Research Article

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Classification and irreducibility of a class of integer polynomials

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Abstract: We find all integer polynomials of degree d that take the values ±1 at exactly d integer arguments, and determine the irreducibility of these polynomials by means of an elementary approach.

Keywords: irreducibility, integer polynomial

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1 Introduction

In this article, all polynomials are integer polynomials (i.e., belonging to $\mathbb{Z}[x]$), and factorization and irreducibility are in $\mathbb{Z}[x]$. There have been lots of results on the irreducibility of certain classes of integer polynomials (see, e.g., [1] for a review of some classic irreducibility criteria for integer polynomials). Some more recent results in this direction can be found in [2–5] for instance.

Suppose $P = P(x) \in \mathbb{Z}[x]$ has degree $d(P) = d$. Denote

$$S(P) := (\ldots, P(-1), P(0), P(1), \ldots),$$

the infinite sequence of values of P at integers. Then, the primes and units in $S(P)$ determine the irreducibility of P in some sense. For example, Murty [6] proved that if $P(m)$ is a prime for a “large” number m, then P is irreducible[1]; Brown and Graham [7] proved that if $p + 2u - d > 4$, then P is irreducible, where p and u are, respectively, the number of primes and units (i.e., ±1) in $S(P)$. However, it is not always easy to find a prime represented by a polynomial. For example ([8, pp. 42, 172]), $P(x) = x^6 + 1091$ is not prime for $x = 1, 2, \ldots, 3905$.

In this article, we characterize all *fit polynomials* (Definition 2.2), i.e., polynomials with $u = d$, and determine the irreducibility of fit polynomials by means of an elementary approach. We note that Dorwart and Ore [9] also studied fit polynomials (without using the term “fit”), and our results give a more detailed characterization.

We say P is *equivalent* to $Q \in \mathbb{Z}[x]$, denoted by $P \sim Q$, if $P = \pm Q(\pm x + b)$ for some $b \in \mathbb{Z}$. Now, we can present our main results, which give a complete classification of fit polynomials up to the equivalence relation ~.

1 More precisely, let $f(x) = a_0 + a_1x + \ldots + a_nx^n$ be an integer polynomial of degree n and set $H = \max\{|a_i/a_n| | 0 \le i \le n-1\}$, if $f(m)$ is a prime for some integer $m \ge H + 2$, then $f(x)$ is irreducible in $\mathbb{Z}[x]$.

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Theorem 1.1. *Each fit polynomial of degree* $d \geq 1$ *is equivalent to one of the following polynomials, where* $0 \neq c \in \mathbb{Z}$.

(i) $cx(x - c_1)(x - c_2)\ldots(x - c_{d-1}) + 1$, *where* $0 < c$, $0 < c_1 < c_2 \ldots < c_{d-1}$, *and moreover,* $c \geq 3$ *if* $d = 1$, $(c, c_1) \neq (1, 3)$ *or* $(2, 2)$ *if* $d = 2$, $(c, c_1, c_2) \neq (1, 1, 3)$ *if* $d = 3$;
(ii) $cx(x - c_1)(x - c_2)\ldots(x - c_{d-1}) - 1$, *where* $0 < c$, d *is even,* $0 < c_1 < c_2 \ldots < c_{d-1}$, *and* $(c, c_1) \neq (1, 1)$ *if* $d = 2$;
(iii) $cx^4 + (1 + 2c)x^3 + (2 - c)x^2 - (2c + 1)x - 1$;
(iv) $cx^4 - 2cx^3 - (c + 1)x^2 + (2c + 1)x + 1$;
(v) $cx^3 - x^2 - (1 + c)x + 1, c \neq 1$;
(vi) $cx^3 + 2x^2 - cx - 1, c \geq 2$;
(vii) $cx^3 + (1 + c)x^2 + (1 - 2c)x - 1, c \neq 1$;
(viii) $cx^2 - x - c, c \geq 2$;
(ix) $cx^2 - (c + 2)x + 1, c \geq 3$.

Conversely, each polynomial equivalent to one in the aforementioned list is fit. Furthermore, if P *and* Q *are both in the aforementioned list, then* $P \sim Q$ *if and only if* $P = Q$.

The following theorem gives an irreducibility testing criterion for fit polynomials.

Theorem 1.2. *Suppose* P *is a fit polynomial of degree* $d > 1$. *Then, the following statements are equivalent*:
(*i*) P *is reducible.*
(*ii*) $P = \pm Q^2$ *for some* $Q \in \mathbb{Z}[x]$.
(*iii*) $P = \pm Q^2$ *for some* $Q \sim x^2 + x - 1, 2x - 1$, *or* x.
(*iv*) $P \sim x^4 + 2x^3 - x^2 - 2x + 1, 4x^2 - 4x + 1$, *or* x^2.

The rest of this article is organized as follows. In Section 2, we introduce basic notions and facts on fit polynomials. Then, we study fit polynomials in Section 3 for $d > 3$, in Section 4 for $d = 3$, and in Section 5 for $d < 3$. We prove Theorems 1.1 and 1.2 in Section 6. Finally, we give an example in Section 7.

2 Preliminaries

We always employ P, Q, R, M, and N to denote integer polynomials of *positive* degree and use a, b, and c to represent integers.

Let $u(P), u_+(P), u_-(P)$, and $p(P)$, respectively, be the number of units (i.e., ±1), the number of 1, the number of −1, and the number of primes in $S(P)$. Note that $u < \infty$ since $P(x) \pm 1$ cannot have infinitely many roots, and that it is possible $p = \infty$, e.g., $p(P) = \infty$ for $P(x) = a + bx$ with $(a, b) = 1$ by Dirichlet's theorem about primes in arithmetic progressions.

