DE GRUYTER Open Mathematics 2025; 23: 20250167

Research Article

Roksolana Lakhva, Roza Uteshova*, Oleksandr Stanzhytskyi, and Viktoria Mogylova

Averaging method in optimal control problems for integro-differential equations

https://doi.org/10.1515/math-2025-0167
received February 6, 2025; accepted May 12, 2025

Abstract: The averaging method is applied to the study of optimal control problems for systems of integro-differential equations with rapidly oscillating coefficients and a small parameter. The original problem is associated with an averaged optimal control problem, formulated for a system of ordinary differential equations, which significantly simplifies the analysis. It is proven that as the small parameter tends to zero, the quality criterion, optimal control, and optimal trajectory of the original problem converge to those of the averaged problem.

Keywords: optimal control, weak convergence, averaging, oscillation, quality criterion, weakly compact

MSC 2020: 49J15, 49J21, 34K11, 34K33

1 Introduction

In this study, we apply the averaging method to the study of optimal control problems for systems of integro-differential equations with rapidly oscillating coefficients and a small parameter. The averaging method is one of the most widely used and effective approaches for analyzing nonlinear dynamical systems. Originally proposed by Krylov and Bogolyubov for ordinary differential equations, this method was later developed and applied to various problems. In particular, it has been employed in the context of integro-differential systems in [1,2], and further extended to boundary value problems for such systems in [3].

Moreover, the averaging method has been effectively employed in the study of optimal control problems. The central idea is to replace the original control problem with a simpler averaged problem, whose optimal solutions are "almost" optimal for the original problem. For systems of ordinary differential equations, this approach was developed in [4,5]. For impulsive optimal control systems with both finite and infinite horizons, it was applied in [6,7]. Optimal control problems using the averaging method for systems of functional-differential equations were studied in [8].

In this work, we apply the averaging method to the analysis of optimal control problems for systems of integro-differential equations. Such equations arise as mathematical models for various processes in the natural sciences, including population dynamics [9], chemical kinetics, and fluid dynamics [10,11]. We consider both a nonlinear optimal control problem for a Volterra-type integro-differential system and a linear control problem. A key role in our study is played by lemmas on the averaging of systems of integro-differential

*** Corresponding author: Roza Uteshova,** Institute of Mathematics and Mathematical Modeling, 125 Pushkin St, 050010, Almaty, Kazakhstan, e-mail: r.uteshova@math.kz
Roksolana Lakhva: Taras Shevchenko National University of Kyiv, 64/13 Volodymyrska St, 01601, Kyiv, Ukraine, e-mail: roksolanalakhva@knu.ua
Oleksandr Stanzhytskyi: Taras Shevchenko National University of Kyiv, 64/13 Volodymyrska St, 01601, Kyiv, Ukraine, e-mail: stanzhytskiy@knu.ua
Viktoria Mogylova: National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", 37 Peremohy Ave, 03056, Kyiv, Ukraine, e-mail: mogylova.viktoria@gmail.com

equations, where the right-hand sides depend on a control functional parameter. The proximity estimates obtained for exact and averaged solutions are uniform with respect to control functions from a set of admissible controls. This allows us to establish the closeness between the optimal solutions of the exact and averaged problems. Notably, the averaged system is already a system of autonomous ordinary differential equations, which significantly simplifies its study in the context of optimal control.

The study consists of an introduction and three sections. In Section 2, we present a rigorous formulation of the problem in both the linear and nonlinear cases and state the main results of the work. Section 3 serves an auxiliary purpose, proving the necessary averaging lemmas mentioned above. The main results are proved in Section 4. Finally, examples illustrating the obtained results are provided at the end of the study.

2 Problem statement

2.1 Optimal control problem, nonlinear with respect to the control, for a system of integro-differential equations with rapidly oscillating parameters

We consider the nonlinear control problem for a system of integro-differential equations with rapidly oscillating parameters:

$$\begin{cases} \dot{x}_\varepsilon = X\left(\frac{t}{\varepsilon}, x_\varepsilon(t), \int\limits_0^t \varphi(t,s,x_\varepsilon(s))\mathrm{d}s, u(t)\right), \\ x_\varepsilon(0) = x_0, \end{cases} \tag{1}$$

with the quality criterion

$$J_\varepsilon[u] = \int\limits_0^T L(t, x_\varepsilon(t), u(t))\mathrm{d}t + \Phi(x_\varepsilon(T)) \to \inf, \tag{2}$$

over the interval $[0, T]$, where $\varepsilon > 0$ is a small parameter, $T > 0$ is a given constant, x is the state vector in $\mathbb{R}^d$, $u(t)$ is the m-dimensional control vector such that $u(t) \in W \subset \mathbb{R}^m$, $d, m = 1, 2, 3, \dots$, $\Phi(x)$ is a given function.

The function $x_\varepsilon(t, u)$ denotes the solution of the Cauchy problems (1) and (2), corresponding to the control $u(t)$. For simplicity of notation, in the following discussion, we omit the explicit dependence on u and ε and denote this solution as $x(t)$.

We assume that there exists a function $X_0(x, u)$ such that for all $x \in \mathbb{R}^d$ and $u \in W$, the following limit exists uniformly:

$$\lim_{\varepsilon \to 0} \int\limits_0^t \left[X\left(\frac{\tau}{\varepsilon}, x, \varphi_1(\tau, x), u\right) - X_0(x, u)\right]\mathrm{d}\tau = 0, \tag{3}$$

where

$$\varphi_1(t, x) = \int\limits_0^t \varphi(t, s, x)\mathrm{d}s, \quad t, s \in [0, T], x \in \mathbb{R}^d.$$

The optimal control problems (1) and (2) with rapidly oscillating coefficients correspond to a simpler optimal control problem

$$\begin{cases} \dot{\xi} = X_0(\xi, u(t)), \\ \xi(0) = x_0, \end{cases} \tag{4}$$

with the quality criterion

$$J_0[u] = \int_0^T L(t, \xi(t), u(t))\mathrm{d}t + \Phi(\xi(T)) \to \inf. \tag{5}$$

For problems (1) and (2), we assume that the following conditions hold:

(C1) The admissible controls are considered to be m-dimensional vector functions $u(\cdot)$ such that $u(\cdot) \in U$, where U is a compact set in $L^2(0, T)$;

(C2) The function $X(t, x, y, u)$ is defined and jointly continuous in all its variables in the domain $Q_0 = \{t \geq 0, x \in \mathbb{R}^d, y \in \mathbb{R}^n, u \in W\}$, and satisfies:

(C2a) a linear growth condition with respect to x, y in Q_0; that is, there exists a constant $M > 0$ such that

$$|X(t, x, y, u)| \leq M(1 + |x| + |y|),$$

for any $(t, x, y, u) \in Q_0$;

(C2b) a Lipschitz condition with constant λ in Q_0; that is,

$$|X(t, x, y, u) - X(t, x_1, y_1, u_1)| \leq \lambda(|x - x_1| + |y - y_1| + |u - u_1|),$$

for all $(t, x, y, u), (t, x_1, y_1, u_1) \in Q_0$;

(C3) The function $\varphi(t, s, x)$ is defined and continuous in the domain $Q_1 = \{t \in [0, T], s \in [0, T], x \in \mathbb{R}^d\}$, takes on the values in $\mathbb{R}^n$, and satisfies the linear growth condition and the Lipschitz condition with respect to x, i.e., there exists $L_\varphi > 0$ such that

$$|\varphi(t, s, x)| \leq L_\varphi(1 + |x|) \quad \text{and} \quad |\varphi(t, s, x) - \varphi(t, s, x_1)| \leq L_\varphi|x - x_1|;$$

(C4) There exists the limit (3) uniformly in $x \in \mathbb{R}^d$ and $u \in W$;

(C5) The function $L(t, x, u)$ is defined in the domain $Q_2 = \{t \in [0, T], x \in \mathbb{R}^d, u \in W\}$, and

(C5a) $L(t, x, u)$ is uniformly continuous in $x \in \mathbb{R}^d$ with respect to $t \in [0, T]$ and $u \in W$;

(C5b) $L(t, x, u)$ satisfies the Lipschitz condition with respect to u in Q_2, with constant $\lambda > 0$;

(C5c) The function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is continuous in x.

The conditions (C2), (C3), and Theorems 3.1 [12] and 2.2 [13] imply that for any admissible control $u(t)$, there exists a unique solution $x(t, u)$ of the Cauchy problem on the whole interval $[0, T]$. It is hence obvious that problems (1) and (4) are valid for all admissible controls.

