Research Article

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Tilings, sub-tilings, and spectral sets on *p*-adic space

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Abstract: In this article, we provide a characterization of tilings, sub-tilings, and spectral sets on the p-adic space $\mathbb{Q}_p^d$. Our methods are based on p-adic distributions and p-adic Fourier analysis. We also obtained a sufficient condition and a necessary condition on tiling of functions by translation on $\mathbb{Q}_p^d$.

Keywords: p-adic spaces, tilings, sub-tilings, spectral sets

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1 Introduction

Let $\mathbb{Q}_p^d$ be the d-dimensional p-adic vector space over the field $\mathbb{Q}_p$ of p-adic numbers. Let $f \in L^1(\mathbb{Q}_p^d)$ be a complex-valued Haar-integrable function, and let $T \subset \mathbb{Q}_p^d$ be a discrete set. We define

$$f + T = \sum_{t \in T} f(x - t).$$

We say that f is a translational sub-tiling of $\mathbb{Q}_p^d$ with a discrete set $T \subset \mathbb{Q}_p^d$, if we have

$$f + T \leq 1, \quad \text{a.e. } x \in \mathbb{Q}_p^d. \tag{1.1}$$

In this case, we say that $f + T$ is a sub-tiling. Note that if $f = 1_\Omega$, the indicator function of a Borel set $\Omega \subset \mathbb{Q}_p^d$ with positive finite Haar measure, then condition (1.1) means that the sets $\{\Omega + t : t \in T\}$ are pairwise disjoint up to Haar measure zero set. In this case, we say that $\Omega + T$ is a sub-tiling.

We say a function $f \in L^1(\mathbb{Q}_p^d)$ tiles $\mathbb{Q}_p^d$ at level one by translation with $T \subset \mathbb{Q}_p^d$, if we have

$$f + T = 1, \quad \text{a.e. } x \in \mathbb{Q}_p^d. \tag{1.2}$$

When $f = 1_\Omega$, equation (1.2) implies that, up to a set of Haar measure zero, the collection of sets $\{\Omega + t : t \in T\}$ forms a partition of $\mathbb{Q}_p^d$. In this case, Ω is called a translational tile. The set T is called a *tiling complement* of Ω, and the pair (Ω, T) is called a *tiling pair*.

We can extend the concepts of tiling and sub-tiling to measures. Consider μ as a positive, finite Borel measure on $\mathbb{Q}_p^d$, and $f \in L^1(\mathbb{Q}_p^d)$. Define the convolution of f and μ as

$$f * \mu = \int_{\mathbb{Q}_p^d} f(x - y) \mathrm{d}\mu(y).$$

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We call $f + \mu$ a tiling (respectively, sub-tiling) of $\mathbb{Q}_p^d$ if $f * \mu = 1$ (respectively, $f * \mu \le 1$) almost everywhere with respect to the Haar measure. It is clear that when $f = 1_\Omega$ (the characteristic function of the set Ω) and $\mu = \delta_T = \sum_{t\in T}\delta_t$ (the Dirac measure concentrated at points $t \in T$), $f * \mu = 1$ is equivalent to (Ω, T) being a tiling pair.

We say function $f \in L^1(\mathbb{Q}_p^d)$ tiles at level ℓ, or multi-tiles, by a discrete translation set $T \subset \mathbb{Q}_p^d$, if for almost all $x \in \mathbb{Q}_p^d$, we have

$$f * \delta_T = \ell.$$

Tilings are classical problems in the geometry of the Euclidean space. Compact sets with positive measure that can tile the real number line $\mathbb{R}$ through translations have been thoroughly investigated in [1,2]. The most straightforward scenario involves compact sets made up of a finite number of unit intervals, where all end-points are integers. This tiling problem can be restated in terms of finite subsets of the set of integers $\mathbb{Z}$ that tile the group $\mathbb{Z}$. As a result, there have been extensive studies of tiling problems on the discrete group $\mathbb{Z}^d$ for $d \in \mathbb{N}$. Coven and Meyerowitz [3] proposed conditions (T1) and (T2) that are sufficient for a finite set $A \subset \mathbb{Z}$ to tile $\mathbb{Z}$, and necessary when the cardinality of A has at most two distinct prime factors. Łaba and Londner [4,5] proved that the Coven-Meyerowitz tiling conditions (T1) and (T2) that are sufficient for a finite set A to tile, and necessary when the cardinality of A is of the form $p^2q^2r^2$ for distinct primes p, q, and r. Bhattacharya [6] proved that any finite set $A \subset \mathbb{Z}^2$ that tiles $\mathbb{Z}^2$ by translations admits a periodic tiling. Greenfeld and Tao [7] obtained structural results on translational tilings of periodic functions in $\mathbb{Z}^d$ by finite tiles. They proved that any level one tiling of a periodic set in $\mathbb{Z}^2$ must be weakly periodic. Grebík et al. [8] extended the tiling problems to a measure space X, and they studied the structure of measurable tilings of X by a measurable tile $A \subset X$ translated by a finite set T. They also established a “dilation lemma” and “structure theorem” for Abelian measurable tilings.

On the other hand, tilings by translates of functions on the Euclidean spaces have also been extensively studied (see [9] for surveys on the tilings of functions by translation on the Euclidean space).

Let $\Omega \subset \mathbb{Q}_p^d$ be a Borel set of positive and finite Haar measure. The Hilbert space $L^2(\Omega)$ of square Haar-integrable functions is equipped with the inner product

$$\langle f, g\rangle_\Omega = \int_\Omega f(x)\overline{g(x)}\mathrm{d}x, \quad \forall f, g \in L^2(\Omega).$$

We call Ω a *spectral set* when there exists a set $\Lambda \subset \widehat{\mathbb{Q}_p^d}$, consisting of continuous characters of $\mathbb{Q}_p^d$, that forms an orthogonal basis for the space $L^2(\Omega)$. The set Λ is then called a *spectrum* of Ω, and the pair (Ω, Λ) is known as a *spectral pair*.

In the Euclidean space $\mathbb{R}^d$, Fuglede put forward the following conjecture in [10].

Spectral set conjecture: A Borel set $\Omega \subset \mathbb{R}^d$ of positive and finite Lebesgue measure is a spectral set if and only if it is a translational tile.

We can formulate the generalized Fuglede conjecture for any locally compact Abelian groups (either finite or infinite) G, and we simply call the generalized spectral set conjecture.

Generalized spectral set conjecture: A Borel set $\Omega \subset G$ of positive and finite Haar measure is a spectral set if and only if it is a translational tile.

Both the initial and the generalized **spectral set conjecture** have drawn significant attention in the past few decades.

In the case of $\mathbb{R}^d$, it was proved to be true when the spectra or tiling sets are lattices [10]. But this conjecture was eventually disproved by Tao et al. for dimensions $d \ge 3$ in both directions [11–15]. These counterexamples constructed are non-convex sets, and it is generally believed that the Fuglede conjecture should be true for convex domains. And it was eventually proved that spectral conjecture is true for convex domains in all dimensions [16,17]. However, the conjecture is still open in dimensions $d = 1, 2$ in general.

