DE GRUYTER Open Mathematics 2025; 23: 20250160

Research Article

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Continuity and essential norm of operators defined by infinite tridiagonal matrices in weighted Orlicz and l^∞ spaces

https://doi.org/10.1515/math-2025-0160
received November 14, 2024; accepted May 5, 2025

Abstract: In this article, we provide a comprehensive study on the continuity and essential norm of an operator defined by an infinite tridiagonal matrix, specifically when it operates from a weighted Orlicz sequence space or a weighted l^∞ space into another space of similar nature. Our findings include significant characterizations regarding the compactness of this operator across various contexts of weighted Orlicz and l^∞ sequence spaces.

Keywords: compact operators, essential norm, tridiagonal operator, multiplication operator

MSC 2020: 47B60, 46B45, 46E30, 46B42

1 Introduction

Throughout history, the importance of using numerical matrices to solve problems in the natural sciences and their applications has been noted. In particular, tridiagonal matrices commonly appear in the applications of mathematics and physics. For example, in the discrete approximation using the finite difference method for the linear second-order differential equation, we can establish a relationship between the solution and the boundary conditions through a matrix of this type (see Burden et al. [1] and LeVeque [2]). In this case, we can imagine that an infinite matrix could represent the "*ideal case*" of discretization, where the error of the approximation is zero.

An infinite matrix M is a function $M : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$, which, as is usual, is denoted by $M = (m_{i,j})$, where $m_{i,j}$ represents the entry at row $i \in \mathbb{N}$ and column $j \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of all positive integers and $\mathbb{C}$ is the set of complex numbers. The multiplication of an infinite matrix M by a sequence $\mathbf{x} = (x_k)$ is realized in the usual way. In fact, we have $M \times \mathbf{x} = \mathbf{y} = (y_i)$ with $y_i = \sum_{j=1}^{\infty} m_{i,j} x_j$ whenever this series will be a convergent series for all $i \in \mathbb{N}$. If X and Y are Banach sequence spaces and $\mathbf{y} = M \times \mathbf{x} \in Y$ for all $\mathbf{x} \in X$, then the infinite matrix M define a linear transformation $T_M : X \to Y$ given by $T_M(\mathbf{x}) = M \times \mathbf{x}$, and we are interested in studying its topological properties, such as continuity and the compactness.

The study of the infinite matrices is more or less ancient, dating back to around 1884 with the work of Poincaré, who wanted to provide a rigorous foundation for the use of infinite matrices and the calculus of determinants. The interest increased, and the study of these matrices was continued by Helge Von Koch and

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was further encouraged by David Hilbert, who studied the eigenvalues of integral operators by viewing these as infinite matrices. For a historical overview and recent applications of infinite matrices, we refer to the monographs by Cooke [3], Shivakumar et al. [4], and the article by Bernkopf [5].

The continuity problem of a process defined by an infinite matrix M consists of finding conditions on M such that a Banach space of sequences X will be mapped into another space Y by the application of an infinite matrix M. This problem is related to the stability of the process defined by an infinite matrix and has been studied by numerous mathematicians in different sequence spaces. In [6], Stieglitz and Tietz provide a table with the conditions required for an infinite matrix M to apply a classical sequence space (such as c, c_0, l^p, etc.) into another space.

In [6], we can see that famous mathematicians such as Hahn, Hardy, Littlewood, Lorentz, Orlicz, Schur, Toeplitz, among others have studied this problem, which is still of great interest, specially in particular cases of special matrices such as diagonal, bidiagonal, tridiagonal, triangular, or matrices defined by blocks.

An operator $T : X \to Y$ is said to be compact if T maps bounded subsets of X to relatively compact subsets of Y. Compact operators are rich in properties, and they find applications in the problem of determining the solution of a linear system of infinite equations of the form $(\lambda K + I)\mathbf{x} = \mathbf{y}$, where $\mathbf{x} \in X$ is the unknown sequence, $K : X \to X$ is a compact operator, λ is a non-zero scalar, and $\mathbf{y} \in X$ is given. In the article by González et al. [7], historical remarks about compact operators and their applications can be found. There is a growing interest in establishing or characterizing the compactness of certain operators acting on specific Banach spaces of sequences. The compactness of a diagonal operator acting on very general Köthe sequence spaces was characterized by Ramos-Fernández and Salas-Brown in [8]. More precisely, Ramos-Fernández and Salas-Brown in [8] proved that if X is a Köthe sequence space and $\mathbf{u} = (u_k)$ is a bounded sequence, then the multiplication operator $M_u : X \to X$ defined by $M_u(\mathbf{x}) = u_k \cdot x_k$ is a compact operator if and only if $\lim_{k\to\infty}|u_k| = 0$ (i.e., $\mathbf{u} \in c_0$). In the article by El-Shabrawy [9], we can find characterizations of the compactness of the Rhaly operator (which generalize Cesàro matrix operator) and the generalized difference operator (or bidiagonal operator) between a lot of Banach sequence spaces such as h (Hahn sequence space), cs, l^1 among others. Our intention in this article is to estimate the essential norm of tridiagonal operators acting from weighted Orlicz or weighted l^∞ sequence spaces into another of the same kind, so that we can obtain characterizations of the compactness of this operator in several cases.