If $u_+ = u_+(P) \geq 1$, then there exist integers $a_1 < a_2 < \ldots < a_{u_+}$ such that $P(a_1) = P(a_2) = \ldots = P(a_{u_+}) = 1$. Similarly, if $u_- = u_-(P) \geq 1$, then there exist integers $b_1 < b_2 < \ldots < b_{u_-}$ such that $P(b_1) = P(b_2) = \ldots = P(b_{u_-}) = -1$.

Definition 2.1. Denote $A_+(P) \coloneqq (a_1, a_2, \ldots, a_{u_+})$ and $A_-(P) \coloneqq (b_1, b_2, \ldots, b_{u_-})$.

Motivated by the definition of fat polynomials in [7], we have the following definition.

Definition 2.2.
(i) The difference $f(P) \coloneqq u(P) - d(P)$ is called the *fatness* of P.
(ii) We call P *fat*, *fit*, or *thin*, respectively, if $f(P) > 0$, $f(P) = 0$, or $f(P) < 0$.

It follows from the definition that nontrivial factors (if exist) of a fit polynomial are *fat*.

Denote

$$I(P) := \{P(\pm x + b) : b \in \mathbb{Z}\},$$
$$J(P) := \{\pm P(\pm x + b) : b \in \mathbb{Z}\}.$$

It is clear that $J(P) = I(P) \cup I(-P)$ is the equivalence class consisting of all $Q \sim P$.

Lemma 2.3. *Suppose $Q \in I(P)$, $u = u(P)$, $u_+ = u_+(P)$, and $u_- = u_-(P)$. Then, the following statements hold:*
(*i*) $u_+(Q) = u_+$, $u_-(Q) = u_-$, *and* $u(Q) = u$.
(*ii*) *The leading coefficient of Q equals either that of P or that of $-P$.*
(*iii*) *If $A_+(P) = (a_1, \ldots, a_{u_+})$ and $A_+(Q) = (b_1, \ldots, b_{u_+})$, then*

$$a_i - a_j = b_i - b_j, \quad \text{for all } 1 \le i < j \le u_+.$$

(*iv*) *If $A_-(P) = (a_1, \ldots, a_{u_-})$ and $A_-(Q) = (b_1, \ldots, b_{u_-})$, then*

$$a_i - a_j = b_i - b_j, \quad \text{for all } 1 \le i < j \le u_-.$$

(*v*) *There exists a unique $R \in I(P)$ such that $A_+(R) = (0, c_2, \ldots, c_{u_+})$.*

Proof. The statements are clear since Q and R can be obtained from P by shifting x and/or reflecting about x-axis. □

Similarly, we also have the following:

Lemma 2.4. *Suppose $Q \in J(P)$. Then, the following statements hold:*
(*i*) *P is irreducible (or fit, respectively) if and only if so is Q.*
(*ii*) $u(Q) = u(P)$, $d(Q) = d(P)$, *and thus,* $f(Q) = f(P)$.
(*iii*) *The leading coefficient of Q equals either that of P or that of $-P$.*
(*iv*) *If $Q \in I(-P)$, then $u_+(Q) = u_-(P)$ and $u_-(Q) = u_+(P)$.*

Lemma 2.5. *Suppose $u_+(P) \ge 1$ and $u_-(P) \ge 1$. Then, $u(P) \le 4$.*

Proof. See the proof of [7, Theorem 1]. □

The following lemma is a direct corollary of [7, Theorem 1], which fully characterizes the family of fat polynomials.

Lemma 2.6.
(*i*) *Let P be a fat polynomial. Then, $u(P) \le 4$, $d(P) \le 3$, $f(P) \le 2$, and P belongs to one of the classes in Table* 1.
(*ii*) *All fat polynomials are irreducible.*

Proof. (i) Note that $J(x) = I(x)$ and $J(2x - 1) = I(2x - 1)$. Then, the statement follows from [7, Theorem 1].
(ii) It is easy to see. □

Table 1: Fat polynomials

	u_+	u_-	u	d	f
$I(x)$	1	1	2	1	1
$I(2x - 1)$	1	1	2	1	1
$I(2x^2 - 1)$	2	1	3	2	1
$I(-2x^2 + 1)$	1	2	3	2	1
$J(x^2 + x - 1)$	2	2	4	2	2
$I(x^3 + 2x^2 - x - 1)$	3	1	4	3	1
$I(-x^3 - 2x^2 + x + 1)$	1	3	4	3	1

Lemma 2.7. [7, Theorem 2 and Corollary 2] *Let* $p = p(P)$, $u = u(P)$, *and* $d = d(P)$.
(*i*) *If* $p + 2u > d \geq 2$, *then either P is irreducible or* $P = QR$ *with* $f(Q) + f(R) \geq p + 2u - d$.
(*ii*) *If* $p + 2u - d > 4$, *then P is irreducible.*

The following lemma characterizes the reducibility of fit polynomials.

Lemma 2.8. *Suppose P is fit and reducible,* $d = d(P)$ *and* $u = u(P)$. *Assume* $P = QR$ *with* $f(Q) \geq f(R)$. *Then,* $u = d \leq 4$. *Furthermore,*
(*i*) *if* $u = d = 2$, *then* $f(Q) = f(R) = 1$;
(*ii*) *if* $u = d = 3$, *then* $f(Q) = 2$ *and* $f(R) = 1$;
(*iii*) *if* $u = d = 4$, *then* $f(Q) = f(R) = 2$, $d(Q) = d(R) = 2$ *and* $Q \sim R \sim x^2 + x - 1$.