In order to disprove this conjecture on $\mathbb{R}^d$, counterexamples were first constructed in finite Abelian groups. So, there has been some increasing interest in the tiling to spectral direction in p-groups [18–22].

The spectral set and spectral measure problem for local fields were considered in [23], and we proved the conjecture in local fields when spectra or tiling sets are quasi-lattices [24]. Recently, this conjecture completely settled down in the one-dimensional p-adic space $\mathbb{Q}_p$. In [25], it is proved that any bounded tile of $\mathbb{Q}_p$ is an almost compact open set. Let $\Omega \subset \mathbb{Q}_p$ be Borel set of positive finite Haar measure. In [26,27], it is shown that Ω is a spectral set if and only if Ω is a tile if and only if Ω is an almost-compact open p-homogeneous set. Additionally, a class of singular spectral measures in $\mathbb{Q}_p$ was constructed, some of which are self-similar measures. Shi [28] characterized all spectral measures in $\mathbb{Q}_p$. It was proven in [29] that for all odd primes p, there are spectral sets in $\mathbb{F}_p^4$ that are not tiles. This means that for $d = 4$, there are compact open spectral sets in $\mathbb{Q}_p^d$ that are not tiles. There also exists a compact open spectral set, which is not a tile in $\mathbb{Q}_2^3$. However, this conjecture is still open in $\mathbb{Q}_p^2$.

One of the aims of this article is to study the relationships between sub-tilings, tilings, and orthogonal sets, spectral sets over the d-dimensional p-adic vector space $\mathbb{Q}_p^d$.

Let $\mathbb{Q}_p$ be the field of p-adic numbers, and any $x \in \mathbb{Q}_p$ can be written as

$$x = \sum_{n=v}^{\infty} a_n p^n \quad (v \in \mathbb{Z},\ a_n \in \{0, 1, \ldots, p-1\} \text{ and } a_v \neq 0).$$

The fractional part of x is defined as

$$\{x\} = \sum_{n=v}^{-1} a_n p^n.$$

For $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{Q}_p^d$, the scalar product in $\mathbb{Q}_p^d$ is defined as

$$x \cdot y := x_1 y_1 + \ldots + x_d y_d.$$

The dual group $\widehat{\mathbb{Q}}_p^d$ of $\mathbb{Q}_p^d$ consists of all $\chi_\lambda(\cdot)$ with $\lambda \in \mathbb{Q}_p^d$, where

$$\chi_\lambda(x) = \chi(x \cdot \lambda) = e^{2\pi i \{x \cdot \lambda\}}.$$

Given a discrete set $\Lambda \subset \mathbb{Q}_p^d$, and $\forall x \in \mathbb{Q}_p^d$, we set

$$E(\Lambda) = \{\chi_\lambda(x) = e^{2\pi i \{\lambda \cdot x\}} : \lambda \in \Lambda\}.$$

For a Borel set $\Omega \subset \mathbb{Q}_p^d$ with $0 < |\Omega| < \infty$, the following theorem gives a criterion for the orthogonality and orthogonal basis property of the exponential system $E(\Lambda)$ in $L^2(\Omega)$ by sub-tiling and tiling conditions, respectively.

Theorem 1.1. *Let $\Omega \subset \mathbb{Q}_p^d$ be a Borel set with $0 < |\Omega| < \infty$, and let $f(x) := |\hat{1}_\Omega(x)|^2/|\Omega|^2$.*
(1) *The system of exponential functions $E(\Lambda)$ is orthogonal in $L^2(\Omega)$ if and only if $f + \Lambda$ is a sub-tiling.*
(2) *The system of exponential functions $E(\Lambda)$ is orthogonal basis in $L^2(\Omega)$ if and only if $f + \Lambda$ is a tiling.*

This theorem in the Euclidean case $\mathbb{R}^d$ is due to Kolountzakis [30]. A open set $A \subseteq \mathbb{Q}_p^d$ is called a orthogonal packing region for $\Omega \subset \mathbb{Q}_p^d$, if

$$(A - A) \cap \mathcal{Z}_\Omega = \varnothing,$$

where $\mathcal{Z}_\Omega = \{x : \hat{1}_\Omega(x) = 0\}$, and $A - A = \{x - y : \forall x, y \in A\}$. We say that an orthogonal packing region A for Ω is tight if we have that $|A| = 1/|\Omega|$.

The following theorem provides a very useful criterion to decide whether a sub-tiling of function is actually a tiling of function on $\mathbb{Q}_p^d$, which is the extension of the result in the Euclidean setting [31] to the non-Archimedean setting. Actually, we have proved a special case of this theorem on the vector space $\mathbb{K}^d$ over the general local field $\mathbb{K}$ [24].

Theorem 1.2. *Assume that* $f, g \in L^1(\mathbb{Q}_p^d)$ *are two functions with* $f, g \geq 0$, *and* $\int_{\mathbb{Q}_p^d} f(x)\mathrm{d}x = \int_{\mathbb{Q}_p^d} g(x)\mathrm{d}x = 1$. *Assume that* μ *is a positive Borel measure on* $\mathbb{Q}_p^d$ *such that* $f * \mu$ *and* $g * \mu$ *are sub-tilings. Then,* $f * \mu$ *is a tiling if and only if* $g * \mu$ *is a tiling.*

Suppose that $E(\Lambda)$ is an orthogonal set in $L^2(\Omega)$, and A is a *tight orthogonal packing region* for Ω. As a consequence of Theorem 1.2, it is easy to see that (Ω, Λ) is a *spectral pair* if and only if (A, Λ) is a *tiling pair*.

The following theorem provides a sufficient condition and a necessary condition on tiling of functions by translation on $\mathbb{Q}_p^d$.

Theorem 1.3. *Let* $f \in L^1(\mathbb{Q}_p^d)$ *be a Haar-integrable complex-valued function. Assume that* T *is an uniformly discrete subset of* $\mathbb{Q}_p^d$. *Let* $\delta_T = \sum_{t \in T} \delta_t$ *be the distribution, and,* $\widehat{\delta_T}$ *be its Fourier transform.*

(1) *If* $f * T$ *is a tiling of level* ℓ, *then*

$$\operatorname{supp}(\widehat{\delta_T}) \setminus \{0\} \subseteq \mathcal{Z}_{\hat{f}}. \tag{1.3}$$

(2) *If* $\widehat{\delta_T}$ *is a measure, then* (1.3)*implies that* $f * T$ *is a tiling of constant level.*

The article is structured as follows. In Section 2, we present fundamental definitions and preliminaries regarding the field $\mathbb{Q}_p$ of p-adic numbers, the Fourier transform of $L^1(\mathbb{Q}_p^d)$ functions, quasi-lattices in $\mathbb{Q}_p^d$, p-adic Bruhat-Schwartz distributions, and the Colombeau-Egorov algebra of p-adic generalized functions. Meanwhile, the proofs of our main theorems are given in Section 3.

