DE GRUYTER Open Mathematics 2025; 23: 20250161

Research Article

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Existence and multiplicity of positive solutions for multiparameter periodic systems

https://doi.org/10.1515/math-2025-0161
received October 12, 2024; accepted May 5, 2025

Abstract: We deal with the existence and multiplicity of positive solutions for differential systems depending on two parameters, λ_1, λ_2, subjected to periodic boundary conditions. We establish the existence of a continuous curve Γ that separates the first quadrant into two disjoint unbounded open sets O_1 and O_2. Specifically, we prove that the periodic system has no positive solutions if $(\lambda_1, \lambda_2) \in O_1$, at least one positive solution if $(\lambda_1, \lambda_2) \in \Gamma$, and at least two positive solutions if $(\lambda_1, \lambda_2) \in O_2$. Our approach relies on the fixed point index theory and the method of lower and upper solutions.

Keywords: positive solution, non-existence/existence, periodic systems, lower and upper solutions

MSC 2020: 34B15, 34B18

1 Introduction

In this work, we study the existence and multiplicity of positive solutions for differential systems of form

$$\begin{cases} -u'' + q(x)u = \lambda_1\mu_1(x)g_1(u, v), & x \in (0, T), \\ -v'' + q(x)v = \lambda_2\mu_2(x)g_2(u, v), & x \in (0, T), \\ u(0) = u(T), \quad u'(0) = u'(T), \\ v(0) = v(T), \quad v'(0) = v'(T), \end{cases} \tag{1.1}$$

where $q \in C([0, T], [0, \infty))$ with $q \not\equiv 0$, $\lambda_1, \lambda_2 > 0$ are real parameters, $\mu_1, \mu_2 \in C([0, T], (0, \infty))$ and $g_1, g_2 : [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous.

The periodic problem for a single equation has been studied in many papers over the last several years [1–6]. Using different approaches, [7–10] generalized these results to differential systems, which describe new and special phenomena. In [9], the existence, multiplicity, and nonexistence of positive solutions of systems

$$\begin{cases} u'' + m^2u = \lambda H(x)G(u), & x \in (0, 1), \\ u(0) = u(1), \quad u'(0) = u'(1) \end{cases}$$

have been established, where $u = [u_1, u_2, \dots, u_n]^T$, m is some positive constant, $\lambda > 0$ is a positive parameter, and $H(x) = \operatorname{diag}[h_1(x), h_2(x), \dots, h_n(x)]$, $G(u) = [g_1(u), g_2(u), \dots, g_n(u)]^T$. Chu et al. [11] studied the n-dimensional nonlinear system

$$\begin{cases} u'' + A(x)u = \lambda H(x)G(u), & x \in (0, 1), \\ u(0) = u(1), \quad u'(0) = u'(1), \end{cases} \tag{1.2}$$

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where $A(x) = \operatorname{diag}[a_1(x), a_2(x), \ldots, a_n(x)]$. They provide sufficient conditions ensuring that the integral operator corresponding to (1.2) has a positive fixed point, and they prove that for each λ within a specified eigenvalue interval, (1.2) has at least one positive solution.

In view of the above, it appears as being natural to extend the previous study to more general, multi-parameter, which does not have a variational structure. So, the main goal of this work is to extend a result of non-existence, existence, and multiplicity from [12] for a single equation to the more general two-parameter systems (1.1). Precisely, according to [12], there exist $\lambda^* > \lambda_* > 0$ such that problem

$$\begin{cases} -u'' + q(x)u = \lambda f(x, u), & x \in (0, T), \\ u(0) = u(T), \quad u'(0) = u'(T) \end{cases}$$

has zero, at least one, or at least two positive solutions according to $0 < \lambda < \lambda_*$, $\lambda_* \le \lambda \le \lambda^*$, or $\lambda > \lambda^*$.

Based upon the lower and upper solutions method and fixed point index, we obtain that there exist $\tilde{\lambda}_1, \tilde{\lambda}_2 > 0$, such that for all $\lambda_1 > \tilde{\lambda}_1$ and $\lambda_2 > \tilde{\lambda}_2$, (1.1) has a positive solution (u, v), where both u and v are positive in $[0, T]$. Moreover, we show the existence of a continuous curve Γ that divides the first quadrant into two separate, unbounded, and open regions O_1 and O_2. Specifically, there are zero positive solutions when (λ_1, λ_2) lies in O_1, at least one positive solution when (λ_1, λ_2) is on Γ, and at least two positive solutions when (λ_1, λ_2) is in O_2. Notably, the curve Γ approaches asymptotically to two lines that are parallel to the coordinate axes $0\lambda_1$ and $0\lambda_2$, while O_1 is located below Γ and adjacent to axes $0\lambda_1$ and $0\lambda_2$.

The structure of this work is as follows. Section 2 introduces some preliminary results related to the reformulation of system (1.1) and a theorem of cone expansion/compression type, which plays a crucial role in our proof. The focus of Section 3 lies in the lower and upper solution method. We finally state and prove our main result for a two-parameter periodic system in Section 4.

2 Preliminaries

Throughout this work, let $C \coloneqq C[0, T]$ be endowed with the sup-norm $||u||_\infty = \max_{x\in[0,T]}|u(x)|$. $C^1 = C^1[0, T]$ with the norm $||u||_1 = \max_{x\in[0,T]}|u(x)| + \max_{x\in[0,T]}|u'(x)|$. While the product space $C^1 \times C^1$ will be understood with the norm $||(u, v)|| = \max\{||u||_\infty, ||v||_\infty\} + \max\{||u'||_\infty, ||v'||_\infty\}$.

