we can write $\Omega = \Sigma_1 + \Sigma_2$, where

$$\Sigma_1 = \sum_{i=1}^{d^n}\sigma_i\xi_i,$$

with

$$\sigma_i = \sum_{j=1}^{k-1}\beta^n_{k,k-j}\operatorname{sgn}(\xi_{i-jd^{n-1}}) + \sum_{j=1}^{d-k}\beta^n_{k,k+j}\operatorname{sgn}(\xi_{i+jd^{n-1}}), \quad \text{if } i \bmod d^n \in S^k_n,$$

and

$$\Sigma_2 = \sum_{i=1}^{d^n}\tau_i\xi_i\operatorname{sgn}(\xi_i) = \sum_{i=1}^{d^n}\tau_i|\xi_i|,$$

with

$$\tau_i = \beta^n_{k,k}, \quad \text{if } i \bmod d^n \in S^k_n.$$

Applying (3.4) in Σ_1 gives

$$\Omega = \Sigma_1 + \Sigma_2 \le \sum_{i=1}^{d^n} \widetilde{\sigma}_i|\xi_i| + \sum_{i=1}^{d^n} \tau_i|\xi_i|,$$

where

$$\widetilde{\sigma}_i = \sum_{j=1}^{k-1} \beta^n_{k,k-j} + \sum_{j=1}^{d-k} \beta^n_{k,k+j}, \quad \text{if } i \bmod d^n \in S^k_n.$$

The final step is to notice that $\widetilde{\sigma}_i = -\tau_i$, see (2.10). Therefore, we have

$$\Omega \le \sum_{i=1}^{d^n} (\widetilde{\sigma}_i + \tau_i)|\xi_i| = 0,$$

which completes the proof of (3.6) and (3.5). □

Lemma 3.2. *Let $P_n(t) = e^{tB_n}$, for $n \ge 1$, where B_n is given by* (2.12). *Then*

$$P_m(t)P_n(s)x = P_n(s)P_m(t)x, \tag{3.7}$$

for any $m, n \in \mathbb{N}$, $t, s \ge 0$, and $x \in l^1(\mathbb{N})$.

Proof. Because the $B_n's$ are bounded, the condition (3.7) is equivalent to

$$B_mB_nx = B_nB_mx, \tag{3.8}$$

see e.g. [10, p. 19]. So we prove (3.8) and it suffices to consider the case for $m < n$. A consequence of this assumption is that $m \le n-1$. Let $x = (\xi_i)_{i\in\mathbb{N}} \in l^1(\mathbb{N})$ be fixed and denote

$$(\eta_i)_{i\in\mathbb{N}} = B_m(\xi_i)_{i\in\mathbb{N}}, \quad (\zeta_i)_{i\in\mathbb{N}} = B_n(\eta_i)_{i\in\mathbb{N}}, \quad (\eta_i')_{i\in\mathbb{N}} = B_n(\xi_i)_{i\in\mathbb{N}}, \quad (\zeta_i')_{i\in\mathbb{N}} = B_m(\eta_i')_{i\in\mathbb{N}}.$$

It can be seen from (2.13) that in order to prove (3.8), it is enough to show

$$\zeta_i = \zeta_i', \quad \text{for } i \in [1, d^n]. \tag{3.9}$$

Fix $i \in [1, d^n]$. Then, for some $k_1, k_2 \in [1, d]$, we have

$$i \bmod d^m \in S^{k_1}_m, \quad i \bmod d^n \in S^{k_2}_n. \tag{3.10}$$

As a result, for some $l \in \{0, 1, \dots, d^{n-1-m}\}$ we also have

$$i \in S_l = S^{k_1}_m + ld^m + (k_2 - 1)d^{n-1}.$$

We begin to calculate ζ_i, i.e., the ith coordinate of B_nB_mx. From (2.12) and (3.10) we obtain