2 Preliminaries

In this section, we shall present some basic preliminaries on the field $\mathbb{Q}_p$ of p-adic numbers, the Fourier transformations of integrable functions $L^1(\mathbb{Q}_p^d)$, quasi-lattices in $\mathbb{Q}_p$, p-adic Bruhat-Schwartz distributions, and Colombeau-Egorov algebra of p-adic generalized functions, a criterion of spectral sets and uniformly discreteness of spectra in $\mathbb{Q}_p^d$, mostly based on the books [32,33].

2.1 p-adic space $\mathbb{Q}_p^d$

We begin by briefly reviewing p-adic numbers. Let $p \geq 2$ be a prime number and $\mathbb{Q}$ be the field of rational numbers. For any non-zero rational number $x \in \mathbb{Q}$, it can be expressed as $x = p^v \frac{a}{b}$, where $v, a, b \in \mathbb{Z}$ and the greatest common divisor of p, a and b is 1. Due to the uniqueness of factorizations in $\mathbb{Z}$, the integer v depends solely on x. We define $v_p(x) = v$ for $x \neq 0$ and $v_p(0) = +\infty$, and the p-adic absolute value of x as $|x|_p = p^{-v_p(x)}$. Then, the p-adic absolute value $|\cdot|_p$ is a non-Archimedean absolute value. This implies:

(i) $|x|_p \geq 0$ and equality holds if and only if $x = 0$;
(ii) $|xy|_p = |x|_p |y|_p$;
(iii) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.

The field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ under the p-adic absolute value $|\cdot|_p$. Any nonzero p-adic number $x \in \mathbb{Q}_p$ is uniquely represented in the canonical form

$$x = \sum_{n=v}^{\infty} a_n p^n \quad (v \in \mathbb{Z}, a_n \in \{0, 1, \ldots, p-1\} \text{ and } a_v \neq 0), \tag{2.1}$$

where, $v(x) \coloneqq v$ is called the p-valuation of x, and $|x|_p = p^{-v}$. Since the p-adic norm has a discrete set of values $\{p^\gamma : \gamma \in \mathbb{Z}\} \cup \{0\}$, we need only consider balls of radiuses $r = p^\gamma$, $\gamma \in \mathbb{Z}$. We denote by $\mathbb{Z}_p \coloneqq \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ the ring of p-adic integers of $\mathbb{Q}_p$. Due to the definition of the p-adic absolute value, $\mathbb{Z}_p$ consists of p-adic numbers

$$x = \sum_{n=0}^{\infty} a_n p^n.$$

The fractional part of a p-adic number $x \in \mathbb{Q}_p$ is denoted by

$$\{x\} = \sum_{n=v}^{-1} a_n p^n.$$

Denote the closed ball of radius p^γ with the center at $a \in \mathbb{Q}_p$ by

$$B_\gamma(a) \coloneqq \{x \in \mathbb{Q}_p : |x - a|_p \leq p^\gamma\},$$

and $B_\gamma(0) \coloneqq p^{-\gamma}\mathbb{Z}_p$. It is the ball centered at 0 of radius p^γ.

We fix the character $\chi \in \widehat{\mathbb{Q}}_p$:

$$\chi(x) = e^{2\pi i\{x\}}.$$

From this character, we can obtain all characters of $\mathbb{Q}_p$ by defining for any $\lambda \in \mathbb{Q}_p$:

$$\chi_\lambda(x) = \chi(\lambda x).$$

We note that each $\chi_\lambda(\cdot)$ is uniformly locally constant, meaning

$$\chi_\lambda(x_1) = \chi_\lambda(x_2), \quad \text{if } |x_1 - x_2|_p \leq \frac{1}{|\lambda|_p}.$$

Actually, the map $y \mapsto \chi_y$ from $\mathbb{Q}_p$ onto $\widehat{\mathbb{Q}}_p$ is an isomorphism. We thus identify a point $\lambda \in \mathbb{Q}_p$, with $\chi_\lambda \in \widehat{\mathbb{Q}}_p$, and we write $\mathbb{Q}_p \cong \widehat{\mathbb{Q}}_p$.

Because $\mathbb{Q}_p$ is a locally compact Abelian group with respect to addition, there is an additive Haar measure on $\mathbb{Q}_p$, denoted by $\mathfrak{m}$ or $\mathrm{d}x$. This $\mathrm{d}x$ is a positive measure that is invariant under translations, i.e., $d(x + a) = \mathrm{d}x$ for all $a \in \mathbb{Q}_p$. When we normalize the Haar measure such that $\mathfrak{m}(\mathbb{Z}_p) = 1$, the Haar measure $\mathfrak{m}$ is unique.

Let $\mathbb{Q}_p^d$ denote the d-dimensional p-adic vector space. We endow $\mathbb{Q}_p^d$ with the norm

$$|x|_p = \max_{1 \leq j \leq d} |x_j|_p, \quad \text{for } x = (x_1, \ldots, x_d) \in \mathbb{Q}_p^d.$$

This norm is non-Archimedean. The space $\mathbb{Q}_p^d$ is a complete metric locally compact and totally disconnected space. For $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{Q}_p^d$, the scalar product in $\mathbb{Q}_p^d$ is defined as

$$x \cdot y \coloneqq x_1 y_1 + \ldots + x_d y_d.$$

We have

$$|x \cdot y|_p \leq |x|_p |y|_p, \quad \text{for } x, y \in \mathbb{Q}_p^d.$$

Denote by $B_\gamma^d(a) \coloneqq \{x \in \mathbb{Q}_p^d : |x - a|_p \leq p^\gamma\}$ the closed ball of radius p^γ with the center at $a = (a_1, \ldots, a_d) \in \mathbb{Q}_p^d$. It is clear that

$$B_\gamma^d(a) = B_\gamma(a_1) \times \ldots \times B_\gamma(a_d),$$

where $B_\gamma(a_k) \coloneqq \{x_k \in \mathbb{Q}_p : |x_k - a_k|_p \leq p^\gamma\}$ is a ball of radius p^γ with the center at $a_k \in \mathbb{Q}_p$, $k = 1, 2, \ldots, d$.

The Haar measure on $\mathbb{Q}_p^d$ is the product measure $\mathrm{d}x_1 \ldots \mathrm{d}x_d$, which is also denoted by $\mathrm{d}x$.

The dual group $\widehat{\mathbb{Q}}_p^d$ of $\mathbb{Q}_p^d$ consists of all $\chi_\lambda(\cdot)$ with $\lambda \in \mathbb{Q}_p^d$, where

$$\chi_\lambda(x) = \chi(x \cdot \lambda) = e^{2\pi i\{x\cdot\lambda\}}.$$

2.2 Quasi-lattices on $\mathbb{Q}_p^d$

In the space $\mathbb{Q}_p^d$, unlike in $\mathbb{R}^d$, there are no lattice groups. This is because finitely generated additive groups in $\mathbb{Q}_p^d$ are bounded. We define quasi-lattices, which will serve as the analogues of lattices in $\mathbb{R}^d$.