The main result of this subsection is the theorem that establishes the relationship between the optimal control and the quality criteria of the exact problems (1), (2) and the averaged problems (4), (5). We set

$$J_\varepsilon^* = \inf_{u(\cdot)\in U} J_\varepsilon[u], \quad J_0^* = \inf_{u(\cdot)\in U} J_0[u].$$

Theorem 2.1. *Let conditions* (C1)–(C5) *hold. Then, problems* (1), (2), *and* (4), (5) *have solutions* $(x_\varepsilon^*(t), u_\varepsilon^*(t))$ *and* $(\xi^*(t), u^*(t))$, *respectively, and*

(*i*) $J_\varepsilon^* \to J_0^*$ *as* $\varepsilon \to 0$;

(*ii*) *for any* $\eta > 0$, *there exists* ε_0 *such that for* $\varepsilon < \varepsilon_0$,

$$|J_\varepsilon^* - J_\varepsilon[u^*]| < \eta,$$

i.e., the optimal control of the averaged problem is nearly optimal for the exact problem;

(*iii*) *there exists a sequence* $\varepsilon_n \to 0$, $n \to \infty$, *such that*

$$x_{\varepsilon_n}^*(t) \to \xi^*(t) \quad \text{uniformly on } [0, T], \tag{6}$$

and

$$u_{\varepsilon_n}^*(\cdot) \to u^*(\cdot) \quad \text{in } L^2(0, T). \tag{7}$$

Furthermore, if the averaged problems (4), (5) *have a unique solution, then convergence* (6) *and* (7) *holds for all* $\varepsilon_n \to 0$.

2.2 Optimal control problem, linear with respect to the control, for a system of integro-differential equations with rapidly oscillating parameters

We also consider the control problem with rapidly oscillating parameters, that is, linear with respect to the control input:

$$\dot{x}_\varepsilon(t) = f\left(\frac{t}{\varepsilon}, x_\varepsilon(t), \int_0^t \varphi(t, s, x_\varepsilon(s))\mathrm{d}s\right) + f_1(x_\varepsilon(t))u(t), \quad x(0) = x_0, \tag{8}$$

with the quality criterion

$$J_\varepsilon[u] = \int_0^T [A(t, x_\varepsilon(t)) + B(t, u(t))]\mathrm{d}t + \Phi(x_\varepsilon(T)) \to \inf, \tag{9}$$

over the interval $[0, T]$, where $\varepsilon > 0$ is a small parameter, $T > 0$ is a given constant, $x \in \mathbb{R}^d$ is the state vector, and $u(t)$ is the m-dimensional control vector belonging to a functional set.

If the following limit exists uniformly with respect to $x \in \mathbb{R}^d$:

$$\lim_{\varepsilon\to 0} \int_0^t \left[f\left(\frac{\tau}{\varepsilon}, x, \varphi_1(\tau, x)\right) - f_0(x)\right]\mathrm{d}\tau = 0, \tag{10}$$

with

$$\varphi_1(t, x) = \int_0^t \varphi(t, s, x)\mathrm{d}s, \quad t, s \in [0, T],$$

then the optimal control problems (8), (9) with rapidly oscillating coefficients correspond to a simpler control problem on the interval $[0, T]$:

$$\dot{\xi} = f_0(\xi) + f_1(\xi)u(t), \quad \xi(0, u(0)) = x_0, \tag{11}$$

with the corresponding quality criterion

$$J_0[u] = \int_0^T [A(t, \xi(t)) + B(t, u(t))]\mathrm{d}t + \Phi(\xi(T)) \to \inf. \tag{12}$$

The main result is the proof of the convergence of the minimal values of the quality criterion, optimal controls, and optimal trajectories of the exact problems (8) and (9) to the corresponding minimal values of the quality criterion, optimal controls, and trajectories of the averaged problems.

We assume that the following conditions are met for problems (8) and (9):

(C6) The admissible control is an m-dimensional vector function $u(\cdot) \in L^p((0, T); V)$, $p > 1$, taking on the values in a closed convex set $V \subset \mathbb{R}^m$;

(C7) The function $f(t, x, y)$ is defined and jointly continuous in all its variables in the domain $Q_3 = \{t \geq 0, x \in \mathbb{R}^d, y \in \mathbb{R}^n\}$; the $n \times m$ matrix function $f_1(x)$ is defined for $x \in \mathbb{R}^d$, and

- (C7a) $f(t, x, y)$ satisfies the linear growth condition with constant M in the domain Q_3, i.e., $|f(t, x, y)| \leq M(1 + |x| + |y|)$ for all $(t, x, y) \in Q_3$;
- (C7b) $f(t, x, y)$ and $f_1(x)$ satisfy, with respect to x, the Lipschitz condition with constant $\lambda > 0$ in their domains;

(C8) Function $\varphi(t, s, x)$ is defined and continuous in the domain $Q_4 = \{t \in [0, T], s \in [0, T], x \in \mathbb{R}^d\}$, takes on the values in the space $\mathbb{R}^n$, and satisfies, with respect to x, the linear growth condition and the Lipschitz condition; that is, there exists some $L_\varphi > 0$ such that

$$|\varphi(t, s, x) - \varphi(t, s, x_1)| \leq L_\varphi|x - x_1| \quad \text{and} \quad |\varphi(t, s, x)| \leq L_\varphi(1 + |x|);$$

(C9) Limit (10) exists uniformly in $x \in \mathbb{R}^d$;

(C10) The scalar functions $A(t, x)$ and $B(t, u)$ are defined for $t \in [0, T]$, $x \in \mathbb{R}^d$, $u \in V$, and jointly continuous in all their variables, and

(C10a) $A(t, x) \geq 0$, $B(t, u) \geq a|u|^p$ with a constant $a > 0$, for all $t \in [0, T]$, and the function $B(t, u)$ is convex with respect to $u \in V$;

(C10b) The function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is non-negative and continuous in x.

The main result here is the following theorem on a relationship between the optimal triples of the exact and averaged problems.

Theorem 2.2. *Let conditions* (*C*6)–(*C*10) *hold. Then, problems* (8), (9) *and* (11), (12) *have solutions* $(x_\varepsilon^*(t), u_\varepsilon^*(t))$ *and* $(\xi^*(t), u^*(t))$, *respectively, and*

(*i*) $J_\varepsilon^* \to J_0^*$ *as* $\varepsilon \to 0$;

(*ii*) *for any* $\eta > 0$, *there exists* ε_0 *such that*

$$|J_\varepsilon^* - J_\varepsilon[u^*]| < \eta$$

holds for $\varepsilon < \varepsilon_0$;

(*iii*) *there exists a sequence* $\varepsilon_n \to 0$, $n \to \infty$, *such that*

$$x_{\varepsilon_n}^*(t) \rightrightarrows \xi^*(t) \quad \textit{uniformly on } [0, T], \tag{13}$$

and

$$u_{\varepsilon_n}^*(\cdot) \xrightarrow{w} u^*(\cdot) \quad \textit{weakly in } L^p(0, T). \tag{14}$$

Furthermore, if the averaged problems (11), (12) *have a unique solution, then convergence* (13) *and* (14) *holds for all* $\varepsilon \to 0$.

3 Averaging lemmas

This section is devoted to proving lemmas on the closeness of solutions of the original optimal control system and the solutions of the corresponding averaged system in both the nonlinear-in-controls case and the linear-in-controls case.

Lemma 3.1. *Let conditions* (*C*1)–(*C*4) *hold. Then, given any* $\eta > 0$, *there exists* $\varepsilon_0 = \varepsilon_0(\eta)$ *such that for* $0 < \varepsilon \leq \varepsilon_0$, *the solutions of the Cauchy problems* (1) *and* (4) *satisfy the estimate*

$$|x(t, u) - \xi(t, u)| \leq \eta, \tag{15}$$

for all $t \in [0, T]$ *and all admissible controls* $u(t)$.

Remark 3.1. In this lemma, it is important that estimate (15) is uniform for all admissible controls u.

Proof. Let us choose an arbitrary $\eta > 0$ and fix it. For any $\varepsilon > 0$ and any admissible control $u(t)$, we estimate the difference between $x(t, u)$ and $\xi(t, u)$. For simplicity, we denote $x(t, u) = x(t)$ and $\xi(t, u) = \xi(t)$. We also omit the dependence of $x(t)$ on ε.

Since U is compact in $L^2(0, T)$, for the given η, there exists a finite $\frac{\eta e^{-\lambda}}{4\lambda}$-net $u_1(t), \ldots, u_N(t)$, where $N = N(\eta)$. Thus, for the chosen control $u(t)$, there exists a representative $u_j(t)$ from the net such that

$$\|u(\cdot) - u_j(\cdot)\|_{L^2} \leq \frac{\eta}{4\lambda} e^{-\lambda}. \tag{16}$$

Again, since U is compact in $L^2(0,T)$, there exists $K>0$ such that all admissible controls $u(t)$ satisfy the inequality

$$\int_0^T |u(t)|\mathrm{d}t \le K. \tag{17}$$

By (C2a) and (C3),

$$|x(t)| \le |x_0| + MT + M\int_0^T \left(|x(s)| + L_\varphi \int_0^s (1+|x(\tau)|)\mathrm{d}\tau\right)\mathrm{d}s.$$

From this, using an analog of the Gronwall-Bellman inequality, we obtain

$$|x(t)| \le C, \tag{18}$$

where $C = C(T)$. Similarly, we obtain the estimate $|\xi(t)| \le C$.