Proof. Suppose $u = d > 4$. Since $p + 2u = p + 2d > d > 4$, by Lemma 2.7 (i), $f(Q) + f(R) \geq p + 2u - d > p + 4 \geq 4$. By Lemma 2.6 (i), fatness of a polynomial is at most 2, and thus, $f(Q) + f(R) \leq 4$, which is a contradiction. Hence, $u = d \leq 4$.

(i) Suppose $u = d = 2$. Then, $d(Q) = d(R) = 1$. By Lemma 2.6 (i), $f(Q) = f(R) = 1$.

(ii) Suppose $u = d = 3$. By Lemmas 2.7 (i) and 2.6 (i), we have that $p + 3 \leq f(Q) + f(R) \leq 4$. Thus, either $f(Q) = 2$ and $f(R) = 1$, or $f(Q) = f(R) = 2$. If $f(Q) = f(R) = 2$, then $d(Q) = d(R) = 2$ and $d(P) = d(Q) + d(R) = 4$, which is a contradiction. Therefore, $f(Q) = 2$ and $f(R) = 1$.

(iii) Similar to (ii), we have $f(Q) + f(R) = 4$, and thus, $f(Q) = f(R) = 2$. By Lemma 2.6 (i), we obtain that $d(Q) = d(R) = 2$ and $Q \sim R \sim x^2 + x - 1$. □

3 Fit polynomials with $d > 3$

In this section, we consider fit polynomials of degree $d > 3$.

3.1 Case $d > 4$

Lemma 3.1. *Suppose P is a fit polynomial with* $u = d > 4$. *Then, P is irreducible and equivalent to one of the following polynomials*:

$$cx(x - c_1)(x - c_2)\dots(x - c_{d-1}) + 1, \quad c > 0, \tag{1}$$

$$cx(x - c_1)(x - c_2)\dots(x - c_{d-1}) - 1, \quad c > 0, d \text{ is even}, \tag{2}$$

where $0 < c_1 < c_2 \dots < c_{d-1}$. *Moreover, each polynomial in* (1) *or* (2) *is fit.*

Proof. By Lemma 2.8, we obtain that P is irreducible. It follows from Lemma 2.5 that either $u_+ = u = d$ or $u_- = u = d$. Thus, by Lemma 2.3,

$$P(x) \sim Q(x) \coloneqq cx(x - c_1)(x - c_2)\dots(x - c_{d-1}) + 1$$

or

$$P(x) \sim R(x) \coloneqq cx(x - c_1)(x - c_2)\dots(x - c_{d-1}) - 1,$$

where $c \neq 0$, and c_i's are distinct nonzero integers. If $c < 0$ and d is odd, then

$$P(x) \sim Q(-x) = (-c)x(x + c_1)(x + c_2)\dots(x + c_{d-1}) + 1,$$

which has the form (1), or

$$P(x) \sim -R(x) = (-c)x(x - c_1)(x - c_2)\ldots(x - c_{d-1}) + 1,$$

which has the form (1). If $c < 0$ and d is even, then

$$P(x) \sim -Q(x) = (-c)x(x - c_1)(x - c_2)\ldots(x - c_{d-1}) - 1,$$

which has the form (2), or

$$P(x) \sim -R(x) = (-c)x(x - c_1)(x - c_2)\ldots(x - c_{d-1}) + 1,$$

which has the form (1). Hence, P is equivalent to a polynomial in (1) or (2).

Conversely, it is clear that each polynomial in (1) or (2) is fit. □

3.2 Case $d = 4$

Now, we consider the case $d = 4$ and $u_+ = 0$ or 4.

Lemma 3.2. *Suppose P is a fit polynomial with $d = u = 4$.*

(*i*) *Suppose $u_+ = 0$ or 4. Then, P is equivalent to*

$$cx(x - c_1)(x - c_2)(x - c_3) \pm 1, \quad 0 < c, 0 < c_1 < c_2 < c_3. \tag{3}$$

Conversely, each polynomial in (3)*is fit.*

(*ii*) *Suppose $u_+ = 4$. Then, P is reducible if and only if $P = Q^2$ for some $Q \sim x^2 + x - 1$.*

(*iii*) *Suppose $u_+ = 0$. Then, P is reducible if and only if $P = -Q^2$ for some $Q \sim x^2 + x - 1$.*

Proof. (i) It is similar to the proof of Lemma 3.1.

(ii) Suppose $u_+ = 4$ and $A_+(P) = (a_1, a_2, a_3, a_4)$. Then, $P(x) = c(x - a_1)(x - a_2)(x - a_3)(x - a_4) + 1$ for some $c \neq 0$. Assume P is reducible and $P = QR$. By Lemma 2.8 (iii), $Q \sim R \sim x^2 + x - 1$. Since $Q(a_i)R(a_i) = P(a_i) = 1$ for $i = 1, 2, 3, 4$, and all factors on the left-hand sides of the aforementioned equations are integers, we have that, for $i = 1, 2, 3, 4$, either $Q(a_i) = R(a_i) = 1$ or $Q(a_i) = R(a_i) = -1$. Hence, the polynomial equation $Q(x) - R(x) = 0$ has four solutions. But $d(Q(x) - R(x)) < 4$. It follows that $Q(x) = R(x)$. Thus, $P = Q^2$ for $Q \sim x^2 + x - 1$.