The unit ball $\mathbb{Z}_p^d$ is an additive subgroup of $\mathbb{Q}_p^d$. Let $\mathbb{L}^d \subset \mathbb{Q}_p^d$ be a complete set of coset representatives of $\mathbb{Z}_p^d$. Then,

$$\mathbb{Q}_p^d = \mathbb{L}^d + \mathbb{Z}_p^d = \bigsqcup_{\gamma\in\mathbb{L}^d} (\gamma + \mathbb{Z}_p^d).$$

We call $\mathbb{L}^d$ a *standard quasi-lattice* in $\mathbb{Q}_p^d$. Recall that $\mathbb{Z}^d$ is the standard lattice in $\mathbb{R}^d$, which is a finitely generated subgroup of $\mathbb{R}^d$.

If $\mathbb{L}$ is a standard quasi-lattice in $\mathbb{Q}_p$, then $\mathbb{L}^d$ is a standard quasi-lattice of $\mathbb{Q}_p^d$. If $\mathbb{L}^d$ is a standard quasi-lattice of $\mathbb{Q}_p^d$, so is $\{\gamma + \eta_\gamma : \gamma \in \mathbb{L}^d\}$, where $\{\eta_\gamma\}_{\gamma\in\mathbb{L}^d}$ is any set in $\mathbb{Z}_p^d$.

Now, we present a standard quasi-lattice in $\mathbb{Q}_p$. For any $n \geq 1$, let

$$V_n = \{1 \leq k < p^n : (k, p) = 1\}.$$

The set V_n is precisely the set of invertible elements of the ring $\mathbb{Z}/p^n\mathbb{Z}$, i.e., $V_n = (\mathbb{Z}/p^n\mathbb{Z})^\times$. Then,

$$\{0\} \sqcup p^{-1}V_1 \sqcup p^{-2}V_2 \sqcup \ldots \sqcup p^{-n}V_n \sqcup \ldots$$

is the standard quasi-lattice of $\mathbb{Q}_p$.

Lemma 2.1. [34] *The set of characters of the group $\mathbb{Z}_p$ is $\{\chi(\gamma x)\}_{\gamma\in\mathbb{L}}$, where $\chi(x) = e^{2\pi i\{x\}}$. Moreover, it is an orthonormal basis for $L^2(\mathbb{Z}_p)$.*

As a result, we can directly obtain the characters of $\mathbb{Z}_p^d$, i.e., the set $\{\chi(\gamma \cdot x)\}_{\gamma\in\mathbb{L}^d}$ constitutes the characters of $\mathbb{Z}_p^d$. It serves as a Fourier basis for $L^2(\mathbb{Z}_p^d)$ and also for $L^2(\mathbb{Z}_p^d + a)$ for any $a \in \mathbb{Q}_p^d$. In essence, $(\mathbb{Z}_p^d + a, \mathbb{L}^d)$ forms a spectral pair.

For the group $p^{-n}\mathbb{Z}_p$ ($n \in \mathbb{Z}$), its characters are represented by $p^n\mathbb{L}$. More generally, if $M \in GL_d(\mathbb{Q}_p)$ is a non-singular $d \times d$ matrix, the characters of the group $M\mathbb{Z}_p^d$ are given by $(M^{-1})^t\mathbb{L}^d$. We term the set $(M^{-1})^t\mathbb{L}^d$ a quasi-lattice of $\mathbb{Q}_p^d$.

A set $E \subset \mathbb{Q}_p^d$ is said to be uniformly discrete when E is countable, and there exists a $\delta \in \mathbb{Z}$ such that for any two distinct points $x, y \in E$, the p-adic absolute value $|x - y|_p \geq p^\delta$. The largest constant δ with this property is known as the separation constant of E, denoted as $\delta(E)$.

Quasi-lattices are separated sets. Standard quasi-lattice $\mathbb{L}^d$ in $\mathbb{Q}_p^d$ has a separation constant $\delta(\mathbb{L}^d) = p$ [23].

2.3 Fourier transform of $L^1(\mathbb{Q}_p^d)$-functions

We denote by $L^1(\mathbb{Q}_p^d)$ the space of Haar-integrable complex-valued functions on $\mathbb{Q}_p^d$ and by $L^2(\mathbb{Q}_p^d)$ the space of Haar-square integrable complex-valued functions on $\mathbb{Q}_p^d$.

The Fourier transformation of $f \in L^1(\mathbb{Q}_p^d)$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{Q}_p^d} f(x)\overline{\chi_\xi(x)}\, \mathrm{d}x, \quad (\xi \in \mathbb{Q}_p^d).$$

Note that

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_p^d} f(x)\overline{\chi(\xi \cdot x)}\mathrm{d}x = \int_{\mathbb{Q}_p^d} f(x)\chi(-\xi \cdot x)\mathrm{d}x.$$

The Fourier transform has the following properties:

(1) The map $f \to \widehat{f}$ is a bounded linear transformation of $L^1(\mathbb{Q}_p^d)$ into $L^\infty(\mathbb{Q}_p^d)$, and $\|\widehat{f}\|_\infty \le \|f\|_1$.

(2) If $f \in L^1(\mathbb{Q}_p^d)$, then $\widehat{f}$ is uniformly continuous.

(3) If $f \in L^1(\mathbb{Q}_p^d) \cap L^2(\mathbb{Q}_p^d)$, then $\|\widehat{f}\|_2 = \|f\|_2$.

2.4 Convolution and Fourier-Stieltjies transform of finite measures on $\mathbb{Q}_p^d$

Let us denote by $\mathcal{M}(\mathbb{Q}_p^d)$ the set of all finite regular measures on $\mathbb{Q}_p^d$. It is obvious that $\mathcal{M}(\mathbb{Q}_p^d)$ is a normed linear space. For $\mu, \nu \in \mathcal{M}(\mathbb{Q}_p^d)$, let $\mu \times \nu$ be their product measure on the product space $\mathbb{Q}_p^d \times \mathbb{Q}_p^d$, and associate with each Borel set A in $\mathbb{Q}_p^d$ the set

$$A_+ = \{(x, y) \in \mathbb{Q}_p^d \times \mathbb{Q}_p^d : x + y \in A\}.$$

Then, the set A_+ is a Borel set in $\mathbb{Q}_p^d \times \mathbb{Q}_p^d$. For any Borel set A in $\mathbb{Q}_p^d$, we define the convolution of μ and ν by

$$(\mu * \nu)(A) = (\mu \times \nu)(A_+).$$

It is well known that $(\mu * \nu) \in \mathcal{M}(\mathbb{Q}_p^d)$ and that if μ and ν are the probability measures, then so is $\mu * \nu$. Let 1_A be the characteristic function of the Borel set A in $\mathbb{Q}_p^d$, then the definition of $(\mu * \nu)(A)$ is equivalent to the equation

$$\int_{\mathbb{Q}_p^d} 1_A d(\mu * \nu) = \int_{\mathbb{Q}_p^d}\int_{\mathbb{Q}_p^d} 1_A(x + y)\mathrm{d}\mu(x)\mathrm{d}\nu(y).$$