Hence, it follows from conditions (C2) and (C3) that

$$\begin{aligned}
|x(t)-\xi(t)| &\le \left|\int_0^t \left(X\left(\frac{s}{\varepsilon}, x(s), \int_0^s \varphi(s,\tau,x(\tau))\mathrm{d}\tau, u(s)\right) - X_0(\xi(s),u(s))\right)\mathrm{d}s\right| \\
&\le \int_0^t \left|X\left(\frac{s}{\varepsilon}, x(s), \int_0^s \varphi(s,\tau,x(\tau))\mathrm{d}\tau, u(s)\right) - X\left(\frac{s}{\varepsilon}, x(s), \int_0^s \varphi(s,\tau,x(\tau))\mathrm{d}\tau, u_j(s)\right)\right|\mathrm{d}s \\
&\quad + \int_0^t |X_0(\xi(s),u(s)) - X_0(\xi(s),u_j(s))|\mathrm{d}s \\
&\quad + \left|\int_0^t X\left(\frac{s}{\varepsilon}, x(s), \int_0^s \varphi(s,\tau,x(\tau))\mathrm{d}\tau, u_j(s)\right) - X_0(\xi(s),u_j(s))\mathrm{d}s\right| \\
&\le \left|\int_0^t X\left(\frac{s}{\varepsilon}, x(s), \int_0^s \varphi(s,\tau,x(\tau))\mathrm{d}\tau, u_j(s)\right) - X_0(\xi(s),u_j(s))\mathrm{d}s\right| + 2\lambda\left(\int_0^T |u(s)-u_j(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}$$

We then obtain

$$|x(t)-\xi(t)| \le I_1 + \frac{\eta}{2}e^{-\lambda T}. \tag{19}$$

Let us now estimate I_1 using conditions (C2), (C3), and (C4). Note that these conditions imply that the function X_0 satisfies the Lipschitz condition. We have:

$$\begin{aligned}
I_1 &\le \int_0^t \left|X\left(\frac{s}{\varepsilon}, x(s), \int_0^s \varphi(s,\tau,x(\tau))\mathrm{d}\tau, u_j(s)\right) - X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(\tau))\mathrm{d}\tau, u_j(s)\right)\right|\mathrm{d}s \\
&\quad + \left|\int_0^t \left(X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(\tau))\mathrm{d}\tau, u_j(s)\right) - X_0(\xi(s),u_j(s))\right)\mathrm{d}s\right| \\
&\le \int_0^t \left(\lambda|x(s)-\xi(s)| + \int_0^s |x(t)-\xi(t)|L_\varphi \mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \left|\int_0^t \left(X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(\tau))\mathrm{d}\tau, u_j(s)\right) - X_0(\xi(s),u_j(s))\right)\mathrm{d}s\right|.
\end{aligned} \tag{20}$$

Since every function in $L^2(0, T)$ can be approximated in the L^2-norm by a continuous function, and every continuous function on a closed interval can be approximated by a piecewise constant function, we choose for $u_j(t)$ a continuous function $u_c(t)$ and a piecewise constant function $u_p(t)$ such that the following inequalities hold:

$$\|u_j - u_c\|_{L^2} < \frac{\eta}{16\lambda} e^{-\lambda T}, \tag{21}$$

$$\|u_c(t) - u_p(t)\|_{L^2} < \frac{\eta}{16\lambda} e^{-\lambda T}, \tag{22}$$

for $t \in [0, T]$.

Using (21) and (22), we estimate the last integral in (20):

$$\begin{aligned}
&\left|\int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau, u_j(s)\right) - X_0(\xi(s), u_j(s))\right]\mathrm{d}s\right| \\
&= \left|\int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau, u_j(s)\right) - X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau, u_c(s)\right)\right]\mathrm{d}s\right. \\
&\quad + \int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau, u_c(s)\right) - X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau, u_p(s)\right)\right]\mathrm{d}s \\
&\quad + \int_0^t X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau, u_p(s)\right)\mathrm{d}s + \int_0^t (X_0(\xi(s), u_c(s)) - X_0(\xi(s), u_j(s)))\mathrm{d}s \\
&\quad \left. + \int_0^t (X_0(\xi(s), u_p(s)) - X_0(\xi(s), u_c(s)))\mathrm{d}s - \int_0^t X_0(\xi(s), u_p(s))\mathrm{d}s\right| \le \lambda\left(\int_0^T |u_j(s) - u_c(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} \\
&\quad + \lambda\left(\int_0^T |u_c(s) - u_p(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} + \left|\int_0^t \left(X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau, u_p(s)\right) - X_0(\xi(s), u_p(s))\right)\mathrm{d}s\right| + \lambda\left(\int_0^T |u_c(s) - u_j(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} \\
&\quad + \lambda\left(\int_0^T |u_p(s) - u_c(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}} \le \left|\int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau, u_p(s)\right) - X_0(\xi(s), u_p(s))\right]\mathrm{d}s\right| + \frac{\eta}{4} e^{-\lambda T}.
\end{aligned}$$

Let us consider the last integral in this inequality. We have:

$$\begin{aligned}
&\left|\int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau, u_p(s)\right) - X_0(\xi(s), u_p(s))\right]\mathrm{d}s\right| \\
&\quad \le \left|\int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau, u_p(s)\right) - X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(s))\mathrm{d}\tau, u_p(s)\right)\right]\mathrm{d}s\right| \\
&\qquad + \left|\int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s, \tau, \xi(s))\mathrm{d}\tau, u_p(s)\right) - X_0(\xi(s), u_p(s))\right]\mathrm{d}s\right| = I_2 + I_3.
\end{aligned}$$

We estimate the integral I_2 by dividing the interval $[0, T]$ by the points $\{t_k\}_0^R$ ($t_0 = 0$, $t_R = T$) in such a way that all components of the vector function $u_p(t)$ have constant values on each subinterval $[t_k, t_{k+1})$, that is, $u_p(t) = u_p(t_k)$ for $t \in [t_k, t_{k+1})$. Here, the natural $R = R(\eta)$ is fixed for a given choice of η.

Now, we choose a natural n and divide the interval $[0, T]$ into n equal parts by the points $t_i = i/n$ ($i = 0, \ldots, n$). Suppose n is large enough so that each subinterval $[t_k, t_{k+1})$ contains the points t_i. As a result, we obtain n intervals $[t_i, t_{i+1})$. If for some k and i we have $t_i < t_k < t_{i+1}$, the interval $[t_i, t_{i+1})$ is split into two subintervals: $[t_i, t_k)$ and $[t_k, t_{i+1})$. Thus, the interval $[0, T]$ is divided into no more than $n + R$ subintervals, each with length not exceeding $\frac{1}{n}$. The partition points are again denoted by t_i, and the total number of intervals $[t_i, t_{i+1})$ is denoted by $K = K(\eta)$. Clearly, $K \le n + R$, and $u_p(t) = u_p(t_i)$ for $t \in [t_i, t_{i+1})$. Let us denote $\xi_i = \xi(t_i)$ and $u_p(t_i) = u_{pi}$. Then,

$$I_2 \le \sum_{i=0}^{K-1} \left| \int_{t_i}^{t_{i+1}} \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(\tau))\mathrm{d}\tau, u_{pi}\right) - X\left(\frac{s}{\varepsilon}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)\mathrm{d}\tau, u_{pi}\right)\right]\mathrm{d}s \right|$$
$$+ \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} \left| X\left(\frac{s}{\varepsilon}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)\mathrm{d}\tau, u_{pi}\right) - X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(s))\mathrm{d}\tau, u_{pi}\right)\right| \mathrm{d}s$$
$$\le \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi(s) - \xi_i|\mathrm{d}s + \int_{t_i}^{t_{i+1}} \int_0^s L_\varphi |\xi(\tau) - \xi_i|\mathrm{d}\tau\mathrm{d}s + \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi_i - \xi(s)|\mathrm{d}s + \int_{t_i}^{t_{i+1}} \int_0^s L_\varphi |\xi_i - \xi(s)|\mathrm{d}\tau\mathrm{d}s$$
$$\le 2\sum_{i=0}^{K-1} \lambda \frac{MT(1+C)}{n^2}\left(1 + \int_{t_i}^{t_{i+1}} \mathrm{d}s \int_0^s L_\varphi \mathrm{d}\tau\right) \le \lambda MT(1+C)\frac{n+R}{n^2}\left(1 + L_\varphi \frac{T}{n}\right).$$

Now, for the chosen $\eta > 0$, there exists a number $\eta > 0$ such that for all $\varepsilon > 0$, the following holds:

$$I_2 \le \frac{\eta}{8} e^{-\lambda T}.$$

We now fix the chosen n and estimate the integral I_3. To do this, we split it over the interval $[0, T]$ into a sum of integrals:

$$\left| \int_0^t \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(s))\mathrm{d}\tau, u_p(s)\right) - X_0(\xi(s), u_p(s))\right]\mathrm{d}s \right|$$
$$\le \left| \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} \left[X\left(\frac{s}{\varepsilon}, \xi(s), \int_0^s \varphi(s,\tau,\xi(s))\mathrm{d}\tau, u_{pi}\right) - X\left(\frac{s}{\varepsilon}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)\mathrm{d}\tau, u_{pi}\right)\right]\mathrm{d}s \right|$$
$$+ \left| \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} [X_0(\xi(s), u_{pi}) - X_0(\xi_i, u_{pi})]\mathrm{d}s \right| + \left| \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} \left[X\left(\frac{s}{\varepsilon}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)\mathrm{d}\tau, u_{pi}\right) - X_0(\xi_i, u_{pi})\right]\mathrm{d}s \right|$$
$$\le \sum_{i=0}^{K-1}\left[\lambda \int_{t_i}^{t_{i+1}} |\xi(s) - \xi_i|\mathrm{d}s + \int_{t_i}^{t_{i+1}} \int_0^s L_\varphi |\xi(s) - \xi_i|\mathrm{d}\tau\mathrm{d}s\right] + \sum_{i=0}^{K-1} \lambda \int_{t_i}^{t_{i+1}} |\xi(s) - \xi_i|\mathrm{d}s + I_4.$$