$$\zeta_i = \sum_{j_2=1}^{k_2-1} \beta^n_{k_2-j_2,k_2}\eta_{i-j_2d^{n-1}} + \beta^n_{k_2,k_2}\eta_i + \sum_{j_2=1}^{d-k_2} \beta^n_{k_2+j_2,k_2}\eta_{i+j_2d^{n-1}}, \tag{3.11}$$

where

$$\eta_i = \sum_{j_1=1}^{k_1-1} \beta^m_{k_1-j_1,k_1}\xi_{i-j_1d^{m-1}} + \beta^m_{k_1,k_1}\xi_i + \sum_{j_1=1}^{d-k_1} \beta^m_{k_1+j_1,k_1}\xi_{i+j_1d^{m-1}}.$$

What remains to be calculated in (3.11) is $\eta_{i+j_2d^{n-1}}$ and $\eta_{i-j_2d^{n-1}}$. These should be expressed in terms of k_1 and k_2. We achieve this from $m \le n-1$. Namely,

$$i \bmod d^m \in S^{k_1}_m \quad \Rightarrow \quad (i + jd^{n-1}) \bmod d^m \in S^{k_1}_m,$$

where j is an integer. Therefore,

$$\eta_{i+j_2 d^{n-1}} = \sum_{j_1=1}^{k_1-1} \beta^m_{k_1-j_1,k_1} \xi_{i+j_2 d^{n-1}-j_1 d^{m-1}} + \beta^m_{k_1,k_1} \xi_{i+j_2 d^{n-1}} + \sum_{j_1=1}^{d-k_1} \beta^m_{k_1+j_1,k_1} \xi_{i+j_2 d^{n-1}+j_1 d^{m-1}},$$

and, in a similar way,

$$\eta_{i-j_2 d^{n-1}} = \sum_{j_1=1}^{k_1-1} \beta^m_{k_1-j_1,k_1} \xi_{i-j_2 d^{n-1}-j_1 d^{m-1}} + \beta^m_{k_1,k_1} \xi_{i-j_2 d^{n-1}} + \sum_{j_1=1}^{d-k_1} \beta^m_{k_1+j_1,k_1} \xi_{i-j_2 d^{n-1}+j_1 d^{m-1}}.$$

So (3.11) can be written as sum of nine terms, i.e.,

$$\zeta_i = \sum_{j=1}^{9} T_j, \tag{3.12}$$

where $T_1, T_2, \dots, T_9$ are given by

$$\begin{aligned}
T_1 &= \sum_{j_2=1}^{k_2-1} \sum_{j_1=1}^{k_1-1} \beta^n_{k_2-j_2,k_2} \beta^m_{k_1-j_1,k_1} \xi_{i-j_2 d^{n-1}-j_1 d^{m-1}},\\
T_2 &= \beta^m_{k_1,k_1} \sum_{j_2=1}^{k_2-1} \beta^n_{k_2-j_2,k_2} \xi_{i-j_2 d^{n-1}},\\
T_3 &= \sum_{j_2=1}^{k_2-1} \sum_{j_1=1}^{d-k_1} \beta^n_{k_2-j_2,k_2} \beta^m_{k_1+j_1,k_1} \xi_{i-j_2 d^{n-1}+j_1 d^{m-1}},\\
T_4 &= \beta^n_{k_2,k_2} \sum_{j_1=1}^{k_1-1} \beta^m_{k_1-j_1,k_1} \xi_{i-j_1 d^{m-1}},\\
T_5 &= \beta^n_{k_2,k_2} \beta^m_{k_1,k_1} \xi_i,\\
T_6 &= \beta^n_{k_2,k_2} \sum_{j_1=1}^{d-k_1} \beta^m_{k_1+j_1,k_1} \xi_{i+j_1 d^{m-1}},\\
T_7 &= \sum_{j_2=1}^{d-k_2} \sum_{j_1=1}^{k_1-1} \beta^n_{k_2+j_2,k_2} \beta^m_{k_1-j_1,k_1} \xi_{i+j_2 d^{n-1}-j_1 d^{m-1}},\\
T_8 &= \beta^m_{k_1,k_1} \sum_{j_2=1}^{d-k_2} \beta^n_{k_2+j_2,k_2} \xi_{i+j_2 d^{n-1}},\\
T_9 &= \sum_{j_2=1}^{d-k_2} \sum_{j_1=1}^{d-k_1} \beta^n_{k_2+j_2,k_2} \beta^m_{k_1+j_1,k_1} \xi_{i+j_2 d^{n-1}+j_1 d^{m-1}}.
\end{aligned}$$