(iii) If $u_+ = 0$, then $P' \coloneqq -P$ satisfies the conditions in (ii), and thus the statement follows from (ii). □

The following lemma deals with the case $d = 4$ and $u_+ = 1$ or 3.

Lemma 3.3. *Suppose P is a fit polynomial with $d = 4$ and $u_+ = 1$ or 3. Then, the following statements hold:*

(*i*) *P is irreducible.*

(*ii*) *P is equivalent to*

$$cx^4 + (1 + 2c)x^3 + (2 - c)x^2 - (2c + 1)x - 1, \quad c \neq 0.$$

(*iii*) *Each polynomial in* (ii) *is fit.*

Proof. Suppose $u_+ = 3$ (if $u_+ = 1$, consider $P' = -P$), $A_+ = (a_1, a_2, a_3)$, and $A_- = (a_4)$. Then,

$$P(x) = M(x)(x - a_1)(x - a_2)(x - a_3) + 1 = N(x)(x - a_4) - 1,$$

for $M(x), N(x) \in \mathbb{Z}[x]$ such that $M(x)$ does not have integer roots other than a_1, a_2, a_3, and $N(x)$ does not have integer roots other than a_4.

(i) Assume P is reducible and $P = QR$. Then, $Q(a_i)R(a_i) = P(a_i) = 1$ for $i = 1, 2, 3$. Similar to the proof of Lemma 3.2, we have that $P = Q^2$, which contradicts the assumption that $P(a_4) = -1$. The contradiction implies that P is irreducible.

(ii) First, note that

$$M(a_4)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) = -2. \tag{4}$$

Note that the conditions $a_1 < a_2 < a_3$ imply that

$$a_4 - a_3 < a_4 - a_2 < a_4 - a_1.$$

Thus, it follows from equation (4) that we have the following two cases to consider.
(2a) $a_4 - a_1 = 2, a_4 - a_2 = 1, a_4 - a_3 = -1, M(a_4) = 1,$
(2b) $a_4 - a_1 = 1, a_4 - a_2 = -1, a_4 - a_3 = -2, M(a_4) = -1.$

For case (2a), by Lemma 2.3, we may assume $a_1 = 0$. From this, we can immediately deduce $a_4 = 2$, which further leads to $a_2 = 1$ and $a_3 = 3$. Thus, $M(x) = c(x - 2) + 1$ for $c \neq 0$. Now,

$$\begin{aligned} P(x) &= (c(x-2)+1)x(x-1)(x-3)+1 \\ &= cx^4 + (1-6c)x^3 + (11c-4)x^2 + (3-6c)x + 1 \\ &= c(x-2)^4 + (1+2c)(x-2)^3 + (2-c)(x-2)^2 - (2c+1)(x-2) - 1 \\ &\sim cx^4 + (1+2c)x^3 + (2-c)x^2 - (2c+1)x - 1. \end{aligned}$$

Similarly, we obtain for case (2b) that

$$\begin{aligned} P &\in I(cx^4 - (6c+1)x^3 + (11c+5)x^2 - (6c+6)x + 1) \\ &= I(cx^4 - (1+2c)x^3 + (2-c)x^2 + (2c+1)x - 1) \\ &= I(cx^4 + (1+2c)x^3 + (2-c)x^2 - (2c+1)x - 1), \end{aligned}$$

where the last = is obtained by replacing x by $-x$.

(iii) Let $P = cx^4 + (1 + 2c)x^3 + (2 - c)x^2 - (2c + 1)x - 1, c \neq 0$. Then $P(0) = P(1) = P(3) = 1$ and $P(2) = -1$. Thus, $u(P) \geq 4$. By Lemma 2.5, $u(P) = 4 = d(P)$. Therefore, P is fit. □

Now, let us consider the case $d = 4$ and $u_+ = 2$.

Lemma 3.4. *Suppose P is a fit polynomial with $d = 4$ and $u_+ = 2$. Then, the following statements hold:*
(*i*) *$P(x)$ is irreducible.*
(*ii*) *P is equivalent to*

$$P_c(x) := cx^4 - 2cx^3 - (c+1)x^2 + (2c+1)x + 1, \quad c \neq 0. \tag{5}$$

(*iii*) *Each P_c with $c \neq 0$ is fit.*

Proof. Suppose $A_+(P) = (a_1, a_2)$ and $A_-(P) = (a_3, a_4)$. Then,

$$P(x) = M(x)(x - a_1)(x - a_2) + 1 = N(x)(x - a_3)(x - a_4) - 1, \tag{6}$$

where $M(x)$ does not have integer roots other than a_1 and a_2, and $N(x)$ does not have integer roots other than a_3 and a_4.

(i) Assume P is reducible and $P = QR$. By Lemma 2.8 (iii), $Q, R \in J(x^2 + x - 1)$ and $u_+(Q) = u_+(R) = u_-(Q) = u_-(R) = 2$. Note that

$$Q(a_i)R(a_i) = P(a_i) = 1, \quad i = 1, 2; \tag{7}$$

$$Q(a_i)R(a_i) = P(a_i) = -1, \quad i = 3, 4. \tag{8}$$

We claim that $Q(a_1) \neq Q(a_2)$ and $Q(a_3) \neq Q(a_4)$. Otherwise, if $Q(a_1) = Q(a_2) = \pm 1$, then it follows from $u_+(Q) = u_-(Q) = 2$ that $Q(a_3) = Q(a_4) = \mp 1$. Now, equations (7) and (8) imply $R(a_1) = R(a_2) = R(a_3) = R(a_4) = \pm 1$, and thus, $u_+(R) = 4$ or $u_-(R) = 4$, which is a contradiction. Hence, $Q(a_1) \neq Q(a_2)$, and similarly $Q(a_3) \neq Q(a_4)$.