An infinite matrix M is called *tridiagonal* if there exist numerical sequences $\mathbf{a} = (a_k)$, $\mathbf{b} = (b_k)$ and $\mathbf{c} = (c_k)$ such that $m_{k,k} = a_k$, $m_{k-1,k} = b_k$, $m_{k+1,k} = c_k$ for all $k \in \mathbb{N}$ and $m_{i,j} = 0$ otherwise. In this case, the tridiagonal matrix will be denoted by $\Delta_{a,b,c}$. The multiplication of an infinite tridiagonal matrix $\Delta_{a,b,c}$ by a numerical sequence $\mathbf{x} = (x_k)$ is the sequence $\mathbf{y} = (y_k)$ defined by $y_1 = a_1x_1 + c_1x_2$ and

$$y_k = b_{k-1}x_{k-1} + a_kx_k + c_kx_{k+1} \tag{1}$$

for all $k \geq 2$. This product is represented by

$$\begin{pmatrix} a_1 & c_1 & 0 & 0 & \dots \\ b_1 & a_2 & c_2 & 0 & \dots \\ 0 & b_2 & a_3 & c_3 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_1x_1 + c_1x_2 \\ b_1x_1 + a_2x_2 + c_2x_3 \\ b_2x_2 + a_3x_3 + c_3x_4 \\ \vdots \end{pmatrix}$$

and can be seen as a simple process in which the entry $\mathbf{x}$ is converted in the output $\mathbf{y}$ by application of an infinite tridiagonal matrix $\Delta_{a,b,c}$. Thus, this process allows us to define a linear operator $T_{a,b,c}$ acting on a Banach sequence space X. In fact, if $\mathbf{x} = (x_k)$ is a complex sequence and belongs to the Banach sequence space X, then we can define $\mathbf{y} = T_{a,b,c}(\mathbf{x}) = \Delta_{a,b,c} \times \mathbf{x}$, and we are interested in the topological properties of these operators, such as continuity, compactness, and to estimate its essential norm.

Clearly, the operator $T_{a,b,c}$ is linear and generalize various types of linear transformations from several perspectives. The first one is the well-known multiplication operator, which is obtained when the sequences $\mathbf{b}$ and $\mathbf{c}$ are the null sequence $\mathbf{0}$, that is, when $b_k = c_k = 0$ for all $k \in \mathbb{N}$. In this case, it is usual to write M_a instead of $T_{a,0,0}$. Multiplication operators have been studied by several researchers and is still a subject of interest for many mathematicians (see, for instance, [8] and the references therein). In fact, tridiagonal operators can be seen as perturbations of multiplication operators in the following sense: If we denote by L the left shift operator and by R the right shift operator, that is,

$$L(\mathbf{x}) = L(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots),$$
$$R(\mathbf{x}) = R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots),$$

then we have

$$T_{a,b,c}(\mathbf{x}) = R(M_b(\mathbf{x})) + M_a(\mathbf{x}) + M_c(L(\mathbf{x})), \tag{2}$$

where for any sequence $\mathbf{y} = (y_k)$, we define $y_0 = 0$.

If only $\mathbf{c}$ is the null sequence $\mathbf{0}$, then we have *the generalized difference operator* $\Delta_{a,b}$ or bidiagonal operator which has been studied by Akhmedov and El-Shabrawy [10] and by a lot of researchers (see [9] and the references therein). If each of the sequences $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are constants sequences of complex numbers, then the infinite tridiagonal matrix is a special case of the infinite Toeplitz matrix and it is well known that the Toeplitz operator is bounded if and only if the coefficients of the Toeplitz matrix are the Fourier coefficients of some essentially bounded function f. In fact, there has been recent work on weighted Toeplitz operators acting on weighted sequence spaces [11].

The compactness and the estimation of the essential norm of the tridiagonal operator $T_{a,b,c}$ acting on weighted l^2 spaces were studied recently by Caicedo et al. [12]. Our goal is to extend the results obtained by Caicedo et al. [12] to the more general case of weighted Orlicz spaces, which we define in Section 2. Our study also will be an extension and a generalization of recent results due to Ramos-Fernández and Salas-Brown [13] regarding the compactness of multiplication operators acting on Orlicz sequence spaces. In addition, we estimate the essential norm of the tridiagonal operators $T_{a,b,c}$ in the case when a weighted Orlicz spaces is mapped into a weighted l^∞ space and *vice versa*. We also consider the case when $T_{a,b,c}$ maps a weighted l^∞ space into itself. Finally, in the last section, we extend our results to another kind of weighted Orlicz sequence space.

The structure of the article is as follows. In Section 2, we define a class of weighted Orlicz sequence spaces and present some of their properties. In Section 3, we address the problem of the continuity of tridiagonal operators, specifically when they map $l^\varphi(\omega)$ into itself, when they map $l^\infty(\omega)$ into itself, and when they map $l^\varphi(\omega)$ into $l^\infty(\omega)$. In Section 4, we estimate the essential norm of the tridiagonal operators for the cases studied in Section 3: $l^\varphi(\omega) \to l^\varphi(\omega)$, $l^\infty(\omega) \to l^\infty(\omega)$, and $l^\varphi(\omega) \to l^\infty(\omega)$. Finally, in Section 5, we examine other types of weighted Orlicz sequence spaces, we characterize the conditions under which the tridiagonal operator is continuous on these spaces, and we compute its essential norm when acting on these spaces of Orlicz sequences.

2 Some remarks on weighted Orlicz sequence spaces

In this section, we present some properties of the weighted Orlicz sequence spaces. First, we recall that a function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a *Young function* (or $\mathcal{N}$-function) if it is an increasing, continuous and convex function which assume the value zero only at zero and $\varphi(t) \to \infty$ as $t \to \infty$. A Young function φ satisfies the global Δ_2-condition, denoted as $\varphi \in \Delta_2$, if there exists a constant $K > 0$ such that $\varphi(2t) \le K\varphi(t)$ for all $t \ge 0$. A numerical sequence $\boldsymbol{\omega} = (\omega_k)$ such that $\omega_k > 0$ for all $k \in \mathbb{N}$ will be called a *weight*. Given a Young function φ, a weight $\boldsymbol{\omega}$ and a complex sequence $\mathbf{x} = (x_k)$, we set

$$N_\omega^\varphi(\mathbf{x}) = \sum_{k=1}^{\infty} \varphi(|x_k|\omega_k) = \lim_{m\to\infty} \sum_{k=1}^{m} \varphi(|x_k|\omega_k), \tag{3}$$

and then we say that $\mathbf{x}$ belongs to the weighted Orlicz sequence space $l^\varphi(\omega)$ if there exists a $\lambda > 0$ such that $N_\omega^\varphi\left(\frac{\mathbf{x}}{\lambda}\right) \le 1$. That is,

$$l^\varphi(\omega) = \left\{\mathbf{x} = (x_k) : N_\omega^\varphi\left(\frac{\mathbf{x}}{\lambda}\right) \le 1 \text{ for some } \lambda > 0\right\}.$$