Every function $f \in L^1(\mathbb{Q}_p^d)$ generates a measure $\mu_f \in \mathcal{M}(\mathbb{Q}_p^d)$, defined by

$$\mu_f(A) = \int_A f(x)\mathrm{d}x,$$

which is absolutely continuous with respect to the Haar measure of $\mathbb{Q}_p^d$. Hence, for an $f \in L^1(\mathbb{Q}_p^d)$ and $\mu \in \mathcal{M}(\mathbb{Q}_p^d)$, we define the convolution of f and μ by

$$(f * \mu)(x) = \int_{\mathbb{Q}_p^d} f(x - y)\mathrm{d}\mu(y).$$

The Fourier-Stieltjies transform of a measure $\mu \in \mathcal{M}(\mathbb{Q}_p^d)$ is defined as

$$\widehat{\mu}(\xi) = \int_{\mathbb{Q}_p^d} \overline{\chi_\xi(x)}\mathrm{d}x, \quad (\xi \in \widehat{\mathbb{Q}}_p^d \simeq \mathbb{Q}_p^d).$$

2.5 Bruhat-Schwartz distributions on $\mathbb{Q}_p^d$

Here, we give a brief introduction to the theory of Bruhat-Schwartz distributions in $\mathbb{Q}_p^d$, which mainly follows from the literature [32,33].

Definition 2.2. A complex-valued function ϕ defined on $\mathbb{Q}_p^d$ is said to be uniformly locally constant if there is an integer $\gamma \in \mathbb{Z}$ such that for all $x \in \mathbb{Q}_p^d$ and all $y \in B_\gamma^d(0)$, we have

$$\varphi(x + y) = \varphi(x).$$

It is evident that uniformly locally constant functions are continuous. Let $\mathcal{E}(\mathbb{Q}_p^d)$ be the set of uniformly locally constant functions on $\mathbb{Q}_p^d$. By definition, the space $\mathcal{D}(\mathbb{Q}_p^d)$ of Bruhat-Schwartz test functions consists of uniformly locally constant functions that have compact support. In fact, a test function $\psi \in \mathcal{D}(\mathbb{Q}_p^d)$ can be written as a finite linear combination of indicator functions of the form $1_{B_\gamma^d(x)}(\cdot)$, where $\gamma \in \mathbb{Z}$ and $x \in \mathbb{Q}_p^d$. The largest such γ is denoted by $\ell := \ell(\psi)$ and is called the constancy parameter of ψ. Since $\psi \in \mathcal{D}(\mathbb{Q}_p^d)$ has compact support, there exists a smallest integer $m := m(\psi)$ such that the support of ψ is contained in $B_m^d(0)$, and this m is called the compactness parameter of ψ.

It is clear that we have the relation $\mathcal{D}(\mathbb{Q}_p^d) \subset \mathcal{E}(\mathbb{Q}_p^d)$. The space $\mathcal{D}(\mathbb{Q}_p^d)$ is endowed with the topology as follows: a sequence $\{\psi_n\} \subset \mathcal{D}(\mathbb{Q}_p^d)$ is called a *null sequence* if there is a fixed pair of $\gamma, \gamma' \in \mathbb{Z}$ such that:

(1) For every ball $B_\gamma^d(x)$ of radius p^γ, the function ψ_n is constant on it;

(2) The support of each ψ_n is contained within the ball $B_{\gamma'}^d(0)$;

(3) The sequence ψ_n converges uniformly to the zero-function.

With this defined topology, the space $\mathcal{D}(\mathbb{Q}_p^d)$ is a complete locally convex topological vector space.

A *Bruhat-Schwartz distribution* f on $\mathbb{Q}_p^d$ is a continuous linear functional on $\mathcal{D}(\mathbb{Q}_p^d)$. The value of f at $\psi \in \mathcal{D}(\mathbb{Q}_p^d)$ will be denoted by $\langle f, \psi\rangle$. Note that linear functionals on $\mathcal{D}$ are inherently continuous.

Denote by $\mathcal{D}'(\mathbb{Q}_p^d)$ the space of *Bruhat-Schwartz distributions*. The space $\mathcal{D}'(\mathbb{Q}_p^d)$ is provided with the weak topology induced by $\mathcal{D}(\mathbb{Q}_p^d)$, which means $\lim_{k\to\infty} f_k = 0$ in $\mathcal{D}'(\mathbb{Q}_p^d)$ if

$$\lim_{k\to\infty} \langle f_k, \psi\rangle = 0, \quad \forall \psi \in \mathcal{D}(\mathbb{Q}_p^d).$$

Every function $f \in L^1(\mathbb{Q}_p^d)$ defines a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^d)$ by the formula

$$\langle f, \psi\rangle = \int_{\mathbb{Q}_p^d} f(x)\psi(x)\mathrm{d}x, \quad \forall \psi \in \mathcal{D}(\mathbb{Q}_p^d).$$

The correspondence between functions $f \in L^1(\mathbb{Q}_p^d)$ and distributions $f \in \mathcal{D}'(\mathbb{Q}_p^d)$ is one-to-one, and such distributions are called regular distributions.

Let $T \subset \mathbb{Q}_p^d$ be a discrete set such that $\sharp(T \cap K) < \infty$ for any compact subset $K \subset \mathbb{Q}_p^d$, then

$$\delta_T = \sum_{t\in T} \delta_t \tag{2.2}$$

determines a discrete Radon measure, which is also a distribution: for any $\psi \in \mathcal{D}(\mathbb{Q}_p^d)$,

$$\langle \delta_T, \psi\rangle = \sum_{t\in T} \psi(t).$$

Here, for each $\psi \in \mathcal{D}(\mathbb{Q}_p^d)$, the sum is finite because each compact set contains at most a finite number of points in T, and the test functions in $\mathcal{D}(\mathbb{Q}_p^d)$ are uniformly locally constant with compact support.

Proposition 2.3. ([32], Theorem 4.8.2) *The Fourier transform $f \to \widehat{f}$ is an isomorphism from $\mathcal{D}(\mathbb{Q}_p^d)$ onto $\mathcal{D}(\mathbb{Q}_p^d)$.*

Due to the fact $\mathcal{D}(\mathbb{Q}_p^d) \subset \mathcal{E}(\mathbb{Q}_p^d)$, we have that $\widehat{f} \in \mathcal{E}(\mathbb{Q}_p^d)$. The Fourier transform of a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^d)$ is another distribution $\widehat{f} \in \mathcal{D}'(\mathbb{Q}_p^d)$, which is defined by the relation

$$\langle \widehat{f}, \psi\rangle = \langle f, \widehat{\psi}\rangle, \quad \forall \psi \in \mathcal{D}(\mathbb{Q}_p^d).$$

Proposition 2.4. [32, Proposition 4.9.1] *The Fourier transform $f \to \widehat{f}$ is an isomorphism from $\mathcal{D}'(\mathbb{Q}_p^d)$ onto $\mathcal{D}'(\mathbb{Q}_p^d)$ under the weak topology.*

2.6 Zeros of distribution

Let $f \in \mathcal{D}'(\mathbb{Q}_p^d)$ be a distribution. A point $x \in \mathbb{Q}_p^d$ is said to be a zero of f if there is an integer $\gamma_0 \in \mathbb{Z}$ such that

$$\langle f, 1_{B^d_{\gamma_0}(y)} \rangle = 0, \quad \forall y \in B^d_\gamma(x), \forall \gamma \le \gamma_0.$$

Denote by $\mathcal{Z}_f$ the set of all zeros of $f \in \mathcal{D}'(\mathbb{Q}_p^d)$. Note that $\mathcal{Z}_f$ is the largest open set O, where f vanishes, meaning that $\langle f, \psi \rangle = 0$ for all $\psi \in D(\mathbb{Q}_p^d)$ whose support is contained in O.