By the aforementioned claim, equation (7) implies either $Q(a_1) = R(a_1) = 1$ and $Q(a_2) = R(a_2) = -1$ or $Q(a_1) = R(a_1) = -1$ and $Q(a_2) = R(a_2) = 1$, and equation (8) implies either $Q(a_3) = -R(a_3) = 1$ and $Q(a_4) = -R(a_4) = -1$ or $Q(a_3) = -R(a_3) = -1$ and $Q(a_4) = -R(a_4) = 1$. We consider one of the four possibilities to get a contradiction and the other cases are similar. Suppose $Q(a_1) = R(a_1) = 1$, $Q(a_2) = R(a_2) = -1$, $Q(a_3) = -R(a_3) = 1$, and $Q(a_4) = -R(a_4) = -1$ (Table 2). Note that $Q \in J(R) = I(R) \cup I(-R)$. Suppose $Q \in I(R)$. By Lemma 2.3, $a_4 - a_2 = a_3 - a_2$ and thus, $a_3 = a_4$, which is impossible. Similarly, we can obtain a contradiction for the case $Q \in I(-R)$. Therefore, P is irreducible.

(ii) It follows from equation (6) that

$$M(a_i)(a_i - a_1)(a_i - a_2) = -2, \quad i = 3, 4,$$
$$N(a_i)(a_i - a_3)(a_i - a_4) = 2, \quad i = 1, 2.$$

Note that the conditions $a_1 < a_2$ and $a_3 < a_4$ imply that

$$a_3 - a_2 < a_3 - a_1 < a_4 - a_1, \quad a_3 - a_2 < a_4 - a_2 < a_4 - a_1.$$

As a result, $a_3 - a_1 \neq \pm 2$, $a_3 - a_2 \neq 1, 2$, $a_4 - a_1 \neq -2, -1$, and $a_4 - a_2 \neq \pm 2$. Also, we note that $(a_3 - a_1) - (a_3 - a_2) = (a_4 - a_1) - (a_4 - a_2)$. Considering all restrictions mentioned earlier, we have the following two cases to consider.

(2a) $a_3 - a_1 = -1, a_3 - a_2 = -2, a_4 - a_1 = 2, a_4 - a_2 = 1$,
(2b) $a_3 - a_1 = 1, a_3 - a_2 = -2, a_4 - a_1 = 2, a_4 - a_2 = -1$.

For case (2a), by Lemma 2.3, we may assume $a_1 = 0$, which implies that $a_3 = -1$, $a_4 = 2$, $a_2 = 1$, and $M(a_3) = M(a_4) = N(a_1) = N(a_2) = -1$. Then, we have

$$M(x) = c(x + 1)(x - 2) - 1 = cx^2 - cx - 2c - 1, \quad c \neq 0.$$

Now,

$$\begin{aligned} P(x) &= (cx^2 - cx - 2c - 1)x(x - 1) + 1 \\ &= cx^4 - 2cx^3 - (c + 1)x^2 + (2c + 1)x + 1 \\ &= P_c(x). \end{aligned}$$

Hence, $P \sim P_c, \quad c \neq 0$.

Similarly, we obtain for case (2b) that, for some integer $c \neq 0$,

$$\begin{aligned} P(x) &= cx^4 - 6cx^3 + (11c + 1)x^2 - (6c + 3)x + 1 \\ &\sim -P(x + 1) \\ &= -cx^4 + 2cx^3 + (c - 1)x^2 + (1 - 2c)x + 1 \\ &\sim cx^4 - 2cx^3 - (c + 1)x^2 + (2c + 1)x + 1 \\ &= P_c(x), \end{aligned}$$

where the last ~ is obtained by replacing c by $-c$.

(iii) Note that $P_c(0) = P_c(1) = 1$ and $P_c(-1) = P_c(2) = -1$. It follows from Lemma 2.5 that $u = 4$. Hence, P_c is fit. □

4 Fit polynomials with $d = 3$

In this section, we study fit polynomials of degree $d = 3$.

Table 2: Values of Q and R at a_i

i	1	2	3	4
$Q(a_i)$	1	−1	1	−1
$R(a_i)$	1	−1	−1	1

Lemma 4.1. *Suppose P is a fit polynomial with $d = 3$. Then, the following statements hold:*
(*i*) *P is irreducible.*
(*ii*) *If $u_+ = 0$ or 3, then*

$$P(x) \sim Q(x) := cx(x - c_1)(x - c_2) + 1, \quad 0 < c, 0 < c_1 < c_2, (c, c_1, c_2) \neq (1, 1, 3).$$

Conversely, each Q in the aforementioned family is fit.
(*iii*) *If $u_+ = 1$ or 2, then P is equivalent to one of the following polynomials:*

$$cx^3 - x^2 - (1 + c)x + 1, \quad c \neq 0, 1;$$
$$cx^3 + 2x^2 - cx - 1, \quad c \geq 2;$$
$$cx^3 + (1 + c)x^2 + (1 - 2c)x - 1, \quad c \neq 0, 1.$$

Conversely, each polynomial that belongs to the aforementioned three families is irreducible and fit with $u_+ = 1$ or 2.