We denote by $G(x, s)$ Green's function corresponding to

$$\begin{cases} -u'' + q(x)u = h(x), & x \in (0, T), \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases}$$

According to Theorem 2.5 of [13], for all $x, s \in [0, T]$, Green's function $G(x, s)$ is positive, and the solution to the problem is given by

$$u(x) = \int_0^T G(x, s)h(s)\mathrm{d}s.$$

Denote

$$m = \min_{0\le x,s\le T} G(x, s), \quad M = \max_{0\le x,s\le T} G(x, s), \quad \sigma = \frac{m}{M}.$$

Obviously, $0 < m < M$ and $0 < \sigma < 1$.

We consider the closed subspace

$$C_M^1 = \{(u, v) \in C^1 \times C^1 : u^{(i)}(0) = u^{(i)}(T), v^{(i)}(0) = v^{(i)}(T), i = 0, 1\}$$

and its closed, convex cone

$$K = \left\{(u,v) \in C_M^1 : u, v \geq 0,\ \min_{0\leq x\leq T} (u(x) + v(x)) \geq \sigma(\|u\|_\infty + \|v\|_\infty)\right\}.$$

Also, we denote $B(\rho) \coloneqq \{(u,v) \in K : \|(u,v)\| < \rho\}$.

We reduce problem (1.1) to an equivalent fixed point problem of the form

$$F_\lambda : K \to K, \quad F_\lambda(u,v) = (F_{1,\lambda}(u,v), F_{2,\lambda}(u,v)),$$

where $F_{i,\lambda}(u,v) = \lambda_i \int_0^T G(x,s)\mu_i(s) g_i(u(s), v(s))\mathrm{d}s$. It is obvious that $F_{i,\lambda}$ is completely continuous.

If A is a subset of K, we set

$$\mathcal{K}(A) \coloneqq \{\mathcal{T} | \mathcal{T} : A \to K \text{ is a compact operator}\}.$$

Also, given a bounded open (in K) subset O of K, we denote by $i(\mathcal{T}, O, K)$ the fixed point index of the operator $\mathcal{T} \in \mathcal{K}(\overline{O})$ on O with respect to K [14]. The following well-known lemma is very crucial in our arguments, refer [15,16] for a proof and further discussion of the fixed point index.

Lemma 2.1. *Let E be a Banach space and P a cone in E. For $r > 0$, define $P_r = \{x \in P : \|x\| < r\}$. Assume that $\mathcal{T} : \overline{P_r} \to P_r$ is completely continuous such that $\mathcal{T}x \neq x$ for $x \in \partial P_r = \{x \in P : \|x\| = r\}$.*

(*i*) *If $\|\mathcal{T}x\| \geq \|x\|$ for $x \in \partial P_r$, then $i(\mathcal{T}, P_r, P) = 0$.*

(*ii*) *If $\|\mathcal{T}x\| \leq \|x\|$ for $x \in \partial P_r$, then $i(\mathcal{T}, P_r, P) = 1$.*

3 Lower and upper solutions

Let us consider

$$\begin{cases} -u'' + q(x)u = f_1(x,u,v), & x \in (0,T), \\ -v'' + q(x)v = f_2(x,u,v), & x \in (0,T), \\ u(0) = u(T), \quad u'(0) = u'(T), \\ v(0) = v(T), \quad v'(0) = v'(T), \end{cases} \tag{3.1}$$

where $f_1, f_2 : [0,T] \times [0,\infty)^2 \to [0,\infty)$ are L^1-Carathéodory functions.

In the terminology of [17,18], if a function $f = f(x,s,t) : [0,T] \times [0,\infty)^2 \to [0,\infty)$ satisfies that for fixed x, s (resp.x, t),

$$f(x,s,t_1) \leq f(x,s,t_2) \text{ as } t_1 \leq t_2 \quad (\text{resp.} f(x,s_1,t) \leq f(x,s_2,t) \text{ as } s_1 \leq s_2).$$

Then, it is said to be quasi-monotone nondecreasing with respect to t (resp. s).

A couple of nonnegative functions $(\alpha_u, \alpha_v) \in C^2 \times C^2$ is a lower solution of (3.1) if

$$\begin{cases} -\alpha_u'' + q(x)\alpha_u \leq f_1(x,\alpha_u,\alpha_v), & x \in (0,T), \\ -\alpha_v'' + q(x)\alpha_v \leq f_2(x,\alpha_u,\alpha_v), & x \in (0,T), \\ \alpha_u(0) = \alpha_u(T), \quad \alpha_u'(0) \geq \alpha_u'(T), \\ \alpha_v(0) = \alpha_v(T), \quad \alpha_v'(0) \geq \alpha_v'(T). \end{cases} \tag{3.2}$$

An upper solution $(\beta_u, \beta_v) \in C^2 \times C^2$ is defined by reversing the first two inequalities in (3.2) and asking $\beta_u'(0) \leq \beta_u'(T)$, $\beta_v'(0) \leq \beta_v'(T)$ instead of $\alpha_u'(0) \geq \alpha_u'(T)$, $\alpha_v'(0) \geq \alpha_v'(T)$.