Let us now estimate the integral I_4. We obtain

$$I_4 = \left| \sum_{i=0}^{K-1} \int_{t_i}^{t_{i+1}} \left[X\left(\frac{s}{\varepsilon}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)\mathrm{d}\tau, u_{pi}\right) - X_0(\xi_i, u_{pi})\right]\mathrm{d}s \right|.$$

In terms of $\varphi_1(t, x)$, we have

$$\left| \int_{t_i}^{t_{i+1}} \left[X\left(\frac{s}{\varepsilon}, \xi_i, \varphi_1(s, \xi_i), u_{pi}\right) - X_0(\xi_i, u_{pi})\right]\mathrm{d}s \right|$$
$$= \left| \int_0^{t_{i+1}} \left[X\left(\frac{s}{\varepsilon}, \xi_i, \varphi_1(s, \xi_i), u_{pi}\right) - X_0(\xi_i, u_{pi})\right]\mathrm{d}s - \int_0^{t_i} \left[X\left(\frac{s}{\varepsilon}, \xi_i, \varphi_1(s, \xi_i), u_{pi}\right) - X_0(\xi_i, u_{pi})\right]\mathrm{d}s \right|.$$

Due to condition (3), each term on the right-hand side of the last equality tends to zero as $\varepsilon \to 0$. Since K is fixed, by choosing a sufficiently small ε, it is possible to achieve the inequality

$$I_4 \le \frac{\eta}{16} e^{-\lambda T}.$$

Hence,

$$I_3 \le \frac{\eta}{8} e^{-\lambda T}.$$

Similarly, for I_2, the following inequality can be obtained:

$$I_1 \le \lambda \left(\int_0^t |x(s) - \xi(s)| \mathrm{d}s + \int_0^t \int_0^s L_\varphi |x(\tau) - \xi(\tau)| \mathrm{d}\tau \mathrm{d}s \right) + \frac{\eta}{4} e^{-\lambda T} \le \frac{\eta}{2} e^{-\lambda T}.$$

The reasoning outlined above can be applied to each function $u_1(t), u_2(t), \ldots, u_n(t)$ from the constructed grid. Due to its finiteness, ε_0 can be chosen uniformly for each function from the grid.

Thus, from inequalities (16)–(19), (22), and the last two estimates for the integrals I_1 and I_2, it follows that inequality (15) holds uniformly for all admissible controls, which proves the lemma. □

Lemma 3.2. *Let conditions* (*C*6)–(*C*9) *hold. If* $u_{\varepsilon_n} \xrightarrow{w} u_0$ *weakly in* $L^p(0, T)$ *as* $\varepsilon \to 0$, *then the solution* $x_\varepsilon(t)$ *of the Cauchy problem* (8) *with* $u(t) = u_\varepsilon(t)$ *converges uniformly on* $[0, T]$ *to the solution* $\xi(t)$ *of the corresponding Cauchy problem* (11) *with control* $u(t) = u_0(t)$, *i.e.,*

$$x_\varepsilon(t) \rightrightarrows \xi(t), \quad \varepsilon \to 0$$

uniformly in $t \in [0, T]$.

Proof. Let us rewrite (8) in the integral form

$$x_\varepsilon(t) = x_0 + \int_0^t f\left(\frac{s}{\varepsilon}, x_\varepsilon(s), \int_0^s \varphi(s, \tau, x_\varepsilon(\tau)) \mathrm{d}\tau \right) \mathrm{d}s + \int_0^t f_1(x_\varepsilon(s)) u_\varepsilon(s) \mathrm{d}s.$$

Without loss of generality we can assume $T = 1$. We have

$$\begin{aligned} |x_\varepsilon(t)| &\le |x_0| + \int_0^t M(1 + |x_\varepsilon(s)| + \int_0^s L_\varphi (1 + |x_\varepsilon(\tau)|) \mathrm{d}\tau) \mathrm{d}s + \int_0^t (|f_1(x_\varepsilon(s)) - f_1(0)| + |f_1(0)|) |u_\varepsilon(s)| \mathrm{d}s \\ &\le |x_0| + \int_0^t (M + L_\varphi + |f_1(0)||u_\varepsilon(s)|) \mathrm{d}s + \int_0^t (M + \lambda |u_\varepsilon(s)|) |x_\varepsilon(s)| \mathrm{d}s + L_\varphi \int_0^t \int_0^s |x_\varepsilon(\tau)| \mathrm{d}\tau \mathrm{d}s. \end{aligned} \tag{23}$$

Applying the generalized Gronwall-Bellman inequality to (23), we obtain

$$|x_\varepsilon(t)| \le (|x_0| + M + L_\varphi + |f_1(0)| \int_0^t |u_\varepsilon(s)| \mathrm{d}s) e^{M + \lambda \int_0^t |u_\varepsilon(s)| \mathrm{d}s + L_\varphi}.$$

Let $M^* = M + L_\varphi$, then

$$|x_\varepsilon(t)| \le (|x_0| + M^* + |f_1(0)| \|u_\varepsilon\|_{L^p}) \cdot e^{M^* + \lambda \|u_\varepsilon\|_{L^p}}. \tag{24}$$

From the weak convergence of u_ε, it follows that u_ε is strongly bounded, i.e., $\sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^p} < \infty$. This, together with (24), implies the existence of a constant $C > 0$ such that

$$|x_\varepsilon(t)| \le C, \tag{25}$$

for all $\varepsilon > 0$ and $t \in [0, 1]$.

Now, for any $t_1 < t_2$, where $t_1, t_2 \in [0, 1]$, we have

$$|x_\varepsilon(t_2) - x_\varepsilon(t_1)| \le \int_{t_1}^{t_2} M(1 + C + L_\varphi \int_0^s (1 + C)\mathrm{d}\tau)\mathrm{d}s + \int_{t_1}^{t_2} (|f_1(0)| + \lambda C)|u_\varepsilon(s)|\mathrm{d}s$$
$$\le M(1 + C)(t_2 - t_1) + ML_\varphi(1 + C)(t_2 - t_1) + (|f_1(0)| + \lambda C)\left(\int_{t_1}^{t_2} |u_\varepsilon(s)|^p \mathrm{d}s\right)^{\frac{1}{p}} (t_2 - t_1)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

From the last inequality, it follows that the family $x_{\varepsilon_n}(t)$ is equicontinuous on $[0, 1]$, and taking into account (25), it is also compact.

Let $x_{\varepsilon_n}(t)$ be a sequence that converges uniformly to some function $\xi(t)$ as $\varepsilon_n \to 0$. We will show that $\xi(t)$ is a solution of the Cauchy problem with the control $u(t) = u_0(t)$. We have

$$x_{\varepsilon_n}(t) = x_0 + \int_0^t f\left(\frac{s}{\varepsilon_n}, x_{\varepsilon_n}(s), \int_0^s \varphi(s, \tau, x_{\varepsilon_n}(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \int_0^t f_1(x_{\varepsilon_n}(s))u_{\varepsilon_n}(s)\mathrm{d}s.$$

Let us consider the following expression:

$$\begin{aligned}
&\left|\int_0^t \left[f\left(\frac{s}{\varepsilon_n}, x_{\varepsilon_n}(s), \int_0^s \varphi(s, \tau, x_{\varepsilon_n}(\tau))\mathrm{d}\tau\right) - f_0(\xi(s)) + f_1(x_{\varepsilon_n}(s))u_{\varepsilon_n}(s) - f_1(\xi(s))u_0(s)\right]\mathrm{d}s\right| \\
&\le \left|\int_0^t \left[f\left(\frac{s}{\varepsilon_n}, x_{\varepsilon_n}(s), \int_0^s \varphi(s, \tau, x_{\varepsilon_n}(\tau))\mathrm{d}\tau\right) - f\left(\frac{s}{\varepsilon_n}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau\right)\right]\mathrm{d}s\right| \\
&+ \left|\int_0^t \left[f\left(\frac{s}{\varepsilon_n}, \xi(s), \int_0^s \varphi(s, \tau, \xi(\tau))\mathrm{d}\tau\right) - f_0(\xi(s))\right]\mathrm{d}s\right| \\
&+ \left|\int_0^t [f_1(x_{\varepsilon_n}(s))u_{\varepsilon_n}(s) - f_1(\xi(s))u_{\varepsilon_n}(s) + f_1(\xi(s))u_{\varepsilon_n}(s) - f_1(\xi(s))u_0(s)]\mathrm{d}s\right|.
\end{aligned} \tag{26}$$