Proof. (i) Suppose P is reducible and $P = MN$ with $f(M) \geq f(N)$. Then, by Lemmas 2.8 (ii) and 2.6 (i), $f(M) = d(M) = 2$. Since $u = 3$, there exist distinct $a_i \in \mathbb{Z}, 1 \leq i \leq 3$, such that

$$M(a_i)N(a_i) = P(a_i) = \pm 1, \quad \text{for } i = 1, 2, 3.$$

Thus, $N(x)$ takes value ± 1 at three different arguments $x = a_1, a_2, a_3$. This is impossible since $d(N) = d(P) - d(M) = 3 - 2 = 1$. Therefore, P is irreducible.

(ii) Similar to the proof of Lemma 3.1, we obtain that $P \sim Q$. Note that Q (with $c > 0$ and $0 < c_1 < c_2$) is fit if and only if $u_-(Q) = 0$, i.e., if and only if $cx(x - c_1)(x - c_2) = -2$ has no integer roots, i.e., if and only if $(c, c_1, c_2) \neq (1, 1, 3)$.

(iii) Suppose $u_+ = 2$ (if $u_+ = 1$ then consider $-P$). Assume $A_+(P) = (a_1, a_2)$ and $A_-(P) = (a_3)$. Then,

$$P(x) = M(x)(x - a_1)(x - a_2) + 1 = N(x)(x - a_3) - 1,$$

such that $M(x)$ does not have integer roots other than a_1, a_2, and $N(x)$ does not have integer roots other than a_3. Hence, we have

$$M(a_3)(a_3 - a_1)(a_3 - a_2) = -2.$$

Thus, we have the following five cases to consider:
(3a) $a_3 - a_1 = 2, a_3 - a_2 = 1, M(a_3) = -1$;
(3b) $a_3 - a_1 = -1, a_3 - a_2 = -2, M(a_3) = -1$;
(3c) $a_3 - a_1 = 1, a_3 - a_2 = -1, M(a_3) = 2$;
(3d) $a_3 - a_1 = 2, a_3 - a_2 = -1, M(a_3) = 1$;
(3e) $a_3 - a_1 = 1, a_3 - a_2 = -2, M(a_3) = 1$.

For case (3a), by Lemma 2.3, we may assume $a_1 = 0$, which implies $a_3 = 2, a_2 = 1$. Since $M(a_3) = M(2) = -1$, we may suppose $M(x) = c(x - 2) - 1$ for some $c \neq 0$. By our assumption, $M(x)$ does not have integer roots other than 0 and 1 and $N(x)$ does not have integer roots other than 2. Thus, $c \neq 1$. Now,

$$\begin{aligned} P(x) &= (c(x - 2) - 1)x(x - 1) + 1 \\ &= cx^3 - (3c + 1)x^2 + (2c + 1)x + 1 \\ &= c(x - 1)^3 - (x - 1)^2 - (1 + c)(x - 1) + 1 \\ &\sim cx^3 - x^2 - (1 + c)x + 1, \quad c \neq 0, 1. \end{aligned}$$

Similar to case (3a), we obtain that

$$\begin{aligned} \text{(3b)}: \quad P(x) &= cx^3 - x^2 + (1 - c)x + 1 \\ &\sim -cx^3 - x^2 - (1 - c)x + 1, \quad c \neq 0, -1, \\ &= c'x^3 - x^2 - (1 + c')x + 1, \quad c' \neq 0, 1, \end{aligned}$$

where the second = is obtained by replacing $-c$ by c'.

$$\begin{aligned}(3c):\quad P(x) &= cx^3 + (2-3c)x^2 + (2c-4)x + 1\\ &= c(x-1)^3 + 2(x-1)^2 - c(x-1) - 1\\ &\sim cx^3 + 2x^2 - cx - 1, \quad c \neq 0, \pm 1.\end{aligned}$$

For $c < 0$,

$$\begin{aligned}P(x) &\sim P(-x) = (-c)x^3 + 2x^2 - (-c)x - 1\\ &= c'x^3 + 2x^2 - c'x - 1, \quad c' \coloneqq -c \geq 2.\\ (3d):\quad P(x) &= cx^3 + (1-5c)x^2 + (6c-3)x + 1\\ &= c(x-2)^3 + (1+c)(x-2)^2 + (1-2c)(x-2) - 1\\ &\sim cx^3 + (1+c)x^2 + (1-2c)x - 1, \quad c \neq 0, 1.\\ (3e):\quad P(x) &= cx^3 + (1-4c)x^2 + (3c-3)x + 1\\ &\sim -cx^3 + (1-4c)x^2 - (3c-3)x + 1, \quad c \neq 0, -1,\\ &\sim cx^3 + (1+4c)x^2 + (3c+3)x + 1, \quad c \neq 0, 1,\\ &\sim c(x-1)^3 + (1+4c)(x-1)^2 + (3c+3)(x-1) + 1\\ &= cx^3 + (1+c)x^2 + (1-2c)x - 1, \quad c \neq 0, 1.\end{aligned}$$

We list all the five cases in Table 3, where

$$\begin{aligned}Q_1(x) &= -(c+1)x - 1,\\ Q_2(x) &= -(1+c)x + 2,\\ Q_3(x) &= (2-2c)x - 2,\\ Q_4(x) &= (1-3c)x - 1,\\ Q_5(x) &= (1-3c)x - 2,\\ P_1(x) &= -x^2 - (1+c)x + 1,\\ P_2(x) &= 2x^2 - cx - 1,\\ P_3(x) &= (1+c)x^2 + (1-2c)x - 1.\end{aligned}$$

The converse direction is easy to check. □

5 Fit polynomials with $d < 3$

In this section, we study fit polynomials of degree $d < 3$. First, we consider the case $d = 2$.