Lemma 3.1. *Suppose that* (3.1) *has an upper solution (β_u, β_v) and a lower solution (α_u, α_v). Let $f_1(x,u,v)$ (resp. $f_2(x,u,v)$) be quasi-monotone nondecreasing with respect to v (resp. u) and define*

$$\mathcal{A}_{\alpha,\beta} = \{(u,v) \in K : \alpha_u \leq u \leq \beta_u, \alpha_v \leq v \leq \beta_v\}.$$

Then,

(i) *there exists at least one solution of problem* (3.1) *in* $\mathcal{A}_{\alpha,\beta}$;

(ii) *if* $(u_0, v_0) \in \mathcal{A}_{\alpha,\beta}$ *is the unique solution of* (3.1) *and there exists* $\rho_0 > 0$ *such that* $B((u_0, v_0), \rho_0) = \{(u, v) \in K : ||(u - u_0, v - v_0)|| \le \rho_0\} \subset \mathcal{A}_{\alpha,\beta}$, *then*

$$i(F, B((u_0, v_0), \rho), K) = 1, \quad \text{for all } 0 \le \rho \le \rho_0,$$

where $F(u, v) = (F_1(u, v), F_2(u, v))$ *and* $F_i : K \to K$ *defined by*

$$F_i(u, v) = \int_0^T G(x, r) f_i(r, u, v) \mathrm{d}r.$$

Proof. (i) We define the continuous functions $\Gamma_1, \Gamma_2 : [0, T] \times [0, \infty)^2 \to [0, \infty)$,

$$\Gamma_1(x, s, t) = f_1(x, \gamma_1(x, s), \gamma_2(x, t)) - s + \gamma_1(x, s),$$
$$\Gamma_2(x, s, t) = f_2(x, \gamma_1(x, s), \gamma_2(x, t)) - t + \gamma_2(x, t),$$

with γ_i given by

$$\gamma_1(x, s) = \max\{\alpha_u(x), s\}, \quad \gamma_2(x, t) = \max\{\alpha_v(x), t\}.$$

And we consider the modified problem

$$\begin{cases} -u'' + q(x)u = \Gamma_1(x, u, v), & x \in (0, T), \\ -v'' + q(x)v = \Gamma_2(x, u, v), & x \in (0, T), \\ u(0) = u(T), & u'(0) = u'(T), \\ v(0) = v(T), & v'(0) = v'(T). \end{cases} \tag{3.3}$$

Next we write (3.3) as a system of integral equations

$$u(x) = \int_0^T G(x, r)\Gamma_1(r, u, v)\mathrm{d}r,$$
$$v(x) = \int_0^T G(x, r)\Gamma_2(r, u, v)\mathrm{d}r.$$

The operator $\overline{F_i} : K \to K$ defined by

$$\overline{F_i}(u, v) = \int_0^T G(x, r)\Gamma_i(r, u, v)\mathrm{d}r$$

is completely continuous and bounded. By Schauder's theorem, $\overline{F}(u, v) = (\overline{F_1}(u, v), \overline{F_2}(u, v))$ has a fixed point, which is a solution of (3.3). We prove that any solution (u, v) of (3.3) satisfies $(u, v) \in \mathcal{A}_{\alpha,\beta}$. Here we only establish the inequality $\alpha_u \le u$ on $[0, T]$ (a similar argument can be made for $\alpha_v \le v$).

Suppose by contradiction that there exists $x_0 \in [0, T]$ such that

$$\max_{0 \le x \le T} (\alpha_u - u) = \alpha_u(x_0) - u(x_0) > 0.$$

If $x_0 \in (0, T)$, then there exists a sequence $\{x_k\} \subset (0, x_0)$ converging to x_0 such that $\alpha_u'(x_0) = u'(x_0)$ and $\alpha_u'(x_k) - u'(x_k) \ge 0$. This implies

$$\alpha_u'(x_k) - \alpha_u'(x_0) \ge u'(x_k) - u'(x_0),$$

which yields

$$\alpha_u''(x_0) \le u''(x_0).$$

Since (α_u, α_v) is a lower solution of (3.1) and f_1 is quasi-monotone nondecreasing with respect to v, we have

$$\begin{aligned}\alpha_u''(x_0) \le u''(x_0) &= q(x_0)u(x_0) - f_1(x_0, \alpha_u(x_0), \gamma_2(x_0, v(x_0))) + u(x_0) - \alpha_u(x_0)\\ &< q(x_0)u(x_0) - f_1(x_0, \alpha_u(x_0), \alpha_v(x_0))\\ &\le q(x_0)\alpha_u(x_0) - f_1(x_0, \alpha_u(x_0), \alpha_v(x_0))\\ &\le \alpha_u''(x_0),\end{aligned}$$

which is a contradiction. If $\max_{0\le x\le T}(\alpha_u - u) = \alpha_u(0) - u(0) = \alpha_u(T) - u(T)$, then $\alpha_u'(0) - u'(0) \le 0, \alpha_u'(T) - u'(T) \ge 0$. Using that $\alpha_u'(0) \ge \alpha_u'(T)$, we deduce that $\alpha_u'(0) - u'(0) = 0 = \alpha_u'(T) - u'(T)$. Applying similar reasoning as for $x_0 = 0$, we have

$$\alpha_u''(0) \le u''(0).$$

Then, using the fact of $\alpha_u'(0) = u'(0)$ and following a similar approach as in the case of $x_0 \in (0, T)$, we can once again get a contradiction. Therefore, $\alpha_u(x) \le u(x)$ for all $x \in [0, T]$ and similarly, we can show that $\beta_u(x) \ge u(x)$ for all $x \in [0, T]$.