The first term in (26), due to conditions (C7) and (C8), admits the following estimate:

$$\lambda \int_0^t \left(|x_{\varepsilon_n}(s) - \xi(s)| + \int_0^s L_\varphi |x_{\varepsilon_n}(\tau) - \xi(\tau)|\mathrm{d}\tau\right)\mathrm{d}s \le \sup_{t\in[0, 1]} |x_{\varepsilon_n}(t) - \xi(t)|\lambda(1 + L_\varphi) \to 0, \quad \varepsilon_n \to 0.$$

For the last term in (26), we obtain the estimate

$$\begin{aligned}
&\left|\int_0^t [(f_1(x_{\varepsilon_n}(s)) - f_1(\xi(s)))u_{\varepsilon_n}(s) + f_1(\xi(s))(u_{\varepsilon_n}(s) - u_0(s))]\mathrm{d}s\right| \\
&\le \left|\int_0^t (f_1(x_{\varepsilon_n}(s)) - f_1(\xi(s)))u_{\varepsilon_n}(s)\mathrm{d}s\right| + \left|\int_0^t f_1(\xi(s))(u_{\varepsilon_n}(s) - u_0(s))\mathrm{d}s\right|.
\end{aligned} \tag{27}$$

Taking into account (25) and the continuity of the function f_1, and using the weak convergence of u_{ε_n} to u_0 in $L^p(0, 1)$, we obtain that the last term in (27) tends to 0.

We now estimate the first term in (27). Under condition (C7b), it holds that

$$\begin{aligned}
\left|\int_0^t (f_1(x_{\varepsilon_n}(s)) - f_1(\xi(s)))u_{\varepsilon_n}(s)\mathrm{d}s\right| &\le \sup_{t\in[0, 1]} |x_{\varepsilon_n}(s) - \xi(s)| \cdot \int_0^t |u_{\varepsilon_n}(s)|\mathrm{d}s \\
&\le \sup_{t\in[0, 1]} |x_{\varepsilon_n}(s) - \xi(s)| \cdot \|u_{\varepsilon_n}\|_{L^p}.
\end{aligned} \tag{28}$$

Taking into account $\sup_{t\in[0,1]}|x_{\varepsilon_n}(s) - \xi(s)| \to 0$, as well as the uniform boundedness of $\|u_{\varepsilon_n}\|_{L^p}$, we conclude that, due to (28), the first term in (27) tends to 0.

Let us now estimate the second term in the right-hand side of (26), which we denote by I_1. We will show that for any $\eta > 0$, there exists ε_{n_0} such that for $\varepsilon_n < \varepsilon_{n_0}$, the following inequality holds:

$$I_1 = \left|\int_0^t \left[f\left(\frac{s}{\varepsilon_n}, \xi(s), \int_0^s \varphi(s,\tau,\xi(\tau))d\tau\right) - f_0(\xi(s))\right]ds\right| < \eta.$$

To show this, we choose a natural number k and divide the interval $[0, 1]$ into k equal parts using the points $t_i = \frac{i}{k}$ $(i = 0, \dots, k)$, i.e. $|t_{i+1} - t_i| \le \frac{1}{k}$. We denote the total number of the intervals $[t_i, t_{i+1})$ by $\kappa = \kappa(\eta)$. Due to the uniform continuity of $\xi(t)$ on $[0, 1]$, for $\eta > 0$ one can specify k such that following estimate will be valid:

$$|\xi(t_{i+1}) - \xi(t_i)| < \frac{\eta}{2\lambda(2 + L_\varphi)}. \tag{29}$$

Let us fix such k and denote $\xi(t_i) = \xi_i$. We have

$$\begin{aligned}
I_1 \le& \left|\sum_{i=0}^{\kappa-1}\int_{t_i}^{t_{i+1}}\left[f\left(\frac{s}{\varepsilon_n}, \xi(s), \int_0^s \varphi(s,\tau,\xi(\tau))d\tau\right) - f\left(\frac{s}{\varepsilon_n}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)d\tau\right)\right.\right.\\
&\left.\left. + f\left(\frac{s}{\varepsilon_n}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)d\tau\right) - f_0(\xi_i) + f_0(\xi_i) - f_0(\xi(s))\right]ds\right|\\
\le& \sum_{i=0}^{\kappa-1}\int_{t_i}^{t_{i+1}}\left[\left|f\left(\frac{s}{\varepsilon_n}, \xi(s), \int_0^s \varphi(s,\tau,\xi(s))d\tau\right) - f\left(\frac{s}{\varepsilon_n}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)d\tau\right)\right| + |f_0(\xi_i) - f_0(\xi(s))|\right]ds\\
&+ \left|\sum_{i=0}^{\kappa-1}\int_{t_i}^{t_{i+1}}\left(f\left(\frac{s}{\varepsilon_n}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)d\tau\right) - f_0(\xi_i)\right)ds\right|\\
\le& \lambda\sum_{i=0}^{\kappa-1}\int_{t_i}^{t_{i+1}}\left(|\xi(s) - \xi_i| + L_\varphi\int_0^t |\xi(\tau) - \xi_i|d\tau + |\xi_i - \xi(s)|\right)ds\\
&+ \left|\sum_{i=0}^{\kappa-1}\int_{t_i}^{t_{i+1}}\left[f\left(\frac{s}{\varepsilon_n}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)d\tau\right) - f_0(\xi_i)\right]ds\right|.
\end{aligned}$$

It follows from (29) that

$$\lambda\sum_{i=0}^{\kappa-1}\int_{t_i}^{t_{i+1}}\left(|\xi(s) - \xi_i| + L_\varphi\int_0^t |\xi(\tau) - \xi_i|d\tau + |\xi_i - \xi(s)|\right)ds \le \frac{\eta}{2}.$$

Let I_{11} denote the following expression:

$$I_{11} = \left|\sum_{i=0}^{\kappa-1}\int_{t_i}^{t_{i+1}}\left[f\left(\frac{s}{\varepsilon_n}, \xi_i, \int_0^s \varphi(s,\tau,\xi_i)d\tau\right) - f_0(\xi_i)\right]ds\right|.$$

In terms of $\varphi_1(t, x)$, we have

$$\begin{aligned}
&\left|\sum_{i=1}^{\kappa-1}\int_{t_i}^{t_{i+1}}\left[f\left(\frac{s}{\varepsilon_n}, \xi_i, \varphi_1(s,\xi_i)\right) - f_0(\xi_i)\right]ds\right|\\
&= \left|\sum_{i=1}^{\kappa-1}\left[\int_0^{t_{i+1}}\left(f\left(\frac{s}{\varepsilon_n}, \xi_i, \varphi_1(s,\xi_i)\right) - f_0(\xi_i)\right)ds - \int_0^{t_i}\left(f\left(\frac{s}{\varepsilon_n}, \xi_i, \varphi_1(s,\xi_i)\right) - f_0(\xi_i)\right)\right]ds\right|.
\end{aligned}$$

Due to (10), for each i, there exists ε_{n_i} such that for $\varepsilon_n < \varepsilon_{n_i}$, the following inequalities hold:

$$\int_0^{t_i}\left[f\left(\frac{s}{\varepsilon_n}, \xi_i, \varphi_1(s, \xi_i)\right) - f_0(\xi_i)\right]\mathrm{d}s \le \frac{\eta}{4k}.$$

Since k is fixed, the number of such integrals is finite. Let $\varepsilon_\eta = \min\{\varepsilon_{n_1}, \ldots, \varepsilon_{n_k}\}$. Then, for $\varepsilon_n < \varepsilon_\eta$, we obtain

$$I_{11} \le \frac{\eta}{2}.$$

Thus,

$$I_1 \le \eta.$$

The latter means that $\xi(t)$ is the solution of the Cauchy problem (8). Consequently, the uniform convergence $x_{\varepsilon_n} \rightrightarrows \xi(t)$ as $\varepsilon_n \to 0$ implies convergence to the solution of the Cauchy problem (8). Since $\xi(t)$ is the unique solution, the entire sequence x_ε converges to $\xi(t)$, which completes the proof of this lemma. □

4 Proof of main theorems

4.1 Nonlinear case

Proof of Theorem 2.1. For simplicity, we will again assume $T = 1$. Let us first prove the existence of solutions. To do this, we will establish the continuity of $J_\varepsilon[u]$ with respect to u for each $\varepsilon > 0$.

Let $u_1(t)$, $u_2(t)$ be any admissible controls for problem (1), (2), and let $x(t, u_1)$, $x(t, u_2)$ be the corresponding trajectories.