Now we calculate ζ_i', i.e., the ith coordinate of $B_m B_n x$. From (2.12) and (3.10), we have

$$\zeta_i' = \sum_{j_1=1}^{k_1-1} \beta^m_{k_1-j_1,k_1} \eta'_{i-j_1 d^{m-1}} + \beta^m_{k_1,k_1} \eta_i' + \sum_{j_1=1}^{d-k_1} \beta^m_{k_1+j_1,k_1} \eta'_{i+j_1 d^{m-1}}, \tag{3.13}$$

where

$$\eta_i' = \sum_{j_2=1}^{k_2-1} \beta^n_{k_2-j_2,k_2} \xi_{i-j_2 d^{n-1}} + \beta^n_{k_2,k_2} \xi_i + \sum_{j_2=1}^{d-k_2} \beta^n_{k_2+j_2,k_2} \xi_{i+j_2 d^{n-1}}.$$

Now notice that

$$(i - jd^{m-1}) \bmod d^n \in S_n^{k_2}, \quad \text{for } j \in [1, k_1 - 1],$$

and, similarly,

$$(i + jd^{m-1}) \bmod d^n \in S_n^{k_2}, \quad \text{for } j \in [1, d - k_1].$$

This allows to express $\eta'_{i-j_1 d^{m-1}}$ and $\eta'_{i+j_1 d^{m-1}}$ in terms of k_1 and k_2, see (3.10), and write ζ'_i in a similar way as ζ_i, i.e., as the sum of nine terms, cf. (3.12). The final form of (3.13) is such that

$$\zeta'_i = \sum_{j=1}^{9} T'_j = \sum_{j=1}^{9} T_j = \zeta_i.$$

In other words, $T_1, \ldots, T_9$ and $T'_1, \ldots, T'_9$ may only differ in order. This completes the proof. □

3.2 Convergence of $\prod_{n=1}^{N} P_n(t)$

Suppose that $i \in \mathbb{N}$. Then, there exists n_0 such that for every $n \geq n_0$

$$i \bmod d^n \in S_n^1 = \{1, 2, \ldots, d^{n-1}\}.$$

For the proof, from $i \leq d^{n-1}$ we obtain $n \geq \log_d(i) + 1$ and n_0 can be written explicitly using the ceiling function, i.e.,

$$n_0 = \lceil \log_d(i) + 1 \rceil. \tag{3.14}$$

To recall, $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$.

In formula (2.12), which defines B_n, n was fixed, and it was not necessary to denote η_i as $\eta_i(n)$. However, in what follows, we sum these terms over n, see (3.15) and (3.19), so the latter notation is used.

As mentioned earlier, the product $\prod_{n=1}^{N} P_n(t)$, for any $N \geq 2$, is a semigroup of contractions and its generator A_N equals $\sum_{n=1}^{N} B_n$ [2, p. 24]. In our case, it means that

$$A_N x = (\zeta_i)_{i \in \mathbb{N}}, \quad \zeta_i = \sum_{n=1}^{N} \eta_i(n), \tag{3.15}$$

where $x = (\xi_i)_{i \in \mathbb{N}} \in l^1(\mathbb{N})$ and $\eta_i(n)$ is given by (2.12). It turns out that A_N still has a manageable form even if $N = +\infty$, cf. [4, Theorem 3.1].