Note that the convexity of φ implies that $l^\varphi(\omega)$ is a vector space. When $\varphi(t) = \frac{1}{p}t^p$ with $p > 1$ fixed, we obtain the weighted l^p space (see, for instance, [14] for the case $p = 2$) and when $\omega \equiv 1$, i.e., $\omega_k = 1$ for all $k \in \mathbb{N}$, then we have the Orlicz sequence space l^{φ}. This kind of weighted Orlicz sequence spaces has been considered by some researchers, for instance, in [15], the authors considered this kind of spaces for functions defined on a group. Note also that $\mathbf{x} \in l^{\varphi}(\omega)$ if and only if $M_{\omega}(\mathbf{x}) = \mathbf{x} \cdot \omega \in l^{\varphi}$. This last relation implies that the multiplication operator $M_{\omega} : l^{\varphi}(\omega) \to l^{\varphi}$ is a linear bijection. Furthermore, we can observe that for $\mathbf{x} \in l^{\varphi}(\omega)$, the set

$$E^{\varphi}_{\omega}(\mathbf{x}) = \left\{\lambda > 0 : N^{\varphi}_{\omega}\left(\frac{\mathbf{x}}{\lambda}\right) \leq 1\right\},$$

is not empty, clearly, it has lower bound 0. Observe that if $\lambda_0 \in E^{\varphi}_{\omega}(\mathbf{x})$ and $\lambda > \lambda_0$, then $\lambda \in E^{\varphi}_{\omega}(\mathbf{x})$. Thus, we can define for $\mathbf{x} \in l^{\varphi}(\omega)$

$$\|\mathbf{x}\|_{\varphi,\omega} = \inf(E^{\varphi}_{\omega}(\mathbf{x})) = \inf\left\{\lambda > 0 : N^{\varphi}_{\omega}\left(\frac{\mathbf{x}}{\lambda}\right) \leq 1\right\}, \tag{4}$$

and $\|\cdot\|_{\varphi,\omega}$ is a norm for $l^{\varphi}(\omega)$, which is known as the *Luxemburg norm* of $\mathbf{x}\omega$. In fact, the pair $(l^{\varphi}(\omega), \|\cdot\|_{\varphi,\omega})$ is a Banach space. In the case that $\omega_k = 1$ for all $k \in \mathbb{N}$, we write $\|\cdot\|_{\varphi}$ instead of $\|\cdot\|_{\varphi,\omega}$. Also, it is important to note that $\|\mathbf{x}\|_{\varphi,\omega} = \|M_{\omega}(\mathbf{x})\|_{\varphi}$ and $M_{\omega} : l^{\varphi}(\omega) \to l^{\varphi}$ is a surjective linear isometry. We refer to the excellent book by Rao and Ren [16] for more properties of Young functions and more properties of Orlicz spaces. Here, we include some facts that we are going to use throughout this article.

(1) If $\mathbf{x} \in l^{\varphi}(\omega)$ and $\mathbf{x}$ is not the null sequence, then

$$N^{\varphi}_{\omega}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_{\varphi,\omega}}\right) \leq 1. \tag{5}$$

That is, $\|\mathbf{x}\|_{\varphi,\omega} \in E^{\varphi}_{\omega}(\mathbf{x})$ and $\|\mathbf{x}\|_{\varphi,\omega}$ is, in fact, a minimum.

(2) For all $\mathbf{x} \in l^{\varphi}(\omega)$, we have

$$\|\mathbf{x}\|_{\varphi,\omega} \leq 1 \;\Leftrightarrow\; N^{\varphi}_{\omega}(\mathbf{x}) \leq 1. \tag{6}$$

(3) If $\mathbf{x} \in l^{\varphi}(\omega)$, then $\lim_{k\to\infty}|x_k|\omega_k = 0$. This relation implies that $l^{\varphi}(\omega)$ is a subspace of the weighted c_0 space, which we denote by $c_0(\omega)$. Observe that when $\omega_k = 1$ for all $k \in \mathbb{N}$, $c_0(\omega)$ coincides with the classical c_0 space.

(4) The space $l^{\varphi}(\omega)$ is continuously embedded in $l^{\infty}(\omega)$. That is,

$$\|\mathbf{x}\|_{\infty,\omega} \leq \varphi^{-1}(1)\|\mathbf{x}\|_{\varphi,\omega}, \tag{7}$$

for all $\mathbf{x} \in l^{\varphi}(\omega)$, where $\|\mathbf{x}\|_{\infty,\omega} = \sup_{k\in\mathbb{N}}|x_k|\omega_k$. Here, $l^{\infty}(\omega)$ is the set of all numerical sequences $\mathbf{x} = (x_k)$ such that $\|\mathbf{x}\|_{\infty,\omega} < \infty$.

(5) For $n \in \mathbb{N}$ fixed, the operator $F_n : l^{\varphi}(\omega) \to \mathbb{C}$ defined by $F_n(\mathbf{x}) = x_n$ (evaluation functional at n) is bounded and

$$\|F_n\| = \frac{1}{\|\mathbf{e_n}\|_{\varphi,\omega}}, \tag{8}$$

where $\|\cdot\|$ is the operator norm and $\mathbf{e_n} = (e_{n,k})$ is the nth canonical sequence defined as $e_{n,n} = 1$ and $e_{n,k} = 0$ for $k \neq n$.