Using condition (C2) and Gronwall's inequality, we obtain

$$\sup_{t\in[0,1]} |x(t, u_1) - x(t, u_2)| \le \lambda\|u_1 - u_2\|_{L^2} e^{\lambda}. \tag{30}$$

Therefore,

$$\begin{aligned} |J_\varepsilon[u_1] - J_\varepsilon[u_2]| &\le \int_0^1 |L(t, x(t, u_1), u_1(t)) - L(t, x(t, u_2), u_1(t)) + L(t, x(t, u_2), u_1(t)) - L(t, x(t, u_2), u_2(t))|\mathrm{d}t \\ &\quad + |\Phi(x(1, u_1)) - \Phi(x(1, u_2))| \\ &\le \lambda\|u_1 - u_2\|_{L^2} + \int_0^1 |L(t, x(t, u_1), u_1(t)) - L(t, x(t, u_2), u_1(t))|\mathrm{d}t + |\Phi(x(1, u_1)) - \Phi(x(1, u_2))|. \end{aligned} \tag{31}$$

Now, using estimate (18), which is uniform for all admissible $u(t)$, we conclude that $x(t, u)$ remains within the ball B_C of radius C centered at zero for all $t \in [0, 1]$.

According to assumption (C5a) and Cantor's theorem, the function $L(t, x, u)$ is uniformly continuous in $x \in B_c$, uniformly with respect to $t \in [0, 1]$ and $u \in W$. Similarly, Φ is uniformly continuous in $x \in B_C$. Therefore, from (30) and (31), it follows that $J_\varepsilon[u]$ is continuous in the L^2-norm.

A similar argument establishes the continuity of the functional $J_0[u]$ with respect to u.

Now, considering the compactness of the set of admissible controls, we establish the existence of optimal solutions $(x_\varepsilon^*(t), u_\varepsilon^*(t))$ and $(\xi^*(t), u^*(t))$ of problems (1), (2) and (4), (5), respectively. This proves the existence of optimal solutions for both the exact and the averaged problems.

Let us now prove statement (i), namely, that $J_\varepsilon^* \to J_0^*$ as $\varepsilon \to 0$. We choose an arbitrary $\eta > 0$ and fix it. Then, we have

$$J_\varepsilon^* \le J_\varepsilon[u^*] = J_0^* + J_\varepsilon[u^*] - J_0[u^*]. \tag{32}$$

However,

$$|J_\varepsilon[u^*] - J_0[u^*]| \leq \int_0^1 |L(t, x(t, u^*), u^*(t)) - L(t, \xi(t), u^*(t))| dt + |\Phi(x(1, u^*)) - \Phi(\xi(1))|. \tag{33}$$

By Lemma 3.1, we have

$$\max_{t\in[0,1]} |x(t, u^*) - \xi^*(t)| \to 0, \quad \varepsilon \to 0. \tag{34}$$

Taking into account the uniform continuity of the function $L(t, x, u)$ with respect to $x \in B_c$, uniformly in $t \in [0, 1]$ and $u \in W$, it follows from (33), (34), and condition (C5) that there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, we have

$$|J_\varepsilon[u^*] - J_0| < \eta.$$

Hence, from (32) we obtain

$$J_\varepsilon^* < J_0^* + \eta. \tag{35}$$

On the other hand, for $\varepsilon < \varepsilon_0$, we obtain

$$J_0^* \leq J_0[u_\varepsilon^*] = J_\varepsilon^* + (J_0[u_\varepsilon^*] - J_\varepsilon[u_\varepsilon^*]).$$

However, similarly to (35), we have

$$|J_\varepsilon[u_\varepsilon^*] - J_0[u_\varepsilon^*]| < \eta.$$

Consequently,

$$J_0^* < J_\varepsilon^* + \eta. \tag{36}$$

It follows from (35) and (36) that $J_\varepsilon^* \to J_0^*$ as $\varepsilon \to 0$, which proves statement (i) of Theorem 2.1.

Statement (ii) of Theorem 2.1 follows directly from the fact that

$$|J_\varepsilon^* - J_\varepsilon[u^*]| \leq |J_\varepsilon^* - J_0^*| + |J_0[u^*] - J_\varepsilon[u^*]|.$$

We proceed to the proof of statement (iii). Since U is compact in $L^2(0, 1)$, we can extract a subsequence $u_{\varepsilon_n}^*$ that converges in $L^2(0, 1)$. Let

$$\lim_{\varepsilon_n \to 0} u_{\varepsilon_n}^* = u_0. \tag{37}$$

Let us now consider the auxiliary systems

$$\begin{cases} \dot{z}_{\varepsilon_n} = X\left(\dfrac{t}{\varepsilon_n}, z_{\varepsilon_n}(t), \displaystyle\int_0^t \varphi(t, s, z_{\varepsilon_n}(s)) ds, u_0(t)\right), \\ z_{\varepsilon_n}(0) = x_0, \end{cases}$$

and

$$\begin{aligned} \dot{\xi} &= X_0(\xi, u_0(t)), \\ \xi(0) &= x_0. \end{aligned} \tag{38}$$

By (30), we have

$$\sup_{t\in[0,1]} |x_{\varepsilon_n}^*(t) - z_{\varepsilon_n}(t)| \to 0, \quad \varepsilon_n \to 0 \tag{39}$$

and, by Lemma 3.1,

$$\sup_{t\in[0,1]} |z_{\varepsilon_n}(t) - \xi(t)| \to 0, \quad \varepsilon_n \to 0.$$

Hence, it follows from (38) and (39) that

$$\sup_{t\in[0,1]} |x^*_{\varepsilon_n}(t) - \xi(t)| \to 0, \quad \varepsilon_n \to 0. \tag{40}$$

Therefore,

$$\begin{aligned} J^*_{\varepsilon_n} = J_{\varepsilon_n}[u^*_{\varepsilon_n}] &= \int_0^1 L(t, x^*_{\varepsilon_n}(t), u^*_{\varepsilon_n}(t))\,\mathrm{d}t + \Phi(x^*_{\varepsilon_n}(1)) \\ &= \int_0^1 L(t, x^*_{\varepsilon_n}(t), u_0(t))\mathrm{d}t + \Phi(x^*_{\varepsilon_n}(1)) + \int_0^1 [L(t, x^*_{\varepsilon_n}(t), u^*_{\varepsilon_n}(t)) - L(t, x^*_{\varepsilon_n}(t), u_0(t))]\mathrm{d}t. \end{aligned} \tag{41}$$

Condition (C5b) and (37) imply that the last term in (41) approaches 0 as $\varepsilon_n \to 0$.

By letting $\varepsilon_n \to 0$ in (41), and using (40), we obtain

$$J^*_0 = \int_0^1 L(t, \xi(t), u_0(t))\mathrm{d}t + \Phi(\xi(1)).$$

Hence, $(\xi(t), u_0(t))$ is the optimal solution of the averaged problems (4), (5), which proves statement (*iii*).

If problems (4), (5) has a unique solution, then the above reasoning implies that any converging sequence $(u^*_{\varepsilon_n}(t), x^*_{\varepsilon_n}(t))$ tends to the same limit. This completes the proof of the final statement of the theorem. □

4.2 Linear case

Proof. We again set $T = 1$ and consider the problem on $[0, 1]$.

The existence of an optimal solution $(x^*_\varepsilon(t), u^*_\varepsilon(t))$ for each $\varepsilon > 0$ is established in a standard way by extracting a weakly convergent minimizing sequence $u^{(n)}_\varepsilon(t)$, converging to $u^*_\varepsilon(t)$, and then passing to the limit. This approach relies on the lower semicontinuity of the integral $\int_0^1 B(t, u(t))\mathrm{d}t$ with respect to u, which follows from the convexity of $B(t, u)$.

The fact that $u^*_\varepsilon(t)$ belongs to the set V for each $t \in [0, 1]$ follows from Mazur's lemma [14], as well as from the convexity and closedness of the set V.

The existence of an optimal pair $(\xi^*(t), u^*(t))$ for problems (11), (12) is proved in a similar manner.

Thus,

$$J^*_\varepsilon = J_\varepsilon[u^*_\varepsilon] = \int_0^1 [A(t, x^*_\varepsilon(t)) + B(t, u^*_\varepsilon(t))]\mathrm{d}t + \Phi(x^*_\varepsilon(1)).$$

Let $\bar{u}$ be an arbitrary constant vector from V. Clearly, the control $u(t) \equiv \bar{u}$ is admissible for problems (8), (9). Then, for each $\varepsilon > 0$, we have

$$J^*_\varepsilon = J_\varepsilon[u^*_\varepsilon] \le J_\varepsilon[\bar{u}].$$

Similarly to the derivation of estimate (18), one can show the existence of a constant C_1, independent of ε, such that

$$|x_\varepsilon(t, \bar{u})| \le C_1$$

for $t \in [0, 1]$. Then, from the continuity of A, B, and Φ, it follows that there exists a constant C_2, independent of ε, such that $J^*_\varepsilon \le C_2$. Therefore,

$$J^*_\varepsilon \le C_2 \tag{42}$$

for all positive ε. From condition (C10) and (42), we obtain

$$\int_0^1 |u_\varepsilon^*(t)|^p \mathrm{d}t \le \frac{C_2}{a}.$$

Thus, the set u_ε^* is weakly compact in $L^p(0,1)$. Let $u_{\varepsilon_n}^*(t)$ be a sequence of optimal controls that weakly converges to $u_0(t)$. From Mazur's lemma, it follows that $u_0(t) \in V$ for $t \in [0,1]$, meaning that $u_0(t)$ is an admissible control.