Theorem 3.3. *Suppose that the sequences* $(\beta_{i,j}^n)_{n \in \mathbb{N}}$ *satisfy*

$$\sum_{i,j=1;\ i \neq j}^{d} \sum_{n=1}^{\infty} \beta_{i,j}^n < \infty, \tag{3.16}$$

and let B_n *be given by* (2.12). *Then the strong limit of* $\prod_{n=1}^{N} e^{tB_n}$, *denoted as* $(T(t))_{t \geq 0}$, *i.e.,*

$$T(t)x \coloneqq \lim_{N \to \infty} \prod_{n=1}^{N} e^{tB_n} x, \quad x \in l^1(\mathbb{N}), \tag{3.17}$$

is a semigroup of contractions, and its generator A *is bounded and*

$$Ax = \lim_{N \to \infty} \sum_{n=1}^{N} B_n x, \quad x \in l^1(\mathbb{N}).$$

Furthermore, denote $(\zeta_i)_{i \in \mathbb{N}} = Ax$. *Then*

$$\zeta_1 = \sum_{n=1}^{\infty} \left(\beta_{1,1}^n \xi_1 + \sum_{j=1}^{d-1} \beta_{1+j,1}^n \xi_{1+jd^{n-1}}\right), \tag{3.18}$$

and for $i \geq 2$, *we have*

$$\zeta_i = \sum_{n=1}^{n_0-1} \eta_i(n) + \sum_{n=n_0}^{\infty} \left(\beta_{1,1}^n \xi_i + \sum_{j=1}^{d-1} \beta_{1+j,1}^n \xi_{i+jd^{n-1}} \right), \tag{3.19}$$

where $\eta_i(n)$ *and* n_0 *are given by* (2.12) *and* (3.14), *respectively*.

Proof. We use [3, Proposition 2.7]. This proposition says that if $(e^{tB_n})_{n\in\mathbb{N}}$ is a family of commuting semigroups of contractions, in a Banach space $(X, ||\cdot||)$, and the set

$$D_1 = \left\{ x \in \bigcap_{n=1}^{\infty} D(B_n) : \sum_{n=1}^{+\infty} ||B_n x|| < \infty \right\}$$

is dense in X, then the semigroup given by (3.17) is well defined. Moreover, its generator is the closure of A, where $A = \lim_{N\to\infty} \sum_{n=1}^{N} B_n$, with $D(A)$ being D_1. In this proposition, the generators may not be bounded, so $D(B_n)$ denotes the domain of B_n.

In our case, $X = l^1(\mathbb{N})$ and $D(B_n) = X$, since B_n is bounded, see (2.14). Furthermore, $D_1 = l^1(\mathbb{N})$, because by (2.14) and (3.16), we have

$$\sum_{n=1}^{+\infty} ||B_n x||_{l^1(\mathbb{N})} \leq d||x||_{l^1(\mathbb{N})} \cdot \sum_{i,j=1;\ i\neq j}^{d} \sum_{n=1}^{\infty} \beta_{i,j}^n < \infty,$$

for every $x \in l^1(\mathbb{N})$. In particular, A is bounded and

$$||A|| \leq d \cdot \sum_{i,j=1;\ i\neq j}^{d} \sum_{n=1}^{\infty} \beta_{i,j}^n.$$

Since the norm convergence in $l^1(\mathbb{N})$ implies convergence in coordinates, components of Ax are limits of components of $A_N x$, where A_N is given by (3.15). Thus, (3.18) and (3.19) follow, and this completes the proof. □

3.3 Convergence of $\prod_{n=1}^{N} P_n^*(t)$

As mentioned earlier, the dual space of $l^1(\mathbb{N})$ can be identified with $l^\infty(\mathbb{N})$ [7, p. 207]. Since B_n is bounded, it induces a linear map $B_n^* : l^\infty(\mathbb{N}) \to l^\infty(\mathbb{N})$, called the adjoint of B_n [11, p. 15]. Moreover, $||B_n|| = ||B_n^*||$, so if B_n is a contraction, then B_n^* is also a contraction, see also (A1).