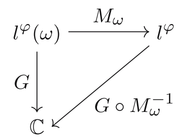
Given a Young function φ, its complementary function $\psi : [0,\infty) \to [0,\infty)$ is defined by

$$\psi(t) = \sup\{\tau t - \varphi(\tau) : \tau \geq 0\}. \tag{9}$$

We can see that ψ is a Young function. Furthermore, the Young inequality, given by $\tau t \leq \varphi(\tau) + \psi(t)$ for all $t, \tau \geq 0$, is immediately evident. Also it is known that if $\varphi \in \Delta_2$, then the dual space of l^{φ} is precisely l^{ψ} in the sense that for each $F \in (l^{\varphi})'$, the dual of l^{φ}, there exists $\mathbf{y} \in l^{\psi}$ such that $F(\mathbf{x}) = \sum_{k=1}^{\infty} x_k y_k$, for all $\mathbf{x} \in l^{\varphi}$. Thus, using the fact that the multiplication operator $M_{\omega} : l^{\varphi}(\omega) \to l^{\varphi}$ defined by $M_{\omega}(\mathbf{x}) = \mathbf{x} \cdot \omega$ is a surjective linear isometry, we obtain $(l^{\varphi}(\omega))' = l^{\psi}(\omega)$ in the sense that for each $G \in (l^{\varphi}(\omega))'$, there exists $\mathbf{y} \in l^{\psi}(\omega)$ such that

$$G(\mathbf{x}) = \sum_{k=1}^{\infty} x_k y_k \,\omega_k^2, \tag{10}$$

for all $\mathbf{x} \in l^{\varphi}(\omega)$. This last affirmation can be seen with the following diagram:



$$\begin{aligned}
&G \in (l^{\varphi}(\omega))' \\
&\quad \Rightarrow G \circ M_\omega^{-1} \in (l^{\varphi})' \\
&\quad \Rightarrow \exists \mathbf{v} \in l^{\psi} \quad \forall \mathbf{u} \in l^{\varphi} : \; G \circ M_\omega^{-1}(\mathbf{u}) = \sum_{k=1}^{\infty} u_k v_k \\
&\quad \Rightarrow \exists \mathbf{y} \in l^{\psi}(\omega) \quad \forall \mathbf{x} \in l^{\varphi}(\omega) : \; G(\mathbf{x}) = \sum_{k=1}^{\infty} x_k y_k \, \omega_k^2.
\end{aligned}$$

We refer this last fact as Riesz's representation theorem for $l^{\varphi}(\omega)$. We would like recommend the excellent book by Rao and Ren [16] for more properties of Orlicz spaces.

3 On the continuity of tridiagonal operators between $l^{\varphi}(\omega)$ and $l^{\infty}(\omega)$ spaces

In this section, we shall characterize all sequences **a**, **b**, **c**, which define continuous tridiagonal operator acting between $l^{\varphi}(\omega)$ and $l^{\infty}(\omega)$ spaces. The results presented in this section are genuine extensions and generalizations of those found in [12,13]. Herein, we use the notation $A \simeq B$ if there exists a constant $K > 0$, independent of A and B, such that $K^{-1}A \leq B \leq KA$. We note that

$$\|\mathbf{e_n}\|_{\varphi,\omega} = \frac{1}{\varphi^{-1}(1)} \omega_n = \frac{1}{\varphi^{-1}(1)} \|\mathbf{e_n}\|_{\infty,\omega},$$

where $\mathbf{e_n}$ is the nth canonical sequences. Also, we can see that

$$T_{a,b,c}(\mathbf{e_n}) = (0, \ldots, 0, c_{n-1}, a_n, b_n, 0, \ldots), \tag{11}$$

that is, if $T_{a,b,c}(\mathbf{e_n}) = \mathbf{y} = (y_k)$, then $y_{n-1} = c_{n-1}$, $y_n = a_n$, $y_{n+1} = b_n$ and $y_k = 0$ otherwise (recall that, for convenience, we have defined $y_0 = 0$ for any sequence $\mathbf{y} = (y_k)$, although our sequences always start at index 1). Hence, for each $n \geq 1$, we have

$$\|T_{a,b,c}(\mathbf{e_n})\|_{\infty,\omega} = \max\{|c_{n-1}|\omega_{n-1}, |a_n|\omega_n, |b_n|\omega_{n+1}\},$$

and we have the following result about the continuity of the tridiagonal operator.

Theorem 1. *Let φ be a Young function and let ω be a weight. For fixed sequences $\mathbf{a} = (a_k)$, $\mathbf{b} = (b_k)$ and $\mathbf{c} = (c_k)$, the following are equivalent:*
(1) *The tridiagonal operator $T_{a,b,c}$ is continuous from $l^{\varphi}(\omega)$ into itself.*
(2) *The tridiagonal operator $T_{a,b,c}$ is continuous from $l^{\varphi}(\omega)$ into $l^{\infty}(\omega)$.*
(3) *The tridiagonal operator $T_{a,b,c}$ is continuous from $l^{\infty}(\omega)$ into itself.*
(4) $\mathbf{a} \in l^{\infty}$, $\mathbf{b} \in l^{\infty}(\boldsymbol{\beta})$ *and* $\mathbf{c} \in l^{\infty}(\boldsymbol{\gamma})$, *where*

$$\beta_k = \frac{\omega_{k+1}}{\omega_k} \quad \text{and} \quad \gamma_k = \frac{\omega_k}{\omega_{k+1}}$$

for all $k \in \mathbb{N}$.