The support of a distribution f is defined as the complementary set of $\mathcal{Z}_f$ and is denoted by $\operatorname{supp}(f)$.

2.7 Convolution and multiplication of distributions

Denote $\Delta_\gamma \coloneqq 1_{B^d_\gamma(0)}$ the characteristic function of the ball $B^d_\gamma(0)$ and $\theta_\gamma \coloneqq \widehat{\Delta}_\gamma = p^{d\gamma} 1_{B^d_{-\gamma}(0)}$ the Fourier transform of the characteristic function of the ball $B^d_\gamma(0)$.

For two distributions $f, g \in \mathcal{D}'(\mathbb{Q}_p^d)$, we define the *convolution* of f and g by

$$\langle f * g, \psi \rangle = \lim_{\gamma \to \infty} \langle f(x), \langle g(\cdot), \Delta_\gamma(x) \psi(x + \cdot) \rangle \rangle,$$

if the limit exists for all $\psi \in \mathcal{D}(\mathbb{Q}_p^d)$.

Proposition 2.5. [32, Proposition 4.7.3] *Let f be a distribution in the space $\mathcal{D}'(\mathbb{Q}_p^d)$. Then, when f is convolved with θ_γ, the resulting function $f * \theta_\gamma$ belongs to the space $\mathcal{E}(\mathbb{Q}_p^d)$, and it has a constancy parameter of at least $-\gamma$.*

We define the product of two distributions f and g as follows. For a test function $\psi \in D(\mathbb{Q}_p^d)$, the action $\langle f \cdot g, \psi \rangle$ is given by the limit $\lim_{\gamma \to \infty} \langle g, (f * \theta_\gamma) \psi \rangle$, provided that this limit exists for all such test functions ψ. This definition of convolution aligns with the standard convolution of two integrable functions, and the definition of multiplication aligns with the standard multiplication of two locally integrable functions. Additionally, the following proposition will show that both convolution and multiplication of distributions are commutative when they are well defined, and the convolution of two distributions is well defined precisely when the multiplication of their Fourier transforms is well defined.

Proposition 2.6. [33, Sections 7.1 and 7.5] *Let $f, g \in \mathcal{D}'(\mathbb{Q}_p^d)$ be two distributions. Then,*
(1) *If $f * g$ is well defined, so is $g * f$, and $f * g = g * f$.*
(2) *If $f \cdot g$ is well defined, so is $g \cdot f$, and $f \cdot g = g \cdot f$.*
(3) *$f * g$ is well defined if and only if $\widehat{f} \cdot \widehat{g}$ is well defined. In this case, we have $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$ and $\widehat{f \cdot g} = \widehat{f} * \widehat{g}$.*

Proposition 2.7. [27, Proposition 2.12] *Let f and g be two distributions in $\mathcal{D}'(\mathbb{Q}_p^d)$. If the intersection of their supports is empty, i.e.,* $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \varnothing$, *then the product $g \cdot f$ is well defined, and it equals zero.*

The product of certain special distributions can be expressed in a straightforward manner. This is particularly true when multiplying a uniformly locally constant function by a distribution.

Proposition 2.8. [33, Section 7.5, Example 2] *Suppose $f \in \mathcal{E}(\mathbb{Q}_p^d)$ and $g \in \mathcal{D}'(\mathbb{Q}_p^d)$. Then, for any test function $\varphi \in \mathcal{D}(\mathbb{Q}_p^d)$, the duality relation $\langle f \cdot g, \varphi \rangle = \langle g, f\varphi \rangle$ holds.*

2.8 Colombeau-Egorov algebra of p-adic generalized functions

Let f_γ be the *regularization* of the distribution $f \in \mathcal{D}'(\mathbb{Q}_p^d)$, i.e.,

$$f_\gamma \coloneqq \Delta_\gamma \cdot (f * \theta_\gamma) \in \mathcal{D}'(\mathbb{Q}_p^d).$$

Then, we have the following proposition.

Proposition 2.9. [32, Lemma 14.3.1] *Let f_γ be the regularization of the distribution $f \in \mathcal{D}'(\mathbb{Q}_p^d)$. Then, $\lim_{\gamma\to\infty} f_\gamma = f$ in $D'(\mathbb{Q}_p^d)$. Additionally, for any test function $\varphi \in \mathcal{D}(\mathbb{Q}_p^d)$, when $\gamma \geq -\ell$ (where ℓ is the constancy parameter of the function φ), $\langle f_\gamma, \varphi\rangle = \langle f, \varphi\rangle$.*

By approximating distributions with test functions, we can create a space that exceeds the space of distributions. This expanded space is the Colombeau-Egorov algebra. Remember that in the space of Bruhat-Schwartz distributions, convolution and multiplication are not universally well defined for all pairs of distributions. However, in the Colombeau-Egorov algebra, both convolution and multiplication are well defined and are associative operations.

Let us take into account the set $\mathcal{P}(\mathbb{Q}_p^d)$ that consists of all sequences $\{f_\gamma\}_{\gamma\in\mathbb{N}}$ of test functions. We define an algebraic structure on $P(\mathbb{Q}_p^d)$, where the operations are defined componentwise as

$$\{f_\gamma\} + \{g_\gamma\} = \{f_\gamma + g_\gamma\},$$
$$\{f_\gamma\} \cdot \{g_\gamma\} = \{f_\gamma \cdot g_\gamma\},$$

where $\{f_\gamma\}, \{g_\gamma\} \in \mathcal{P}(\mathbb{Q}_p^d)$. Define $\mathcal{N}(\mathbb{Q}_p^d)$ as the subalgebra consisting of elements $\{f_\gamma\}_{\gamma\in\mathbb{N}} \in \mathcal{P}(\mathbb{Q}_p^d)$ with the following property: for every compact set K in $\mathbb{Q}_p^d$, there exists a natural number N such that for all $\gamma > N$ and for all x in K, $f_\gamma(x) = 0$. Clearly, $\mathcal{N}(\mathbb{Q}_p^d)$ is an ideal within the algebra $\mathcal{P}(\mathbb{Q}_p^d)$.