Lemma 5.1. *Suppose P is a fit polynomial with $d = 2$.*

(*i*) *Suppose $u_+ = 0$ or 2. Then, P is equivalent to one of the following polynomials*:

$$\begin{aligned}&cx(x-c_1) + 1, \quad c > 0, c_1 > 0, (c, c_1) \neq (1,3) \text{ or } (2,2);\\ &cx(x-c_1) - 1, \quad c > 0, c_1 > 0, (c, c_1) \neq (1,1).\end{aligned}$$

Conversely, each polynomial in the aforementioned list is fit.

(*ii*) *Suppose $u_+ = 0$ or 2. Then, P is reducible if and only if $P = \pm Q^2$, $Q \in J(2x-1) \cup J(x)$.*

(*iii*) *If $u_+ = 1$, then P is irreducible and equivalent to one of the following polynomials*:

$$\begin{aligned}&cx^2 - x - c, \quad c \geq 2,\\ &cx^2 - (c+2)x + 1, \quad c \geq 3.\end{aligned}$$

Conversely, each polynomial in the aforementioned two families is irreducible and fit.

Table 3: Fit polynomials with $d = 3$ and $u_+ = 2$

	(3a)	(3b)	(3c)	(3d)	(3e)
$a_3 - a_1$	2	−1	1	2	1
$a_3 - a_2$	1	−2	−1	−1	−2
$M(a_3)$	−1	−1	2	1	1
a_1, a_2, a_3	0, 1, 2	0,1 −1	0, 2, 1	0, 3, 2	0, 3, 1
$M(x)$	$cx - 2c - 1$	$cx + c - 1$	$cx - c + 2$	$cx - 2c + 1$	$cx - c + 1$
$N(x)$	$cx^2 + Q_1$	$cx^2 + Q_2$	$cx^2 + Q_3$	$cx^2 + Q_4$	$cx^2 + Q_5$
$P(x)$	$cx^3 + P_1$		$cx^3 + P_2$	$cx^3 + P_3$	
c	$c \neq 0,1$		$c \geq 2$	$c \neq 0,1$	

Proof. (i)–(ii) It is similar to Lemmas 3.2 and 4.1.

(iii) Suppose P is reducible and $P = QR$. Assume that

$$P(x) = M(x)(x - a_1) + 1 = N(x)(x - a_2) - 1, \tag{9}$$

where $a_1 < a_2$, $M(x)$ does not have integer roots other than a_1, and $N(x)$ does not have integer roots other than a_2. Then, $Q(a_1)R(a_1) = P(a_1) = 1$ and $Q(a_2)R(a_2) = P(a_2) = -1$. Similar to the proof of Lemma 4.1, we have the following four cases:

(3a) $a_2 - a_1 = 2$, $M(a_2) = -1$, $N(a_1) = -1$;
(3b) $a_2 - a_1 = -2$, $M(a_2) = 1$, $N(a_1) = 1$;
(3c) $a_2 - a_1 = 1$, $M(a_2) = -2$, $N(a_1) = -2$;
(3d) $a_2 - a_1 = -1$, $M(a_2) = 2$, $N(a_1) = 2$.

In a similar way to the proof of Lemma 4.1, we can deal with the aforementioned cases and the results are listed in Table 4, where the last row (the restriction on c) follows from the assumption on the roots of M and N in equation (9). The converse direction is easy to check. □

Similar to Lemma 4.1, we have the following lemma for fit polynomials of degree 1.

Lemma 5.2. *Suppose P is a fit polynomial with $d = 1$. then $P(x) \sim cx + 1$ for some integer $c \geq 3$. Conversely, $cx + 1$ with $c \geq 3$ is fit.*

6 Proofs of theorems

Now we are ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Suppose P is a fit polynomial of degree d. We gather the nine items (i)–(ix) into five groups:

(G1): (i)–(ii). It follows from Lemmas 3.1, 3.2, 4.1, and 5.1 that P is equivalent to a polynomial in (i)–(ii) if $u_+ = d$ or $u_- = d$.
(G2): (iii). If $u_+ = 3$ and $u_- = 1$, or $u_+ = 1$ and $u_- = 3$, then, by Lemma 3.3, P is equivalent to a polynomial in (iii).
(G3): (iv). If $u_+ = u_- = 2$, by Lemma 3.4, P is equivalent to a polynomial in (iv).
(G4): (v)–(vii). If $u_+ = 2$ and $u_- = 1$, or $u_+ = 1$ and $u_- = 2$, by Lemma 4.1, P is equivalent to a polynomial in (v)–(vii).
(G5): (viii)–(ix). If $u_+ = u_- = 1$, by Lemma 5.1, P is equivalent to a polynomial in (viii)–(ix).

Conversely, it follows from Lemmas 3.1, 3.2, 3.3, 3.4, 4.1, 5.1, and 2.4 (i) that each polynomial that is equivalent to one listed in (i)–(ix) is fit.

Now suppose P and Q are both in (i)–(ix) and $P \sim Q$. Then, it follows from Lemma 2.4 that $d(P) = d(Q)$, either $u_+(P) = u_+(Q)$ or $u_+(P) = u_-(Q)$, and either P and Q or P and $-Q$ have the same leading coefficient. Thus, $P \nsim Q$ if they are in different groups. It remains to consider the case where P and Q are in the same group.