(ii) Observe that the operator $\overline{F} = F$ on $\mathcal{A}_{\alpha,\beta}$ and, by the result of (i), any fixed point (u, v) of $\overline{F}$ satisfies $(u, v) \in \mathcal{A}_{\alpha,\beta}$. In particular, it is also a fixed point of F. Therefore, (u_0, v_0) is the unique fixed point of $\overline{F}$. Since

$$(0,0) \notin (I - \overline{F})(\overline{B(d)}\backslash B((u_0, v_0), \rho_0))$$

for sufficiently large d, and combining this fact with the excision property and [19], we obtain

$$1 = i(\overline{F}, B(d), K) = i(\overline{F}, B((u_0, v_0), \rho), K), \quad \text{for all } 0 < \rho \le \rho_0.$$

Since $\overline{F} = F$ on $\mathcal{A}_{\alpha,\beta}$ and $\overline{B}((u_0, v_0), \rho_0) \subset \mathcal{A}_{\alpha,\beta}$, the conclusion is immediate. □

4 Non-existence, existence, and multiplicity

Now, we suppose that g_1, g_2 satisfy

(H1) $g_1(u, v)$(resp. $g_2(u, v)$) is quasi-monotone nondecreasing with respect to v (resp. u).

(H2) $g_{i,0} := \lim_{(u,v)\to 0}\frac{g_i(u,v)}{u+v} = 0$, $g_{i,\infty} := \lim_{(u,v)\to\infty}\frac{g_i(u,v)}{u+v} = 0$. Setting

$$\Sigma := \{(\lambda_1, \lambda_2) | \lambda_1, \lambda_2 > 0 \text{ and (1.1) has at least one positive solution}\}.$$

Lemma 4.1. *Assume that* (*H*1) *and* (*H*2) *hold. Then, the following are true*:

(*i*) *there exist* $\Lambda_1, \Lambda_2 > 0$ *such that* $\Sigma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$ *and for all* $(\lambda_1, \lambda_2) \in (0, +\infty)^2\backslash([\Lambda_1, +\infty) \times [\Lambda_2, +\infty))$, *problem* (1.1) *has no positive solution*;

(*ii*) *if* $(\underline{\lambda_1}, \underline{\lambda_2}) \in \Sigma$, *then* $[\underline{\lambda_1}, +\infty) \times [\underline{\lambda_2}, +\infty) \subset \Sigma$;

(*iii*) *if* $(\underline{\lambda_1}, \underline{\lambda_2}) \in \Sigma$, *then for all* $(\lambda_1, \lambda_2) \in (\underline{\lambda_1}, +\infty) \times (\underline{\lambda_2}, +\infty)$, *there exist at least two positive solutions of problem* (1.1).

Proof. (i) For $(u, v) \in K$ and $||(u, v)|| = p$, let

$$m(p) = \min\left\{\int_0^T G(x, s)\mu_1(s)g_1(u, v)\mathrm{d}s, \int_0^T G(x, s)\mu_2(s)g_2(u, v)\mathrm{d}s\right\}.$$

Choose a number $r_1 > 0$, let $\lambda_0 = \frac{r_1}{2m(r_1)}$ and set

$$\Omega_{r_1} = \{(u, v) : (u, v) \in K, ||(u, v)|| < r_1\}.$$

Then, for $\lambda_1, \lambda_2 \geq \lambda_0$ and $(u, v) \in K \cap \partial\Omega_{r_1}$, we have

$$\begin{aligned} F_{i,\lambda}(u, v) &= \lambda_i \int_0^T G(x, s)\mu_i(s)g_i(u, v)\mathrm{d}s \\ &\geq \lambda_0 \int_0^T G(x, s)\mu_i(s)g_i(u, v)\mathrm{d}s \\ &\geq \lambda_0 m(r_1), \end{aligned}$$

which implies

$$\|F_\lambda(u, v)\| \geq r_1 = \|(u, v)\|$$

for $(u, v) \in K \cap \partial\Omega_{r_1}$. Hence, Lemma 2.1 implies

$$i(F_\lambda, \Omega_{r_1}, K) = 0. \tag{4.1}$$

Since $g_{i,0} = 0$, we may choose $r_2 \in (0, r_1)$ so that $g_i(u, v) \leq \eta(u + v)$ for $0 < u, v < r_2$, where the constant $\eta > 0$ satisfies

$$2\lambda_i \eta M \int_0^T \mu_i(s)\mathrm{d}s \leq 1.$$

Set $\Omega_{r_2} = \{(u, v) : (u, v) \in K, \|(u, v)\| < r_2\}$. If $(u, v) \in K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned} F_{i,\lambda}(u, v) &= \lambda_i \int_0^T G(x, s)\mu_i(s)g_i(u, v)\mathrm{d}s \\ &\leq \lambda_i \eta \int_0^T G(x, s)\mu_i(s)(u + v)\mathrm{d}s \\ &\leq \lambda_i \eta \int_0^T G(x, s)\mu_i(s)\mathrm{d}s \|(u, v)\| \\ &\leq \frac{\|(u, v)\|}{2}. \end{aligned}$$