Now, we introduce the p-adic *Colombeau-Egorov* algebra

$$\mathcal{G}(\mathbb{Q}_p^d) = \mathcal{P}(\mathbb{Q}_p^d)/\mathcal{N}(\mathbb{Q}_p^d).$$

The elements $\widetilde{f}$ of this algebra will be called the p-adic *Colombeau-Egorov generalized functions*. The equivalence class of sequences that defines the element $\widetilde{f}$ will be denoted by $\widetilde{f} = [f_\gamma]$. For any $\widetilde{f} = [f_\gamma]$, $\widetilde{g} = [g_\gamma]$, the addition and multiplication are defined as

$$\widetilde{f} + \widetilde{g} = [f_\gamma + g_\gamma], \quad \widetilde{f} \cdot \widetilde{g} = [f_\gamma \cdot g_\gamma].$$

Clearly, $(\mathcal{G}(\mathbb{Q}_p^d), +, \cdot)$ is an associative and commutative algebra.

Theorem 2.10. [32, Theorem 14.3.3] *There is a linear embedding*

$$\mathcal{D}'(\mathbb{Q}_p^d) \subset \mathcal{G}(\mathbb{Q}_p^d),$$

i.e., any distribution on $\mathbb{Q}_p^d$ is a generalized function on $\mathbb{Q}_p^d$. This linear embedding is given by the map

$$\mathcal{D}'(\mathbb{Q}_p^d) \ni f \mapsto \widetilde{f} = [\Delta_\gamma \cdot (f * \theta_\gamma)] \in \mathcal{G}(\mathbb{Q}_p^d).$$

Any distribution f in the space $\mathcal{D}'(\mathbb{Q}_p^d)$ can be embedded into $\mathcal{G}(\mathbb{Q}_p^d)$ through a mapping. This mapping links f to a generalized function that comes from the regularization of f. As a result, we can conclude that the multiplication defined on $\mathcal{D}'(\mathbb{Q}_p^d)$ is associative in the sense described in the following.

Lemma 2.11. [27, Proposition 2.16] *Suppose f, g, and h are the elements of $\mathcal{D}'(\mathbb{Q}_p^d)$. When $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are well defined as products of distributions, we have $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.*

2.9 Spectral set criterion

We denote by $|\Omega|$ the Haar measure of a set Ω. Here is a criterion for a Borel set $\Omega \subset \mathbb{Q}_p^d$, with $0 < |\Omega| < \infty$ being a spectral set [23]. Recall that

$$E(\Lambda) = \{e^{2\pi i\{\lambda \cdot x\}} : \lambda \in \Lambda\}.$$

Lemma 2.12. [23] *A Borel set $\Omega \subset \mathbb{Q}_p^d$ with positive finite Haar measure is a spectral set with Λ as a spectrum if and only if*

$$\sum_{\lambda\in\Lambda} |\hat{1}_\Omega|^2(\xi - \lambda) = |\Omega|^2, \quad \forall \xi \in \mathbb{Q}_p^d. \tag{2.3}$$

3 Proof of theorems

For a function $f : \mathbb{Q}_p^d \to \mathbb{C}$, we write

$$\mathcal{N}_f = \{x \in \mathbb{Q}_p^d : f(x) = 0\}.$$

Let f be a continuous function in $\mathbb{Q}_p^d$. Then, the set $\mathcal{N}_f$ is closed, and $\mathcal{Z}_f$ consists of the interior points of $\mathcal{N}_f$. Moreover, the support of f considered as a continuous function is the same as the support of f considered as a distribution.

The following lemma confirms the conditions that the points of an orthogonal set Λ must satisfy.

Lemma 3.1. *If $\Omega \subset \mathbb{Q}_p^d$ is a Borel set with $0 < |\Omega| < \infty$, then the set $E(\Lambda)$ is orthogonal in $L^2(\Omega)$ if and only if*

$$\Lambda - \Lambda \subset \mathcal{N}_{\hat{1}_\Omega} \cup \{0\}. \tag{3.1}$$

Proof. For distinct $\lambda, \lambda' \in \Lambda$, we have

$$\begin{aligned}\hat{1}_\Omega(\lambda - \lambda') &= \int_{\mathbb{Q}_p^d} e^{-2\pi i\{\langle\lambda-\lambda',x\rangle\}} 1_\Omega(x)\mathrm{d}x \\ &= \int_\Omega e^{-2\pi i\{\langle\lambda,x\rangle\}} e^{2\pi i\{\langle\lambda',x\rangle\}}\mathrm{d}x \\ &= \langle\chi_\lambda, \chi_{\lambda'}\rangle_\Omega.\end{aligned} \tag{3.2}$$

If (3.1) holds, then $\langle\chi_\lambda, \chi_{\lambda'}\rangle_\Omega = 0$, for distinct $\lambda, \lambda' \in \Lambda$, which means that Λ is an orthogonal set. Conversely, if Λ is an orthogonal set in $L^2(\Omega)$, then $\langle\chi_\lambda, \chi_{\lambda'}\rangle_\Omega = 0$, for distinct $\lambda, \lambda' \in \Lambda$, which is equivalent to (3.1). □

Lemma 3.2. *Let $\Omega \subset \mathbb{Q}_p^d$ be a Borel set with $0 < |\Omega| < \infty$. If (Ω, Λ) is a spectral pair, then Λ is uniformly discrete.*

Proof. By taking into account the fact that $\hat{1}_\Omega(0) = |\Omega| > 0$ and the continuity of the function $\hat{1}_\Omega(x)$, there exists a $n_0 \in \mathbb{Z}$, such that $\hat{1}_\Omega(x) \neq 0$ for all $x \in B(0, p^{n_0})$. This, together Lemma 3.1, implies that $|\lambda - \lambda'|_p \geq p^{n_0}$ for different $\lambda, \lambda' \in \Lambda$, which means that Λ is uniformly discrete. □

We set

$$\gamma_0 \coloneqq \min\{\gamma : \exists\xi \in B(0, p^\gamma), \hat{1}_\Omega(\xi) = 0\}.$$

Then, Lemma 3.2 indicates that if Λ is a spectrum for Ω, then it is a uniformly discrete set with separation constant $\delta(\Lambda) \geq \gamma_0$.

Proof of theorem 1.1. (1) By the Bessel inequality, we have that the set $E(\Lambda)$ is orthogonal in $L^2(\Omega)$ if and only if

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = \sum_{\lambda\in\Lambda} |\hat{1}_\Omega(x-\lambda)|^2/|\Omega|^2 \le 1. \tag{3.3}$$

For every $x \in \mathbb{Q}_p^d$, inequality (3.3) holds if and only if $f + \Lambda$ is a sub-tiling.

(2) By Lemma (2.12), we have that (Ω, Λ) is a spectral pair if and only if

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = \sum_{\lambda\in\Lambda} |\hat{1}_\Omega(x-\lambda)|^2/|\Omega|^2 = 1. \tag{3.4}$$

For every $x \in \mathbb{Q}_p^d$, Equation (3.4) holds if and only if $f + \Lambda$ is a tiling. □

Proof of theorem 1.2. It suffices to prove one side of the equivalence. Assume that $f * \mu = 1$. Since $g \ge 0$, $\int_{\mathbb{Q}_p^d} g(x)\mathrm{d}x = 1$, and convolution is associative, we have

$$1 = f*\mu \Rightarrow 1 = 1*g = g*f*\mu = f*g*\mu.$$

Let $h = g*\mu$, and we have that $0 \le h \le 1$ and $f*h = 1$. We need to show that $h = 1$.