In all these cases, we have, $\|T_{a,b,c}\| \simeq \max\{\|\mathbf{a}\|_{\infty}, \|\mathbf{b}\|_{\infty,\beta}, \|\mathbf{c}\|_{\infty,\gamma}\}$, where $\|\mathbf{b}\|_{\infty,\beta} = \sup_{k\in\mathbb{N}} |b_k|\beta_k$ and $\|\mathbf{c}\|_{\infty,\gamma}$ is defined in a similar way.

Proof. This result follows from the fact that the propositions (1), (2), and (3) are all equivalent to (4). That is, we have noted that the statements (1) $\Leftrightarrow$ (4), (2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (4) are true. We shall prove that (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2), while the proofs of (2) $\Rightarrow$ (4), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (1), and (4) $\Rightarrow$ (3) are similar.

(1) $\Rightarrow$ **(4)** Let us suppose that $T_{a,b,c} : l^{\varphi}(\omega) \to l^{\varphi}(\omega)$ is continuous. There exists a constant $M > 0$ such that $\|T_{a,b,c}(\mathbf{x})\|_{\varphi,\omega} \leq M\, \|\mathbf{x}\|_{\varphi,\omega}$ for all $\mathbf{x} \in l^{\varphi}(\omega)$. In particular, for the nth canonical sequence $\mathbf{e_n}$, we can write

$$\left\| \frac{T_{a,b,c}(\mathbf{e_n})}{M\, \|\mathbf{e_n}\|_{\varphi,\omega}} \right\|_{\varphi,\omega} \leq 1$$

for all $n \in \mathbb{N}$. Thus, by the equivalence (6), we have

$$N_{\omega}^{\varphi}\left(\frac{T_{a,b,c}(\mathbf{e_n})}{M\, \|\mathbf{e_n}\|_{\varphi.\omega}} \right) \leq 1.$$

Hence, the definition of $T_{a,b,c}(\mathbf{e_n})$ (expression (11)) and the definition of N_{φ}^{ω} in (3) implies that

$$\varphi\left(\frac{|c_{n-1}|}{M\, \|\mathbf{e_n}\|_{\varphi,\omega}} \omega_{n-1} \right) + \varphi\left(\frac{|a_n|}{M\, \|\mathbf{e_n}\|_{\varphi,\omega}} \omega_n \right) + \varphi\left(\frac{|b_n|}{M\, \|\mathbf{e_n}\|_{\varphi,\omega}} \omega_{n+1} \right) \leq 1.$$

Then since φ is an increasing function, from the first term of this inequality, we obtain

$$|c_{n-1}| \frac{\|\mathbf{e_{n-1}}\|_{\varphi,\omega}}{\|\mathbf{e_n}\|_{\varphi,\omega}} \leq M,$$

where we define $c_0 = 0$ and $\omega_0 = 0$. This means that $\mathbf{c} \in l^{\infty}(\gamma)$ with $\gamma = (\gamma_k)$ defined by

$$\gamma_k = \frac{\|\mathbf{e_k}\|_{\varphi,\omega}}{\|\mathbf{e_{k+1}}\|_{\varphi,\omega}} = \frac{\omega_k}{\omega_{k+1}},$$

where $k \geq 1$. In similar way, we obtain $|a_n| \leq M$, which tells us that $\mathbf{a} \in l^{\infty}$. Furthermore,

$$|b_n| \frac{\|\mathbf{e_{n+1}}\|_{\varphi,\omega}}{\|\mathbf{e_n}\|_{\varphi,\omega}} \leq M$$

for all $n \in \mathbb{N}$, which means that $\mathbf{b} \in l^{\infty}(\beta)$ with $\boldsymbol{\beta} = (\beta_k)$ defined as follows:

$$\beta_k = \frac{\|\mathbf{e_{k+1}}\|_{\varphi,\omega}}{\|\mathbf{e_k}\|_{\varphi,\omega}} = \frac{\omega_{k+1}}{\omega_k}.$$

This conclude the proof of the implication. Observe that also we have

$$\max\{\|\mathbf{a}\|_{\infty}, \|\mathbf{b}\|_{\infty,\beta}, \|\mathbf{c}\|_{\infty,\gamma}\} \leq \|T_{a,b,c}\|.$$

This proves the implication. The proofs of the implications (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) are similar.

(4) $\Rightarrow$ **(2)** Suppose now that $\mathbf{a} \in l^{\infty}$, $\mathbf{b} \in l^{\infty}(\beta)$ and $\mathbf{c} \in l^{\infty}(\gamma)$. We set

$$M = \max\{\|\mathbf{a}\|_{\infty}, \|\mathbf{b}\|_{\infty,\beta}, \|\mathbf{c}\|_{\infty,\gamma}\}.$$

Then for any $\mathbf{x} \in l^{\varphi}(\omega)$, we have $T_{a,b,c}(\mathbf{x}) = \mathbf{u} + \mathbf{v} + \mathbf{w}$, where $u_k = b_{k-1}x_{k-1}$, $v_k = a_k x_k$ and $w_k = c_k x_{k+1}$ with $b_0 = x_0 = 0$. Thus, for any $n \geq 2$, we have

$$|u_n|\omega_n = |b_{n-1}| \frac{\omega_n}{\omega_{n-1}} |x_{n-1}|\omega_{n-1} \leq \|\mathbf{b}\|_{\infty,\beta} \|\mathbf{x}\|_{\infty,\omega} \leq \varphi^{-1}(1)\|\mathbf{b}\|_{\infty,\beta} \|\mathbf{x}\|_{\varphi,\omega}$$

in virtue of the inequality (7). Hence, $\|\mathbf{u}\|_{\infty,\omega} \leq \varphi^{-1}(1)\|\mathbf{b}\|_{\infty,\beta} \|\mathbf{x}\|_{\varphi,\omega}$. In a very similar way, we also obtain

$$\|\mathbf{v}\|_{\infty,\omega} \leq \varphi^{-1}(1)\|\mathbf{a}\|_{\infty} \|\mathbf{x}\|_{\varphi,\omega} \quad \text{and} \quad \|\mathbf{w}\|_{\infty,\omega} \leq \varphi^{-1}(1)\|\mathbf{c}\|_{\infty,\gamma} \|\mathbf{x}\|_{\varphi,\omega},$$

and therefore,

$$\|T_{a,b,c}(\mathbf{x})\|_{\infty,\omega} \leq 3\varphi^{-1}(1) \max\{\|\mathbf{a}\|_{\infty}, \|\mathbf{b}\|_{\infty,\beta}, \|\mathbf{c}\|_{\infty,\gamma}\}\|\mathbf{x}\|_{\varphi,\omega}.$$