Hence, $\|F_\lambda(u, v)\| = \|F_{1,\lambda}(u, v)\| + \|F_{2,\lambda}(u, v)\| \leq \|(u, v)\|$ for $(u, v) \in K \cap \partial\Omega_{r_2}$. Using Lemma 2.1 once again, we have

$$i(F_\lambda, \Omega_{r_2}, K) = 1. \tag{4.2}$$

Now, it follows from (4.1), (4.2), and the additivity of the fixed point index that for $\lambda_i > \lambda_0$,

$$i(F_\lambda, \Omega_{r_1} \backslash \Omega_{r_2}, K) = -1.$$

Consider now the nonempty sets

$$\begin{aligned} \Sigma_1 &:= \{\lambda_1 > 0 \mid \exists \lambda_2 > 0 \text{ such that } (\lambda_1, \lambda_2) \in \Sigma\}, \\ \Sigma_2 &:= \{\lambda_2 > 0 \mid \exists \lambda_1 > 0 \text{ such that } (\lambda_1, \lambda_2) \in \Sigma\}, \end{aligned}$$

and let

$$\Lambda_i := \inf \Sigma_i (< +\infty) \quad (i = 1, 2).$$

It follows that $\Sigma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$ and for all $(\lambda_1, \lambda_2) \in (0, +\infty)^2 \backslash ([\Lambda_1, +\infty) \times [\Lambda_2, +\infty))$, system (1.1) has no positive solution.

(ii) Let $(\lambda_1^0, \lambda_2^0) \in [\underline{\lambda}_1, +\infty) \times [\underline{\lambda}_2, +\infty)$ be arbitrarily chosen and suppose that (α_u, α_v) is a positive solution of (1.1) when $\lambda_1 = \underline{\lambda}_1$, and $\lambda_2 = \underline{\lambda}_2$. Then, for fixed $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$, (α_u, α_v) is a lower solution of (1.1).

Similarly, let $(\overline{\lambda}_1, \overline{\lambda}_2) \in [\lambda_1^0, +\infty) \times [\lambda_2^0, +\infty)$ be arbitrarily chosen and suppose that (β_u, β_v) is a positive solution for (1.1) when $\lambda_1 = \overline{\lambda}_1$, and $\lambda_2 = \overline{\lambda}_2$. Then, for fixed $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$, (β_u, β_v) is an upper solution of (1.1). According to Lemma 3.1 (i) and the positivity of (α_u, α_v), we conclude that $(\lambda_1^0, \lambda_2^0) \in \Sigma$.

(iii) From (ii) we obtain that $(\underline{\lambda}_1, +\infty) \times (\underline{\lambda}_2, +\infty) \subset \Sigma$ and let

$$(\lambda_1^0, \lambda_2^0) \in (\underline{\lambda}_1, +\infty) \times (\underline{\lambda}_2, +\infty) \backslash [\overline{\lambda}_1, +\infty) \times [\overline{\lambda}_2, +\infty).$$

It remains to show that system (1.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$ has a second positive solution. For this, we define (α_u, α_v) as the lower solution and (β_u, β_v) as the upper solution, both constructed as above. We fix (u_0, v_0) a positive solution of (1.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$ such that $(u_0, v_0) \in \mathcal{A}_{\alpha,\beta}$.

Now, we claim that there exists $\varepsilon > 0$ such that $\overline{B}((u_0, v_0), \varepsilon) \subset \mathcal{A}_{\alpha,\beta}$. For all $x \in [0, T]$, we have

$$\begin{aligned} \alpha_u(x) &= \underline{\lambda}_1 \int_0^T G(x, s)\mu_1(s)g_1(u, v)\mathrm{d}s \\ &< \lambda_1^0 \int_0^T G(x, s)\mu_1(s)g_1(u, v)\mathrm{d}s \\ &= u_0(x). \end{aligned}$$

Analogously, we obtain that $\alpha_v(x) < v_0(x)$ on $[0, T]$. So, choose an $\varepsilon_1 > 0$ such that if $(u, v) \in K$, then

$$||u - u_0||_\infty \le \varepsilon_1 \Rightarrow \alpha_u \le u \quad \text{and} \quad ||v - v_0||_\infty \le \varepsilon_1 \Rightarrow \alpha_v \le v \quad \text{on } [0, T]. \tag{4.3}$$

Alternatively, there is some $\varepsilon_2 \in (0, \varepsilon_1)$ such that if $(u, v) \in K$, then

$$||u - u_0||_\infty \le \varepsilon_2 \Rightarrow u \le \beta_u \quad \text{and} \quad ||v - v_0||_\infty \le \varepsilon_2 \Rightarrow v \le \beta_v \quad \text{on } [0, T]. \tag{4.4}$$

The claim is a consequence of (4.3) and (4.4), by taking $\varepsilon \in (0, \varepsilon_2)$.

Furthermore, if $\mathcal{A}_{\alpha,\beta}$ contains a second solution of (1.1), then it is nontrivial, thereby concluding the proof. Alternatively, if this is not the case, by Lemma 3.1 we infer that

$$i(F_{(\lambda_1^0,\lambda_2^0)}, B((u_0, v_0), \rho_1), K) = 1 \quad \text{for all } 0 < \rho_1 \le \varepsilon,$$

where $F_{(\lambda_1^0,\lambda_2^0)}$ stands for the fixed point operator corresponding to (1.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$. Also, from the proof of (i) and $g_{i,0}, g_{i,\infty} = 0$, we have

$$i(F_{(\lambda_1^0,\lambda_2^0)}, \Omega_{\rho_2}, K) = 1 \quad \text{for all } \rho_2 > 0 \text{ sufficiently large},$$

$$i(F_{(\lambda_1^0,\lambda_2^0)}, \Omega_{\rho_3}, K) = 1 \quad \text{for all } \rho_3 > 0 \text{ sufficiently small}.$$