Let $M = \{x \in \mathbb{Q}_p^d : h(x) < 1\}$, $N = M^c$ be the complement of M. Then, we have

$$f*h(\xi) = \int_{\mathbb{Q}_p^d} h(x)f(\xi-x)\mathrm{d}x = \int_M h(x)f(\xi-x)\mathrm{d}x + \int_N h(x)f(\xi-x)\mathrm{d}x.$$

If $|M| > 0$, then, by the Fubini theorem, we have

$$\int_{\mathbb{Q}_p^d}\int_M f(\xi-x)\mathrm{d}x\mathrm{d}\xi = \int_M\int_{\mathbb{Q}_p^d} f(\xi-x)\mathrm{d}\xi\mathrm{d}x = |M| > 0.$$

Hence, there exists a subset $O \subset M$, with $|O| > 0$, such that

$$\int_M f(\xi-x)\mathrm{d}\xi > 0, \quad \forall x \in O.$$

So, if $x \in O$, we have

$$\begin{aligned}\int_M h(x)f(\xi-x)\mathrm{d}x + \int_N h(x)f(\xi-x)\mathrm{d}x &< \int_M f(\xi-x)\mathrm{d}x + \int_N f(\xi-x)\mathrm{d}x \\ &= \int_{\mathbb{Q}_p^d} f(\xi-x)\mathrm{d}x = 1*f = 1.\end{aligned} \tag{3.5}$$

Thus, we obtain a contradiction to the fact that $f*h = 1$ is almost everywhere. So $|O| = 0$, and $h = 1$. □

Before proving Theorem 1.3, we give the following lemma, which is very useful to prove it.

Lemma 3.3. *Consider a continuous function $g \in C(\mathbb{Q}_p^d)$ and a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^d)$. Assume that the product $F = g \cdot f$ is well defined. Then,*

$$\mathcal{Z}_F \backslash \mathcal{Z}_g \subseteq \mathcal{Z}_f.$$

Proof. Take an arbitrary *Bruhat-Schwartz test function* $\psi \in \mathcal{D}(\mathbb{Q}_p^d)$, with $\mathrm{supp}(\psi) \subset \mathcal{Z}_F\backslash\mathcal{Z}_g$. Since $\mathrm{supp}(\psi) \subset \mathcal{Z}_F\backslash\mathcal{Z}_g$, we can find a function $h(x)$, which is defined on $\mathrm{supp}(\psi)$, such that for $\forall x \in \mathrm{supp}(\psi)$, we have

$$h(x) \cdot g(x) = 1_{\mathrm{supp}(\psi)}.$$

Define

$$h(x) = \begin{cases} \dfrac{1}{g(x)}, & x \in \operatorname{supp}(\psi), \\ 0, & \text{elsewhere}. \end{cases}$$

Then, h is the required function, and bounded and compactly supported, so $h \in L^1(\mathbb{Q}_p^d)$. Thus, by taking into account Proposition 2.8, we obtain that

$$\langle f, \psi \rangle = \langle f, 1_{\operatorname{supp}(\psi)} \cdot \psi \rangle = \langle 1_{\operatorname{supp}(\psi)} \cdot f, \psi \rangle = \langle h \cdot g \cdot f, \psi \rangle.$$

Since $\operatorname{supp}(h) \subset \mathcal{Z}_F$, by Proposition 2.7, the product $h \cdot F$ is well defined and $h \cdot F = 0$. By Proposition 2.11,

$$\langle f, \psi \rangle = \langle h \cdot F, \psi \rangle = 0.$$

We note that $\operatorname{supp}(h) \subset \mathcal{Z}_F$, by Proposition 2.7, the product $h \cdot F$ is well defined and $h \cdot F = 0$. Taking into account Proposition 2.11, we obtain that

$$\langle f, \psi \rangle = \langle h \cdot F, \psi \rangle = 0.$$

Thus, we obtained the desired result. □

Proof of Theorem 1.3. (1) Let $f + T$ be a tiling of level ℓ, which means

$$\sum_{t \in T} f(x - t) = \ell, \quad \text{a.e. } x \in \mathbb{Q}_p^d. \tag{3.6}$$

By the definition of convolution of discrete measures with $L^1(\mathbb{Q}_p^d)$ functions, equation (3.6) is equivalent to

$$f * \delta_T = \ell, \tag{3.7}$$

where $\delta_T = \sum_{t \in T} \delta_t$ is the Dirac measure concentrated at points $t \in T$.

Taking Fourier transform of two sides of equation (3.7), we have, by Proposition 2.6, that

$$\widehat{f * \delta_T} = \widehat{f} \cdot \widehat{\delta_T} = \ell \delta_0.$$

Since the Fourier transform of $f \in L^1(\mathbb{Q}_p^d)$ is continuous and $\mathcal{Z}_{\delta_0} = \mathbb{Q}_p^d \backslash \{0\}$, by Lemma 3.3, we obtain

$$\mathbb{Q}_p^d \backslash \mathcal{Z}_{\widehat{\delta_T}} \subseteq \mathcal{Z}_{\widehat{f}} \cup \{0\},$$

which is equivalent to

$$\operatorname{supp}(\widehat{\delta_T}) \backslash \{0\} \subseteq \mathcal{Z}_{\widehat{f}}.$$

(2) Let $\psi \in \mathcal{D}(\mathbb{Q}_p^d)$ be a *Bruhat-Schwartz test function*, then by definition of the Fourier transform of the distribution, and by Propositions 2.6 and 2.8, we have

$$\langle f * \delta_T, \psi \rangle = \langle \widehat{f} \cdot \widehat{\delta_T}, \widehat{\psi} \rangle = \langle \widehat{\delta_T}, \widehat{\psi} \widehat{f} \rangle.$$

Since $\operatorname{supp}(\widehat{\delta_T}) \backslash \{0\} \subseteq \mathcal{Z}_{\widehat{f}}$, and $\widehat{\delta_T}$ is a measure, any continuous function vanishing on $\operatorname{supp}(\widehat{\delta_T})$ becomes zero under $\widehat{\delta_T}$. Thus, we have that

$$\langle \widehat{\delta_T}, \hat{\psi} \hat{f} \rangle = \lambda \hat{\psi}(0),$$

where $\lambda = \widehat{\delta_T}(\{0\}) \widehat{f}(0)$. This means that $\langle f * \delta_T, \psi \rangle = \lambda \widehat{\psi}(0)$.

On the other hand, $\langle \lambda 1_{\mathbb{Q}_p^d}, \psi \rangle = \lambda \widehat{\psi}(0)$. So, we have that

$$f * \delta_T = \lambda 1_{\mathbb{Q}_p^d}.$$

We obtained the desired result. □

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