This proves that the operator $T_{a,b,c}$ is continuous from $l^{\varphi}(\omega)$ into $l^{\infty}(\omega)$. The proofs of the implications (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (3) are similar. □

We can also characterize the continuity of the tridiagonal operator from $l^{\infty}(\omega)$ to $l^{\varphi}(\omega)$. In the following result, we present the necessary and sufficient conditions:

Theorem 2. *Let φ be a Young function and let ω be a weight. For fixed sequences $\mathbf{a} = (a_k)$, $\mathbf{b} = (b_k)$, and $\mathbf{c} = (c_k)$, the tridiagonal operator $T_{a,b,c}$ is continuous from $l^{\infty}(\omega)$ into $l^{\varphi}(\omega)$ if and only if $\mathbf{a} \in l^{\varphi}$, $\mathbf{b} \in l^{\varphi}(\beta)$, and $\mathbf{c} \in l^{\varphi}(\gamma)$, where $\beta_k = \frac{\omega_{k+1}}{\omega_k}$ and $\gamma_k = \frac{\omega_k}{\omega_{k+1}}$ for all $k \in \mathbb{N}$. In this case,*

$$\|T_{a,b,c}\| \simeq \max\{\|\mathbf{a}\|_{\varphi}, \|\mathbf{b}\|_{\varphi,\beta}, \|\mathbf{c}\|_{\varphi,\gamma}\}.$$

where $\|\mathbf{b}\|_{\varphi,\beta}$ is the $\ell^{\varphi}(\beta)$-norm, which is defined similarly to $\ell^{\varphi}(\omega)$, but with the weight ω replaced by the weight β. Similarly, $\|\mathbf{c}\|_{\infty,\gamma}$ is defined in a comparable manner.

Proof. Let us suppose first that $\mathbf{a} \in l^{\varphi}$, $\mathbf{b} \in l^{\varphi}(\beta)$ and $\mathbf{c} \in l^{\varphi}(\gamma)$. As before, we set

$$M = \max\{\|\mathbf{a}\|_{\varphi}, \|\mathbf{b}\|_{\varphi,\beta}, \|\mathbf{c}\|_{\varphi,\gamma}\}.$$

Then for any non-null $\mathbf{x} \in l^{\infty}(\omega)$ we have $T_{a,b,c}(\mathbf{x}) = \mathbf{u} + \mathbf{v} + \mathbf{w}$, where $u_k = b_{k-1}x_{k-1}$, $v_k = a_k x_k$ and $w_k = c_k x_{k+1}$ with $b_0 = x_0 = 0$. Thus,

$$N_{\omega}^{\varphi}\left(\frac{\mathbf{u}}{M\,\|\mathbf{x}\|_{\infty,\omega}}\right) \le \sum_{k=2}^{\infty} \varphi\left(\frac{|b_{k-1}|}{\|\mathbf{b}\|_{\varphi,\beta}} \frac{\omega_k}{\omega_{k-1}} \frac{|x_{k-1}|\omega_{k-1}}{\|\mathbf{x}\|_{\infty,\omega}}\right) \le \sum_{k=2}^{\infty} \varphi\left(\frac{|b_{k-1}|}{\|\mathbf{b}\|_{\varphi,\beta}} \beta_{k-1}\right) \le 1$$

in virtue of relation (5) since $\mathbf{b} \in l^{\varphi}(\beta)$. This prove that $\|\mathbf{u}\|_{\varphi,\omega} \le M\,\|\mathbf{x}\|_{\infty,\omega}$ for all $\mathbf{x} \in l^{\infty}(\omega)$, thanks to the equivalence in (6). In a very similar way, we can see that $\|\mathbf{v}\|_{\varphi,\omega} \le M\,\|\mathbf{x}\|_{\infty,\omega}$ and $\|\mathbf{w}\|_{\varphi,\omega} \le M\,\|\mathbf{x}\|_{\infty,\omega}$ for all $\mathbf{x} \in l^{\infty}(\omega)$, which tells us that $T_{a,b,c}$ is a continuous operator from $l^{\infty}(\omega)$ into $l^{\varphi}(\omega)$.

Suppose now that $T_{a,b,c} : l^{\infty}(\omega) \to l^{\varphi}(\omega)$ is a continuous operator, then there exists $M > 0$ such that $\|T_{a,b,c}(\mathbf{x})\|_{\varphi,\omega} \le M\,\|\mathbf{x}\|_{\infty,\omega}$ for all $\mathbf{x} \in l^{\infty}(\omega)$. We consider the sequence $\mathbf{x} = (x_k)$ defined by

$$x_k = \begin{cases} \dfrac{1}{\omega_k}, & \text{if } k \equiv 1 \pmod 3, \\ 0, & \text{otherwise}, \end{cases}$$

where $k \equiv 1 \pmod 3$ means that the remainder when divide k by 3 is 1. Clearly, $\mathbf{x} \in l^{\infty}(\omega)$, in fact, $\|\mathbf{x}\|_{\infty,\omega} = 1$. Hence, $\|T_{a,b,c}(\mathbf{x})\|_{\varphi,\omega} \le M$ and then

$$\sum_{k=0}^{\infty} \varphi\left(\frac{|a_{3k+1}|}{M}\right) \le 1, \quad \sum_{k=0}^{\infty} \varphi\left(\frac{|b_{3k+1}|}{M}\beta_{3k+1}\right) \le 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \varphi\left(\frac{|c_{3k}|}{M}\gamma_{3k}\right) \le 1. \tag{12}$$