Let $y(t)$ be the solution of the Cauchy problem (11) with $u(t) = u_0(t)$. By Lemma 3.2, the solution $x_{\varepsilon_n}(t, u_{\varepsilon_n}^*)$ of the Cauchy problem (8) converges uniformly, with respect to $t \in [0,1]$, to $y(t)$ for $\varepsilon_n \to 0$.

For any $\eta > 0$, we have

$$J_{\varepsilon_n}^* \le J_{\varepsilon_n}[u^*] = J_0[u^*] + J_{\varepsilon_n}[u^*] - J_0[u^*] = J_0^* + J_{\varepsilon_n}[u^*] - J_0[u^*]. \tag{43}$$

Again, according to Lemma 3.2, the solution $x_{\varepsilon_n}(t, u^*)$ of the Cauchy problem (8) converges uniformly, with respect to $t \in [0,1]$, to $\xi^*(t)$ as $\varepsilon_n \to 0$. Hence,

$$|J_{\varepsilon_n}[u^*] - J_0[u^*]| \le \int_0^1 |A(t, x_{\varepsilon_n}(t, u^*) - A(t, \xi^*(t)))|\mathrm{d}t + |\Phi(x_{\varepsilon_n}(1, u^*)) - \Phi(\xi(1))| \to 0, \quad \varepsilon_n \to 0.$$

Thus, for any $\eta > 0$, there exists $\bar{\varepsilon}$ such that, for $\varepsilon_n < \bar{\varepsilon}$,

$$|J_{\varepsilon_n}[u^*] - J_0[u^*]| < \eta. \tag{44}$$

This, together with (43), implies

$$J_{\varepsilon_n}^* \le J_0^* + \eta. \tag{45}$$

On the other hand, we have

$$J_0^* \le J_0[u_{\varepsilon_n}^*] = J_{\varepsilon_n}^* + J_0[u_{\varepsilon_n}^*] - J_{\varepsilon_n}[u_{\varepsilon_n}^*]. \tag{46}$$

Let us consider an auxiliary system

$$\dot{z}_n = f_0(z_n) + f_1(z_n)u_{\varepsilon_n}^* \tag{47}$$

and system

$$\dot{y} = f_0(y) + f_1(y)u_0. \tag{48}$$

Applying Lemma 3.2 to systems (47) and (48), we obtain

$$\sup_{t\in[0,1]} |z_n(t) - y(t)| \to 0, \quad n \to \infty.$$

From this, taking into account the uniform convergence of $x_{\varepsilon_n}^*$ to y, it follows that

$$\sup_{t\in[0,1]} |x_{\varepsilon_n}^*(t) - z_n(t)| \to 0, \quad n \to \infty.$$

Hence,

$$\begin{aligned}|J_{\varepsilon_n}[u_{\varepsilon_n}^*] - J_0[u_{\varepsilon_n}^*]| \le &\int_0^1 |A(t, x_{\varepsilon_n}^*(t)) - A(t, z_n(t))|\mathrm{d}t + \int_0^1 |A(t, z_n(t)) - A(t, y(t))|\mathrm{d}t \\ &+ |\Phi(x_{\varepsilon_n}^*(1)) - \Phi(y(1))| + |\Phi(x_{\varepsilon_n}^*(1)) - \Phi(y(1))| \to 0, \quad n \to \infty,\end{aligned}$$

due to the uniform continuity of $A(t,x)$ on the compact and the obvious estimates

$$\sup_{t\in[0,1]} |x_{\varepsilon_n}^*(t)| \le C_3, \quad \sup_{t\in[0,1]} |z_n(t)| \le C_3$$

for some constant $C_3 > 0$ independent of n.

Thus, for an arbitrary $\eta > 0$, there exists $\bar{\varepsilon}$ such that

$$|J_{\varepsilon_n}[u^*_{\varepsilon_n}] - J_0[u^*_{\varepsilon_n}]| < \eta.$$

Consequently, by (46), we obtain

$$J_0^* \le J^*_{\varepsilon_n} + \eta, \tag{49}$$

for $\varepsilon_n < \bar{\varepsilon}_1$.

Then, if $\varepsilon_n < \min\{\bar{\varepsilon}, \bar{\varepsilon}_1\}$, it follows from (45) and (49) that $|J_0^* - J^*_{\varepsilon_n}| < \eta$, which means

$$J^*_{\varepsilon_n} \to J_0^*, \quad \varepsilon_n \to 0. \tag{50}$$

Since a convergent subsequence $\{u^*_{\varepsilon_m}\}$ can be chosen from any sequence in the family of controls $\{u^*_\varepsilon\}$, for which relation (50) holds analogously to the above, we obtain

$$J^*_\varepsilon \to J_0^*, \quad \varepsilon \to 0, \tag{51}$$

which proves statement (i) of the theorem.

Now, let us prove statement (ii). Since $x_\varepsilon(t, u^*)$ converges to $\xi^*(t)$, uniformly with respect to for $t \in [0, 1]$, as $\varepsilon \to 0$, we obtain the inequality by arguments similar to those used in the derivation of estimate (44):

$$|J_\varepsilon[u^*] - J_0[u^*]| < \eta, \tag{52}$$

which holds for any $\eta > 0$ for sufficiently small ε. Therefore,

$$|J^*_\varepsilon - J_\varepsilon[u^*]| \le |J^*_\varepsilon - J_0^*| + |J_\varepsilon[u^*] - J_0[u^*]|.$$

From (51) and (52), statement (ii) follows.

Now, let us prove statement (iii). To do so, we will show that $(y(t), u_0(t))$ is indeed the optimal solution of problems (8) and (9). We have

$$J^*_{\varepsilon_n} = \int_0^1 [A(t, x^*_{\varepsilon_n}(t)) + B(t, u^*_{\varepsilon_n}(t))]\mathrm{d}t + \Phi(x^*_{\varepsilon_n}(1)).$$

Letting $n \to \infty$ and taking into account (51) and condition (C10), we obtain

$$\begin{aligned} J_0^* &= \int_0^1 A(t, y(t))\mathrm{d}t + \lim_{\varepsilon_n \to 0} \int_0^1 B(t, u^*_{\varepsilon_n}(t))\mathrm{d}t + \Phi(y(1)) \\ &\ge \int_0^1 [A(t, y(t)) + B(t, u_0(t))]\mathrm{d}t + \Phi(y(1)). \end{aligned}$$

From this, it follows that $(y(t), u_0(t))$ is an optimal pair.

The final statement of the theorem is proved similar to the corresponding statement in Theorem 2.1. □

5 Examples

Example 1 (Weakly nonlinear regulator). Consider the following optimal control problem:

$$\begin{gathered} \dot{x}(t) = f\left(\frac{t}{\varepsilon}\right)x + f_1\left(\frac{t}{\varepsilon}, x(t), \int_0^t \varphi(t, s, x(s))\mathrm{d}s\right) + f_2(t)u(t), \\ x(0) = x_0, \end{gathered} \tag{53}$$

where $t \in [0, T]$, $x \in \mathbb{R}^d$, $u \in \mathbb{R}^m$, with the quality criterion

$$J_\varepsilon[u] = \int_0^T [(C(t)x_\varepsilon(t), x_\varepsilon(t)) + (F(t)u(t), u(t))]dt + (Dx_\varepsilon(T), x(T)) \to \inf, \tag{54}$$

where $C(t)$ and D are symmetric non-negative definite $d \times d$ matrices, $F(t)$ is a positive definite $m \times m$ matrix, $f(t)$ is a $d \times d$ matrix, $f_1(t, x, y)$ is a d-dimensional vector function defined for $t \in [0, T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^n$, and $f_2(t)$ is a $n \times m$ matrix.

Since the terms in functional (54) are quadratic forms, this problem is referred to as an optimal control problem for a weakly nonlinear oscillator. The classical linear case has been studied, for example, in [15].

We assume that the functions f_1 and φ satisfy conditions $C7$ and $C8$, and functions f and f_2 are continuous.

By introducing a small positive parameter, this problem is reduced to an optimal control problem for a weakly nonlinear oscillator. The classical and linear cases have been studied. We consider a function $\varphi \in L^p(\Omega)$. Function $f_1(t, x)$ and φ are assumed to be measurable functions, satisfying conditions (C7) and (C8).

Let $\varphi_1(t, x) = \int_0^t \varphi(t, s, x)ds$. Suppose that the following limits exist uniformly with respect to $x \in \mathbb{R}^d$ and $u \in \mathbb{R}^m$:

$$\lim_{\varepsilon \to 0} \int_0^t \left[f\left(\frac{\tau}{\varepsilon}\right) - A_0 \right] d\tau = 0,$$

$$\lim_{\varepsilon \to 0} \int_0^t f_1\left(\frac{\tau}{\varepsilon}, x, \varphi_1(\tau, x)\right) d\tau = 0.$$

We associate the optimal control problems (53) and (54) with the corresponding averaged problem

$$\begin{gathered} \dot{\xi} = A_0 \xi + f_2(t)u, \\ J_0[u] = \int_0^T [(C(t)\xi(t), \xi(t)) + (F(t)u(t), u(t))]dt + (D\xi(T), \xi(T)) \to \inf. \end{gathered} \tag{55}$$

Problem (55) is a classical linear regulator problem. It is well known that its solution reduces to the matrix Riccati equation. In particular, when f_2, C, and F are constants, this equation is autonomous, and in the one-dimensional case, it can be solved exactly. Consequently, the averaged problem (55) is solvable. The proven theorem then states that the optimal control found for the averaged problem is “almost” optimal for the original problem.