Table 4: Fit polynomials with $d = 2$ and $u_+ = 1$

	(3a)	(3b)	(3c)	(3d)
$a_2 - a_1$	2	−2	1	−1
$M(a_2)$	−1	1	−2	2
$N(a_1)$	−1	1	−2	2
a_1, a_2	0, 2	0, −2	0, 1	0, −1
$M(x)$	$cx - 2c - 1$	$cx + 2c + 1$	$cx - c - 2$	$cx + c + 2$
$N(x)$	$cx - 1$	$cx + 1$	$cx - 2$	$cx + 2$
$P(x)$	$cx^2 - x - c$		$cx^2 - (c + 2)x + 1$	
c	$c \geq 2$		$c \geq 3$	

Table 5: Values of $P(x)$ at arguments around 0

x	0	1	−1	2	−2	3	−3	4	−4	5	−5
$P(x)$	−1	−1	1	1	77	77	371	371	1099	1099	2549

Suppose P and Q are both in (G1). Note that $P \sim Q$ if and only if either $P \in I(Q)$ or $P \in I(-Q)$. Since all polynomials in (G1) have positive leading coefficients, $P \notin I(-Q)$. Thus, $P \in I(Q)$. By Lemma 2.3 (iv), we have that $P = Q$.

Suppose P and Q are both in (G2). Assume $P(x) = P_c(x) \coloneqq cx^4 + (1 + 2c)x^3 + (2 - c)x^2 - (2c + 1)x - 1$ for some $c \neq 0$. If $P \neq Q$, then $Q(x) = P_{-c}(x)$. Now we have that $P_c {\sim} P_{-c}$. Since $d(P) = 4$ is even, we have that $P_c = -P_{-c}(\pm x + b)$, for some $b \in \mathbb{Z}$. It is easy to check that $P_c \neq -P_{-c}(\pm x + b)$ for all $c \in \mathbb{Z}\backslash\{0\}$ and $b \in \mathbb{Z}$. This contradiction shows that $P = Q$.

In a similar way, one can prove that if P and Q are both in (G3) (resp. (G4), (G5)), then $P = Q$. □

Proof of Theorem 1.2. It follows from Lemmas 3.2, 3.3, 3.4, 4.1, and 5.1 that a fit polynomial P is irreducible if and only if $P = \pm Q^2$ for some $Q \in \mathbb{Z}[x]$. Furthermore, if it is the case, then $Q \sim x^2 + x - 1, 2x - 1$, or x. That means (i) ⇔ (ii) ⇔ (iii). It is obvious that (iii) ⇔ (iv). □

7 Example

Example 7.1. Let us investigate the irreducibility of

$$P(x) = 3x^4 - 6x^3 - 2x^2 + 5x - 1.$$

First, by solving $P(x) = 1$ and $P(x) = -1$, we obtain that $u_+ = u_- = 2$, and thus, P is fit. Since the leading coefficient of P is 3, we have that $P \neq \pm Q^2$ for all $Q \in \mathbb{Z}[x]$, and thus P is irreducible by Theorem 1.2. It is worth mentioning that $P(x) = -P_{-3}(x)$, where $P_c(x)$ is defined in equation (5).

Note that Eisenstein's criterion and Perron's criterion cannot be applied directly to P here. Brown-Graham's criterion [7] works for P, while the computation is not obvious. Table 5 gives the values of $P(x)$ with $|x| \leq 5$, where the only prime is $P(-5) = 2{,}549$. Hence, we obtain $p \geq 1$, $u \geq 4$, and thus, $p + 2u - d > 4$. Therefore, P is irreducible by Lemma 2.7. We also note that Murty's criterion [6] is also applicable to P in the example.

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References

[1] S. Kumar and J. Singh, A study of some recent irreducibility criteria for polynomials having integer coefficients, 2023, arXiv: https://doi.org/10.48550/arXiv.2310.02860.

[2] J. Singh and S. Kumar, *A new class of irreducible polynomials*, Comm. Algebra **49** (2021), no. 6, 2722–2727, DOI: https://doi.org/10.1080/00927872.2021.1881789.

[3] W. Zhang and P. Yuan, *On two conjectures of irreducible polynomials*, Comm. Algebra **51** (2023), no. 11, 4879–4884, DOI: https://doi.org/10.1080/00927872.2023.2222406.

[4] N. C. Bonciocat, R. Garg, and J. Singh, *Some factorization results for bivariate polynomials*, Comm. Algebra **53** (2024), no. 1, 328–341, DOI: https://doi.org/10.1080/00927872.2024.2377390.

[5] J. Singh and R. Garg, *Some factorization results on polynomials having integer coefficients*, Comm. Algebra **52** (2024), no. 6, 2467–2474, DOI: https://doi.org/10.1080/00927872.2023.2301530.

[6] M. R. Murty, *Prime numbers and irreducible polynomials*, Amer. Math. Monthly **109** (2002), no. 5, 452–458, DOI: https://doi.org/10.1080/00029890.2002.11919872.

[7] W. S. Brown and R. L. Graham, *An irreducibility criterion for polynomials over the integers*, Amer. Math. Monthly **76** (1969), no. 7, 795–797, DOI: https://doi.org/10.1080/00029890.1969.12000326.

[8] R. Hazrat, *Mathematica: A problem-centered approach*, *Springer Undergraduate Mathematics Series*, vol. 53, Springer, New York, 2010.

[9] H. L. Dorwart and O. Ore, *Criteria for the irreducibility of polynomials*, Ann. of Math. **34** (1933), no. 1, 81–94, DOI: https://doi.org/10.2307/1968341.