Next, we consider the sequence $\mathbf{y} = (y_k)$ defined by

$$y_k = \begin{cases} \dfrac{1}{\omega_k}, & \text{if } k \equiv 2 \pmod 3, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\|\mathbf{y}\|_{\infty,\omega} = 1$ which implies that $\|T_{a,b,c}(\mathbf{y})\|_{\varphi,\omega} \le M$, and hence,

$$\sum_{k=0}^{\infty} \varphi\left(\frac{|a_{3k+2}|}{M}\right) \le 1, \quad \sum_{k=0}^{\infty} \varphi\left(\frac{|b_{3k+2}|}{M}\beta_{3k+2}\right) \le 1 \quad \text{and} \quad \sum_{k=0}^{\infty} \varphi\left(\frac{|c_{3k+1}|}{M}\gamma_{3k+1}\right) \le 1. \tag{13}$$

Consider the sequence $\mathbf{z} = (z_k)$ defined by

$$z_k = \begin{cases} \dfrac{1}{\omega_k}, & \text{if } k \equiv 0 \pmod 3, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\|\mathbf{z}\|_{\infty,\omega} = 1$, $\|T_{a,b,c}(\mathbf{z})\|_{\varphi,\omega} \le M$, and therefore,

$$\sum_{k=1}^{\infty}\varphi\left(\frac{|a_{3k}|}{M}\right) \le 1, \quad \sum_{k=1}^{\infty}\varphi\left(\frac{|b_{3k}|}{M}\beta_{3k}\right) \le 1 \quad \text{and} \quad \sum_{k=1}^{\infty}\varphi\left(\frac{|c_{3k-1}|}{M}\gamma_{3k-1}\right) \le 1. \tag{14}$$

Finally, from inequalities (12), (13), and (14), we conclude that

$$\sum_{k=1}^{\infty}\frac{1}{3}\varphi\left(\frac{|a_k|}{M}\right) \le 1, \quad \sum_{k=1}^{\infty}\frac{1}{3}\varphi\left(\frac{|b_k|}{M}\beta_k\right) \le 1 \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{1}{3}\varphi\left(\frac{|c_k|}{M}\gamma_k\right) \le 1.$$

The convexity of φ, the fact that $\varphi(0) = 0$ and the relation (5) implies that $\|\mathbf{a}\|_{\varphi} \le 3M$, $\|\mathbf{b}\|_{\varphi,\beta} \le 3M$, and $\|\mathbf{c}\|_{\varphi,\gamma} \le 3M$. The proof is now complete. □

We conclude this section with the following remark:

Remark 1. Our results on the continuity of tridiagonal operators acting between weighted l^{∞} and l^{φ} spaces can be applied to establish the "stability" of certain processes defined by infinite tridiagonal matrices. For example, consider a simple process of the form:

$$\boxed{\text{Input: } \mathbf{x} = (x_k) \in l^{\infty}(\boldsymbol{\omega})} \longrightarrow \boxed{\text{Process: Apply} T_{a,b,c}} \longrightarrow \boxed{\text{Output:} \mathbf{y} = (y_k) \in l^{\varphi}(\omega)}.$$

This process is stable in the sense that small perturbations of the entries lead to slight perturbations in the output if and only if $\mathbf{a} \in l^{\varphi}$, $\mathbf{b} \in l^{\varphi}(\beta)$ and $\mathbf{c} \in l^{\varphi}(\gamma)$.

4 Essential norm of $T_{a,b,c}$ between $l^{\varphi}(\omega)$ and $l^{\infty}(\omega)$ spaces

Next, we shall estimate the essential norm of a continuous tridiagonal operator $T_{a,b,c}$ between $l^{\varphi}(\omega)$ and $l^{\infty}(\omega)$ spaces. We recall that if X and Y are Banach spaces, a linear operator $K : X \to Y$ is said to be compact if the image of the closed unit ball $B_X = \{x \in X : \|x\|_X \le 1\}$ is a relatively compact subset of Y; this means that $\overline{K(B_X)}$ is a compact subset of Y. It is known:

- Each compact operator is also a continuous operator.
- If $\dim(K(X)) < \infty$ (i.e., K has finite rank), then $K : X \to Y$ is a compact operator.
- If $\{K_n\}$ is a sequence of compact operators from X into Y, which satisfying $\lim_{n\to\infty}\|T - K_n\| = 0$, then $T : X \to Y$ is a compact operator.
- If $K : X \to Y$ is a compact operator, then for any sequence $\{\mathbf{x_n}\}$ in X such that $\mathbf{x_n} \to \mathbf{x}$ weakly, it follows that $\|K(\mathbf{x_n}) - K(\mathbf{x})\|_Y \to 0$. This means that every compact operator is also completely continuous.

We refer the excellent book by Conway [17] for more properties of compact operators.

The essential norm of a continuous linear operator $T : X \to Y$ is denoted by $\|T\|_e^{X\to Y}$, and it is the distance to the class $\mathcal{K}(X, Y)$ of all compact operators from X into Y, that is,

$$\|T\|_e^{X\to Y} = \inf\{\|T - K\| : K \in \mathcal{K}(X, Y)\}.$$

In particular, a continuous operator $T : X \to Y$ is compact if and only if $\|T\|_e^{X\to Y} = 0$.