Let $(\eta_i^*)_{i\in\mathbb{N}} = B_n^* x$, where $x = (\xi_i)_{i\in\mathbb{N}} \in l^1(\mathbb{N})$. In our case, we have

$$\eta_i^* = \sum_{j=1}^{k-1} \beta_{k,k-j}^n \xi_{i-jd^{n-1}} + \beta_{k,k}^n \xi_i + \sum_{j=1}^{d-k} \beta_{k,k+j}^n \xi_{i+jd^{n-1}}, \quad \text{if } i \bmod d^n \in S_n^k. \tag{3.20}$$

This formula simply says that $M(B_n^*) = M^T(B_n)$, i.e., the matrix of B_n^* is equal to the transpose of $M(B_n)$, see (2.13). Therefore, it can be written as follows:

$$M(B_n^*) = \begin{bmatrix} M(\widetilde{B}_n^*) & 0 & 0 & \dots \\ 0 & M(\widetilde{B}_n^*) & 0 & \dots \\ 0 & 0 & M(\widetilde{B}_n^*) & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} M^T(\widetilde{B}_n) & 0 & 0 & \dots \\ 0 & M^T(\widetilde{B}_n) & 0 & \dots \\ 0 & 0 & M^T(\widetilde{B}_n) & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

where $M(\widetilde{B}_n^*)$ denotes the matrix of B_n^* truncated to the first d^n coordinates. Similar estimate to (2.14) can be obtained for B_n^*. Namely, we have

$$||B_n^* x||_{l^\infty(\mathbb{N})} \leq (d-1) \cdot \sum_{i,j=1;\ i\neq j}^{d} \beta_{i,j}^n ||x||_{l^\infty(\mathbb{N})}. \tag{3.21}$$

Moreover, notice that the equality $(B_nB_m)^* = B_m^*B_n^*$, together with (3.8), implies that the B_n^*'s also commute. In Lemma A.1, we prove that $(e^B)^* = e^{B^*}$. This allows for a correct definition

$$P_n^*(t) := e^{tB_n^*} = (e^{tB_n})^* = (P_n(t))^*, \quad n \geq 1. \tag{3.22}$$

In consequence of (3.22) and (1.1), we obtain

$$||P_n^*(t)|| \leq 1, \quad P_m^*(t)P_n^*(s) = P_n^*(s)P_m^*(t), \tag{3.23}$$

for any $m, n \in \mathbb{N}$ and $t, s \geq 0$. So we are in a similar situation as in Section 3.2. In other words, the product $\prod_{n=1}^N P_n^*(t)$, for any $N \geq 2$, is a semigroup of contractions and its generator A_N^* equals $\sum_{n=1}^N B_n^*$, which means

$$A_N^*x = (\zeta_i^*)_{i\in\mathbb{N}}, \quad \zeta_i^* = \sum_{n=1}^{N} \eta_i^*(n), \tag{3.24}$$

where $x = (\xi_i)_{i\in\mathbb{N}} \in l^\infty(\mathbb{N})$ and $\eta_i^*(n)$ is given by (3.20). We have the following theorem, cf. [4, Theorem 3.2].