Choose ρ_1, ρ_3 to be sufficiently small and ρ_2 to be sufficiently large, such that $\overline{B}((u_0, v_0), \rho_1) \cap \overline{B}(\rho_3) = \varnothing$ and $\overline{B}((u_0, v_0), \rho_1) \cup \overline{B}(\rho_3) \subset B(\rho_2)$. From the additivity-excision property of the fixed point index, it follows that

$$i(F_{(\lambda_1^0,\lambda_2^0)}, B(\rho_2) \backslash [\overline{B}((u_0, v_0), \rho_1) \cup \overline{B}(\rho_3)], K) = -1.$$

Therefore, $F_{(\lambda_1^0,\lambda_2^0)}$ has a fixed point $(u, v) \in B(\rho_2) \backslash [\overline{B}((u_0, v_0), \rho_1) \cup \overline{B}(\rho_3)]$. However, this implies the existence of a second positive solution to (1.1). □

Now, considering $\theta \in (0, \pi/2)$, we define

$$\mathcal{S}(\theta) := \{\lambda > 0 | (\lambda \cos\theta, \lambda \sin\theta) \in \Sigma\},$$

where $\mathcal{S}(\theta)$ is known to be nonempty. Subsequently, we rewrite problem (1.1) as follows:

$$\begin{cases} -u'' + q(x)u = \lambda \cos\theta \mu_1(x)g_1(u, v), & x \in (0, T), \\ -v'' + q(x)v = \lambda \sin\theta \mu_2(x)g_2(u, v), & x \in (0, T), \\ u(0) = u(T), \quad u'(0) = u'(T), \\ v(0) = v(T), \quad v'(0) = v'(T). \end{cases} \tag{4.5}$$

Lemma 4.2. *There exists a continuous function* $\Lambda : (0, \pi/2) \to (0, \infty)$ *such that*

$$\lim_{\theta\to 0} \Lambda(\theta)\sin\theta - \Lambda_2 = 0, \qquad \lim_{\theta\to\pi/2} \Lambda(\theta)\cos\theta - \Lambda_1 = 0. \tag{4.6}$$

Furthermore, for every $\theta \in (0, \pi/2)$, *the following hold true*:

(*i*) $\Lambda(\theta) \in \mathcal{S}$;

(*ii*) *system* (1.1) *has at least two positive solutions for all* $(\lambda_1, \lambda_2) \in (\Lambda(\theta)\cos\theta, +\infty) \times (\Lambda(\theta)\sin\theta, +\infty)$.

Proof. Define

$$\Lambda(\theta) \coloneqq \inf\mathcal{S}(\theta), \qquad \theta \in (0, \pi/2). \tag{4.7}$$

According to Lemma 4.1 (i), $\mathcal{S} \neq \varnothing$ and $0 < \Lambda(\theta) < \infty$.

Step 1. Statements (i) and (ii) hold true.

(i) Suppose on the contrary that for every $\theta \in (0, \pi/2)$, $\Lambda(\theta) \notin \mathcal{S}$. Then, there exists a sequence $\{(u_n, v_n)\}$ of solutions of (4.5) such that $||u_n||, ||v_n|| \to 0,\ n \to \infty$.

Let $z_n = u_n/||u_n||$, $w_n = v_n/||v_n||$, we have

$$\begin{cases} -z_n'' + q(x)z_n = \lambda\cos\theta\mu_1(x)\dfrac{g_1(u_n, v_n)}{||u_n||}, & x \in (0, T), \\ -w_n'' + q(x)w_n = \lambda\sin\theta\mu_2(x)\dfrac{g_2(u_n, v_n)}{||v_n||}, & x \in (0, T), \\ z_n(0) = z_n(T), \quad z_n'(0) = z_n'(T), \\ w_n(0) = w_n(T), \quad w_n'(0) = w_n'(T), \end{cases}$$

that is,

$$z_n(x) = \lambda\cos\theta\int_0^T G(x, s)\mu_1(s)\frac{g_1(w_n, y_n)}{||w_n||}\mathrm{d}s.$$

Since $g_{1,0} = 0$, we have that

$$\lim_{n\to\infty}\frac{g_1(w_n, y_n)}{||w_n||} \leq \lim_{n\to\infty}\frac{g_1(||w_n||, ||y_n||)}{||w_n||} = 0, \quad \text{uniformly in } x \in [0, T].$$

Hence, $\lim_{n\to\infty} z_n = 0$ uniformly, yet this contradicts the fact that $||z_n|| = 1$ for all $n \in \mathbb{N}$.

(ii) This conclusion is a direct consequence of statement (iii) of Lemma 4.1.

Step 2. Λ is continuous at each $\theta_0 \in (0, \pi/2)$.