The following example is illustrative and demonstrates the convergence of the optimal controls and trajectories of the original problem to those of the averaged problem.

Example 2. We consider the optimal control problem

$$\begin{cases} \dot{x}_\varepsilon = \sin\left(\frac{t}{\varepsilon}\right) \int_0^t (x_\varepsilon(s) \cos s) ds + u, \\ x_\varepsilon(0) = 1, \quad t \in [0, 1], \\ J_\varepsilon[u] = \int_0^1 (x_\varepsilon(t) - u(t))^2 dt \to \inf. \end{cases} \tag{56}$$

Here $\varphi_1(t, x) = \int_0^t x \cos s ds = x \sin t$. Then, according to (10), we have

$$\lim_{\varepsilon \to 0} \int_0^t x \sin\left(\frac{s}{\varepsilon}\right) \sin s ds = \frac{1}{2} x \left[\frac{\varepsilon}{1 - \varepsilon} \sin\left(\left(\frac{1}{\varepsilon} - 1\right)t\right) - \frac{\varepsilon}{1 + \varepsilon} \sin\left(\left(\frac{1}{\varepsilon} + 1\right)t\right) \right] = 0.$$

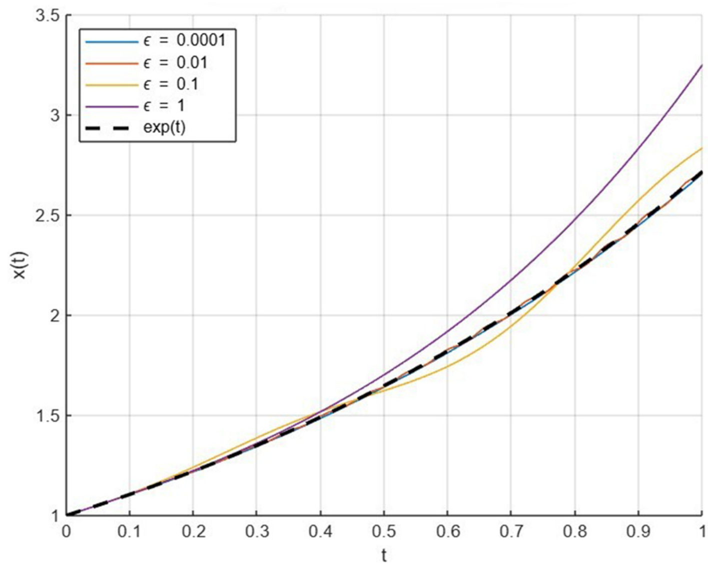

Figure 1: Convergence of the solution $x_\varepsilon(t)$ of the original problem (58) to the solution $\xi^*(t) = e^t$ of the averaged problem (57) as $\varepsilon \to 0$.

So, the averaged problem is as follows:

$$\begin{cases} \dot{\xi} = u, \\ \xi(0) = 1, \\ J[u] = \int_0^1 (\xi(t) - u(t))^2 \mathrm{d}t \to \inf. \end{cases} \tag{57}$$

The optimal control of problem (57) is obviously $u^*(t) = \xi^*(t)$, where $\xi^*(t)$ is the solution of the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}\xi^*}{\mathrm{d}t} = \xi^*, \\ \xi^*(0) = 1. \end{cases}$$

Hence, $u^*(t) = e^t$.

For the initial problem (56), it is also obvious that $x_\varepsilon^*(t) = u_\varepsilon^*(t)$, where $x_\varepsilon^*(t)$ is the solution of the Cauchy problem

$$\begin{cases} \dot{x}_\varepsilon = \sin\left(\frac{t}{\varepsilon}\right) \int_0^t x_\varepsilon(s) \cos s \mathrm{d}s + x_\varepsilon, \\ x_\varepsilon(0) = 1. \end{cases} \tag{58}$$

Table 1: Numerical comparison between the solutions of the original problem (58) and the averaged problem (57): values of $x_\varepsilon(t)$, e^t, and $|x_\varepsilon(t) - e^t|$ at selected points

ε	t	0.20	0.40	0.60	0.80	1.00
	e^t	1.221403	1.491825	1.822119	2.225541	2.718282
$\varepsilon = 10^{-2}$	$x_\varepsilon(t)$	1.220604	1.495096	1.829428	2.226669	2.706371
$\varepsilon = 10^{-4}$	$x_\varepsilon(t)$	1.218997	1.485621	1.813555	2.217434	2.707980
$\varepsilon = 10^{-2}$	$\lvert x_\varepsilon - e^t\rvert$	7.985×10^{-4}	3.272×10^{-3}	7.309×10^{-3}	1.128×10^{-3}	1.191×10^{-2}
$\varepsilon = 10^{-4}$	$\lvert x_\varepsilon - e^t\rvert$	2.405×10^{-3}	6.203×10^{-3}	8.564×10^{-3}	8.107×10^{-3}	1.030×10^{-2}

The graphs and numerical illustrations below demonstrate the convergence of the solution of problem (58) toward the function e^t as $\varepsilon \to 0$ (Figure 1 and Table 1).

Funding information: This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP23485618). The research of O. Stanzhytskyi was partially supported by Grant No. 0124U001412 provided by the Ministry of Education and Science of Ukraine.

Author contributions: All authors have equally contributed to this work. All authors read and approved the final manuscript.

Conflict of interest: The authors state no conflict of interest.

References

[1] A. N. Filatov, *Averaging in systems of integral and integro-differential equations*, in: *Research in Analytical Mechanics*, Nauka, Tashkent, 1965. [in Russian].
[2] A. N. Filatov and A. G. Umarov, *On averaging method in systems of integro-differential equations*, Dokl. Akad. Nauk **327** (1985), no. 6, 12–14. [in Russian].
[3] O. M. Stanzhytskyi, S. G. Karakenova, and R. E. Uteshova, *Averaging method and boundary value problems for systems of Fredholm integro-differential equations*, Nonlinear Dyn. Syst. Theory **21** (2021), no. 1, 100–113.
[4] A. N. Stanzhitskii and T. V. Dobrozdii, *Study of optimal control problems on the half-line by the averaging method*, Differ. Equ. **47** (2011), no. 2, 264–277, DOI: https://doi.org/10.1134/S0012266111020121.
[5] T. V. Nosenko and O. M. Stanzhytskyi, *Averaging method in some problems of optimal control*, Nonlinear Oscil. **48** (2008), no. 11, 539–547, DOI: https://doi.org/10.1007/s11072-009-0049-5.
[6] T. V. Koval'chuk, V. V. Mohyl'ova, and T. V. Shovkoplyas, *Averaging method in problems of optimal control over impulsive systems*, J. Math. Sci. (N.Y.) **247** (2020), no. 2, 314–327, DOI: https://doi.org/10.1007/s10958-020-04804-2.
[7] T. V. Koval'chuk, V. V. Mogylova, O. M. Stanzhytskyi, and T. V. Shovkoplyas, *Application of the averaging method to the problems of optimal control of the impulse systems*, Carpathian Math. Publ. **12** (2020), no. 2, 504–521, DOI: https://doi.org/10.15330/cmp.12.2.504-521.
[8] V. I. Kravets', T. V. Koval'chuk, V. V. Mohyl'ova, and O. M. Stanzhytskyi, *Application of the method of averaging to the problems of optimal control over functional-differential equations*, Ukrainian Math. J. **70** (2018), no. 2, 232–242, DOI: https://doi.org/10.1007/s11253-018-1497-9.
[9] J. F. M. Al-Omari and S. A. Gourley, *A nonlocal reaction-diffusion model for a single species with stage structure and distributed maturation delay*, Eur. J. Appl. Math. **16** (2005), 37–51.
[10] A. Alawneh, K. Al-Khaled, and M. Al-Towaiq, *Reliable algorithms for solving integro-differential equations with applications*, Int. J. Comput. Math. **87** (2010), no. 7, 1538–1554, DOI: https://doi.org/10.1080/00207160802385818.
[11] H. R. Thieme, *A model for the spatio-spread of an epidemic*, J. Math. Biol. **4** (1977), 337–351, DOI: https://doi.org/10.1007/BF00275082.
[12] V. Mogylova, R. Lakhva, and V. Kravets, *Optimal control problem for systems of integro-differential equations*, J. Math. Sci. (N.Y.) **282** (2024), no. 6, 983–1007, DOI: https://doi.org/10.1007/s10958-024-07229-3.
[13] R. Lakhva, Z. Khaletska, and V. Mogylova, *The optimal control problem for systems of integro-differential equations with finite and infinite horizon*, Georgian Math. J. (in press), 2024, DOI: https://doi.org/10.1515/gmj-2024-2065.
[14] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, New York, 1980.
[15] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, vol. 25, Springer Science & Business Media, New York, 2006.