Theorem 3.4. *Suppose that the sequences $(\beta_{i,j}^n)_{n\in\mathbb{N}}$ satisfy* (3.16) *and let B_n^* be given by* (3.20). *Then the strong limit of $\prod_{n=1}^N e^{tB_n^*}$, denoted as $(T^*(t))_{t\geq 0}$, i.e.*

$$T^*(t)x := \lim_{N\to\infty} \prod_{n=1}^{N} e^{tB_n^*}x, \quad x \in l^\infty(\mathbb{N}) \tag{3.25}$$

is a semigroup of contractions, and its generator A^* is bounded and

$$A^*x = \lim_{N\to\infty} \sum_{n=1}^{N} B_n^*x, \quad x \in l^\infty(\mathbb{N}).$$

Furthermore, denote $(\zeta_i^*)_{i\in\mathbb{N}} = A^*x$. Then

$$\zeta_1^* = \sum_{n=1}^{\infty} \left(\beta_{1,1}^n \xi_1 + \sum_{j=1}^{d-1} \beta_{1,1+j}^n \xi_{1+jd^{n-1}} \right), \tag{3.26}$$

and for $i \geq 2$, we have

$$\zeta_i^* = \sum_{n=1}^{n_0-1} \eta_i^*(n) + \sum_{n=n_0}^{\infty} \left(\beta_{1,1}^n \xi_i + \sum_{j=1}^{d-1} \beta_{1,1+j}^n \xi_{i+jd^{n-1}} \right), \tag{3.27}$$

where $\eta_i^*(n)$ and n_0 are given by (3.20) *and* (3.14), *respectively.*

Proof. The proof is analogous to that of Theorem 3.3, i.e., we use [3, Proposition 2.7]. By (3.21), for every $x \in l^\infty(\mathbb{N})$, we have

$$\sum_{n=1}^{+\infty} ||B_n^*x||_{l^\infty(\mathbb{N})} \leq (d-1)||x||_{l^\infty(\mathbb{N})} \cdot \sum_{i,j=1;\ i\neq j}^{d} \sum_{n=1}^{\infty} \beta_{i,j}^n < \infty.$$

Thus, $D_1 = l^\infty(\mathbb{N})$, A^* is bounded and

$$||A^*|| \leq (d-1) \cdot \sum_{i,j=1;\ i\neq j}^{d} \sum_{n=1}^{\infty} \beta_{i,j}^n.$$

As in $l^1(\mathbb{N})$, the norm convergence in $l^\infty(\mathbb{N})$ implies the coordinate-wise convergence, so components of A^*x are limits of components of A_N^*x, where A_N^* is given by (3.24). Thus, (3.26) and (3.27) follow, and this completes the proof. □

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Appendix

It is well known [1, p. 163] that if B and C are bounded linear operators, given in a Banach space X, then

$$\|B\| = \|B^*\|, \quad (B + C)^* = B^* + C^*, \quad (BC)^* = C^*B^*, \tag{A1}$$

where B^* and C^* are adjoint operators to B and C, respectively. Recall also that $\mathcal{L}(X)$ denotes the space of all bounded linear maps on X. To justify (3.22), we prove the following lemma.

Lemma A.1. *Let X be a Banach space and suppose that $B \in \mathcal{L}(X)$. Then*

$$(e^B)^* = e^{B^*}. \tag{A2}$$

Proof. For $N \geq 1$ denote

$$S_N = \sum_{n=0}^{N} \frac{B^n}{n!}, \quad S_N^* = \sum_{n=0}^{N} \frac{(B^*)^n}{n!}.$$

These operators converge uniformly to e^B and e^{B^*}, respectively [1, p. 251]. This means

$$\lim_{N\to\infty} \|e^B - S_N\| = 0, \quad \lim_{N\to\infty} \|e^{B^*} - S_N^*\| = 0.$$

By (A1), we have $(S_N)^* = S_N^*$, for $N \geq 1$, which implies

$$\|(e^B)^* - S_N^*\| = \|(e^B - S_N)^*\| = \|e^B - S_N\|.$$

Therefore, for any $x \in X^*$, we have

$$\lim_{N\to\infty} \|(e^B)^* x - S_N^* x\| \leq \lim_{N\to\infty} \|(e^B)^* - S_N^*\| \cdot \|x\| = 0.$$

In consequence,

$$(e^B)^* x = e^{B^*} x, \quad x \in X^*,$$

and this concludes the proof of (A2). □