The remaining arguments are the same as that of Lemma 4.2 of [17] and Proposition 4.5 of [18]. Suppose by contradiction that Λ is not continuous at some $\theta_0 \in (0, \pi/2)$, then there exists an $\varepsilon \in (0, \Lambda(\theta_0))$ such that for all sufficiently large $n \in \mathbb{N}$, $\theta_n \in (\theta_0 - 1/n, \theta_0 + 1/n) \subset (0, \pi/2)$ with $\Lambda(\theta_n) \notin (\Lambda(\theta_0) - \varepsilon, \Lambda(\theta_0) + \varepsilon)$. Assuming that $\Lambda(\theta_n) \geq \Lambda(\theta_0) + \varepsilon$ holds for infinitely many $n \in \mathbb{N}$. Then, for a subsequence of $\{\theta_n\}$ (also denoted as $\{\theta_n\}$ for simplicity), we have

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right)\cos\theta_n \geq \left(\Lambda(\theta_0) + \frac{\varepsilon}{2}\right)\cos\theta_n,$$

respectively,

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right)\sin\theta_n \geq \left(\Lambda(\theta_0) + \frac{\varepsilon}{2}\right)\sin\theta_n.$$

Furthermore, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $(\Lambda(\theta_0) + \varepsilon/2)\cos\theta_n > \Lambda(\theta_0)\cos\theta_0$, and $(\Lambda(\theta_0) + \varepsilon/2)\sin\theta_n > \Lambda(\theta_0)\sin\theta_0$. As a result, for all $n \geq n_0$, it follows that

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right)\cos\theta_n > \Lambda(\theta_0)\cos\theta_0,$$

respectively,

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right)\sin\theta_n > \Lambda(\theta_0)\sin\theta_0.$$

Using the fact that $\Lambda(\theta_0) \in \mathcal{S}(\theta_0)$ and combining it with Lemma 4.1 (ii), we have that $((\Lambda(\theta_n) - \varepsilon/2)\cos\theta_n, (\Lambda(\theta_n) - \varepsilon/2)\sin\theta_n) \in \Sigma$, so $\Lambda(\theta_n) - \varepsilon/2 \in \mathcal{S}(\theta_n)$. However, this contradicts the definition of $\Lambda(\theta_n)$. Similarly, if we assume that $\Lambda(\theta_n) \leq \Lambda(\theta_0) - \varepsilon$ for infinitely many $n \in \mathbb{N}$, we can employ a similar reasoning to obtain the contradiction.

Step 3. $\lim_{\theta\to 0}\Lambda(\theta)\sin\theta - \Lambda_2 = 0$, $\lim_{\theta\to\pi/2}\Lambda(\theta)\cos\theta - \Lambda_1 = 0$.

Considering a sequence $\{\theta_n\} \subset (0, \pi/2)$ with $\theta_n \to \pi/2$, as $n \to \infty$, we will show that

$$\Lambda(\theta_n)\cos\theta_n \to \Lambda_1, \quad n \to \infty.$$

It suffices to prove that any subsequence of $\{\theta_n\}$ (also denoted by $\{\theta_n\}$ for simplicity), contains a subsequence $\{\theta_{n_k}\}$ such that

$$\Lambda(\theta_{n_k})\cos\theta_{n_k} \to \Lambda_1, \quad k \to \infty.$$

From the definition of Λ_1, there exists a sequence $\{\lambda_1^k\} \subset \Sigma_1$ with $\lambda_1^k \to \Lambda_1$, as $k \to \infty$. Because $\theta_n \to \pi/2$, according to Lemma 4.1 (ii), we can find a sequence $\{r_k\} \subset (0, \infty)$ and a subsequence $\theta_{n_k} \subset \theta_n$, which, for all $k \in \mathbb{N}$, satisfy

$$r_k\cos\theta_{n_k} = \lambda_1^k \tag{4.8}$$

and

$$(r_k\cos\theta_{n_k}, r_k\sin\theta_{n_k}) \in \Sigma.$$

By the definition of the mapping Λ, we obtain $\Lambda(\theta_{n_k}) \leq r_k$. Hence, $\Lambda(\theta_{n_k})\cos\theta_{n_k} \leq r_k\cos\theta_{n_k}$. Because of (4.8) and the definition of Λ_1, we have

$$\Lambda_1 \leq \Lambda(\theta_{n_k})\cos\theta_{n_k} \leq r_k\cos\theta_{n_k} = \lambda_1^k \to \Lambda_1, \quad \text{as } k \to \infty.$$

Analogously, we can show that $\Lambda(\theta_n)\sin\theta_n \to \Lambda_2$ when $\theta_n \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 4.3. *Assume* (*H*1) *and* (*H*2). *Then, there exist positive constants* $\Lambda_1, \Lambda_2 > 0$ *and a continuous function* $\Lambda : (0, \pi/2) \to (0, +\infty)$, *generating the curve*

$$(\Gamma)\begin{cases}\lambda_1(\theta) = \Lambda(\theta)\cos\theta, & \theta \in (0, \pi/2),\\ \lambda_2(\theta) = \Lambda(\theta)\sin\theta, & \theta \in (0, \pi/2),\end{cases}$$

such that

(*i*) $\Gamma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$;

(*ii*) $\lim_{\theta\to\pi/2}\lambda_2(\theta) = +\infty = \lim_{\theta\to 0}\lambda_1(\theta)$, $\lim_{\theta\to 0}\lambda_2(\theta) - \Lambda_2 = 0 = \lim_{\theta\to\pi/2}\lambda_1(\theta) - \Lambda_1$;

(*iii*) *The curve* Γ *divides the first quadrant* $(0, +\infty) \times (0, +\infty)$ *into two disjoint sets* O_1 *and* O_2 *such that system* (1.1) *has zero positive solutions if* $(\lambda_1, \lambda_2) \in O_1$, *at least one positive solution if* $(\lambda_1, \lambda_2) \in \Gamma$, *or at least two positive solutions if* $(\lambda_1, \lambda_2) \in O_2$.

Proof. We have shown the existence of the continuous function Λ in Lemma 4.2 and the constants Λ_1 and Λ_2 in Lemma 4.1 (i).

(i) This result follows from combining Lemma 4.2 (i) with Lemma 4.1 (i).

(ii) The equalities $\lim_{\theta\to\pi/2}\lambda_2(\theta) = +\infty = \lim_{\theta\to 0}\lambda_1(\theta)$ are a direct consequence of the inequalities

$$\Lambda(\theta) \geq \frac{\Lambda_1}{\cos\theta} \quad \text{and} \quad \Lambda(\theta) \geq \frac{\Lambda_2}{\sin\theta},$$

and $\lim_{\theta\to 0}\lambda_2(\theta) - \Lambda_2 = 0 = \lim_{\theta\to\pi/2}\lambda_1(\theta) - \Lambda_1$ is a conclusion of Lemma 4.2.

(iii) Using Lemma 4.2 and the definition of $\Lambda(\theta)$ given in (4.7), we obtain the conclusion. □

Example 4.4. The functions $g_1(u, v) = \min\{u^{p_1}, u^{q_1}\} + \min\{v^{p_2}, v^{q_2}\}$, $g_2(u, v) = \min\{u^{p_2}, u^{q_2}\} + \min\{v^{p_1}, v^{q_1}\}$ satisfy the conditions of Theorem 4.3, where $0 < q_1, q_2 < 1$, $1 < p_1, p_2 < \infty$.

Acknowledgments: The authors are very grateful to the anonymous referees for their very valuable suggestions.

Funding information: This work was supported by Doctoral Research Foundation of Gansu Agricultural University (No. GAU-KYQD-2022-32) and Gansu University Innovation Foundation (No. 2022B-107).

Author contributions: The authors declare that the research was conducted in collaboration, with each author contributing equally. All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results, and approved the final version of the manuscript.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: Data sharing is not applicable to this article, as no datasets were generated.

References

[1] J. R. Graef, L. Kong, and H. Wang, *Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem*, J. Differential Equations **245** (2008), no. 5, 1185–1197, DOI: https://doi.org/10.1016/j.jde.2008.06.012.

[2] D. Jiang, J. Chu, D. O'Regan, and R. Agarwal, *Multiple positive solutions to superlinear periodic boundary value problems with repulsive singular forces*, J. Math. Anal. Appl. **286** (2003), no. 2, 563–576, DOI: https://doi.org/10.1016/S0022-247X(03)00493-1.

[3] R. Ma, *Bifurcation from infinity and multiple solutions for periodic boundary value problems*, Nonlinear Anal. **42** (2000), no. 1, 27–39, DOI: https://doi.org/10.1016/S0362-546X(98)00327-7.

[4] I. Rachünková, *Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems*, J. Differential Equations **176** (2001), 445–469, DOI: https://doi.org/10.1006/jdeq.2000.3995.

[5] P. Torres, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations **190** (2003), no. 2, 643–662, DOI: https://doi.org/10.1016/S0022-0396(02)00152-3.

[6] Z. Zhang and J. Wang, *Positive solutions to a second order three-point boundary value problem*, J. Math. Anal. Appl. **285** (2003), no. 1, 237–249, DOI: https://doi.org/10.1016/S0022-247X(03)00396-2.

[7] R. P. Agarwal, D. O'Regan, and P. J. Y. Wong, *Constant-sign solutions of a system of Fredholm integral equations*, Acta Appl. Math. **80** (2004), 57–94, DOI: https://doi.org/10.1023/B:ACAP.0000013257.42126.ca.

[8] D. Jiang, J. Wei, and B. Zhang, *Positive periodic solutions of functional differential equations and population models*, Electron. J. Differential Equations **2002** (2002), no. 71, 1–13.

[9] D. O'Regan and H. Wang, *Positive periodic solutions of systems of second order ordinary differential equations*, Positivity **10** (2006), 285–298.

[10] H. Wang, *On the number of positive solutions of nonlinear systems*, J. Math. Anal. Appl. **281** (2003), 287–306, DOI: https://doi.org/10.1016/S0022-247X(03)00100-8.

[11] J. Chu, H. Chen, and D. O'Regan, *Positive periodic solutions and eigenvalue intervals for systems of second order differential equations*, Math. Nachr. **281** (2008), no. 11, 1549–1556, DOI: https://doi.org/10.1002/mana.200510695.

[12] X. Hao, L. Liu, and Y. Wu, *Existence and multiplicity results for nonlinear periodic boundary value problems*, Nonlinear Anal. **72** (2010), no. 9-10, 3635–3642, DOI: https://doi.org/10.1016/j.na.2009.12.044.

[13] F. M. Atici and G. Guseinov, *On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions*, J. Comput. Appl. Math. **132** (2001), 341–356, DOI: https://doi.org/10.1016/s0377-0427(00)00438-6.

[14] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[15] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1987.

[16] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, 1988.

[17] D. Gurban and P. Jebelean, *Positive radial solutions for multiparameter Dirichlet systems with mean curvature operator in Minkowski space and Lane-Emden type nonlinearities*, J. Differential Equations **266** (2019), 5377–5396, DOI: https://doi.org/10.1016/j.jde.2018.10.030.

[18] D. Gurban, P. Jebelean, and C. Şerban, *Non-potential and non-radial Dirichlet systems with mean curvature operator in Minkowski space*, Discrete Contin. Dyn. Syst. **40** (2020), no. 1, 133–151, DOI: https://doi.org/10.3934/dcds.2020006.

[19] C. D. Coster and P. Habets, *Upper and Lower Solutions in the Theory of ODE Boundary Value Problems: Classical and Recent Results*, Springer, Vienna, 1996.