Additionally, for the proof of the main result in this section, it is helpful to recall some facts about finitely additive measures. From the classical article of Yosida and Hewitt [18], we know that a function $\mu : 2^{\mathbb{N}} \to \mathbb{R}$ is a finite signed finitely additive measure if $\mu(\emptyset) = 0$, $\sup_{A\subseteq\mathbb{N}}|\mu(A)| < +\infty$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in 2^{\mathbb{N}}$ such that $A \cap B = \emptyset$. From Jordan-Hahn decomposition theorem, we know that $\mu = \mu_+ - \mu_-$, where μ_- and μ_+ are non-negative and mutually singular, and such measures are unique. The measure $|\mu| = \mu_+ + \mu_-$ is called the variation of μ and the value $\|\mu\|_{ba} = |\mu|(\mathbb{N})$ is called the total variation of μ. The set of all finitely additive measures μ defined on $2^{\mathbb{N}}$ is denoted by ba and it is vector space with the usual sum of finitely additive measures and the usual product by scalars. In fact, it is known that $(ba, \|\cdot\|_{ba})$ is a Banach space, which can be identified with the dual space of l^{∞} (Theorem 2.3 in [18]).

Theorem 3. (Theorem 2.3 [18]) *If $F : l^\infty \to \mathbb{R}$ is a bounded linear functional, then there exists a finitely additive measure μ on $2^{\mathbb{N}}$ such that*

$$F(\mathbf{x}) = \int_{\mathbb{N}} \mathbf{x}\,\mathrm{d}\mu \tag{15}$$

for all $\mathbf{x} \in l^\infty$. Conversely, if $\mu \in ba$, then (15) defines a bounded linear functional on l^∞. In this case, $\|F\| = \|\mu\|_{ba} = |\mu|(\mathbb{N})$.

As an important consequence of the aforementioned result, we have:

Lemma 1. *The normalized canonical sequences $\{\hat{\mathbf{e}}_n\}$ defined by*

$$\hat{\mathbf{e}}_n = \frac{\mathbf{e_n}}{\|\mathbf{e_n}\|_{\infty,\omega}} \tag{16}$$

converges weakly to zero in $l^\infty(\omega)$.

Proof. First, we can see that the multiplication operator M_ω is a surjective linear isometry from $l^\infty(\omega)$ onto l^∞ since $\|M_\omega(\mathbf{x})\|_\infty = \|\mathbf{x}\|_{\infty,\omega}$ for all $\mathbf{x} \in l^\infty(\omega)$. Hence, if G is any bounded linear functional on $l^\infty(\omega)$, then $G \circ M_\omega^{-1}$ is a bounded linear functional on l^∞ and there exists a finitely additive measure μ on $2^{\mathbb{N}}$ such that

$$G(\mathbf{x}) = \int_{\mathbb{N}} \mathbf{x} \cdot \omega\,\mathrm{d}\mu$$

for all $\mathbf{x} \in l^\infty(\omega)$. In particular, for $H \in \mathbb{N}$ fixed, we can consider the sequence $\mathbf{x}_H = (x_{H,k})$ defined by

$$x_{H,k} = \begin{cases} \dfrac{1}{\omega_k}\operatorname{sgn}(\mu(\{k\})), & k \le H, \\ 0, & \text{otherwise}. \end{cases}$$

We have $\|\mathbf{x}_H\|_{\infty,\omega} = 1$ and consequently $\|G\| \ge |G(\mathbf{x})| = \sum_{k=1}^{H} |\mu(\{k\})|$, since μ is finitely additive. This last relation tells us that the series $\sum_{k=1}^{\infty} |\mu(\{k\})|$ is convergent, and therefore, $\lim_{n\to\infty} |\mu(\{n\})| = 0$. Thus, for the normalized canonical sequence,

$$\hat{\mathbf{e}}_n = \frac{\mathbf{e_n}}{\|\mathbf{e_n}\|_{\infty,\omega}},$$

we can write $G(\hat{\mathbf{e}}_n) = \mu(\{n\}) \to 0$ as $n \to \infty$ and the sequence $\{\hat{\mathbf{e}}_n\}$ converges weakly to zero in $l^\infty(\omega)$. □

Now we can state and prove the main result of this section.

Theorem 4. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a Young function satisfying a global Δ_2-condition, and let ω be a weight. Suppose further that $\mathbf{a} = (a_k) \in l^\infty$, $\mathbf{b} = (b_k) \in l^\infty(\beta)$, and $\mathbf{c} = (c_k) \in l^\infty(\gamma)$ with $\beta_k = \frac{\omega_{k+1}}{\omega_k}$ and $\gamma_k = \frac{\omega_k}{\omega_{k+1}}$ for all $k \in \mathbb{N}$. Then the following relations hold*

$$\begin{aligned} \|T_{a,b,c}\|_e^{l^\varphi(\omega)\to l^\varphi(\omega)} &\simeq \limsup_{n\to\infty} |b_n| \frac{\omega_{n+1}}{\omega_n} + \limsup_{n\to\infty} |a_n| + \limsup_{n\to\infty} |c_n| \frac{\omega_n}{\omega_{n+1}} \\ &\simeq \|T_{a,b,c}\|_e^{l^\varphi(\omega)\to l^\infty(\omega)} \simeq \|T_{a,b,c}\|_e^{l^\infty(\omega)\to l^\infty(\omega)}. \end{aligned} \tag{17}$$

Proof. First, we note that the hypotheses imposed on the sequences $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ imply that

$$L = \limsup_{n\to\infty} |b_n| \frac{\omega_{n+1}}{\omega_n} + \limsup_{n\to\infty} |a_n| + \limsup_{n\to\infty} |c_n| \frac{\omega_n}{\omega_{n+1}} < \infty.$$

Next, we shall separately analyze the following cases: (1) $T_{a,b,c}: l^{\varphi}(\omega) \to l^{\varphi}(\omega)$, (2) $T_{a,b,c}: l^{\varphi}(\omega) \to l^{\infty}(\omega)$, and (3) $T_{a,b,c}: l^{\infty}(\omega) \to l^{\infty}(\omega)$:

Case (1) $T_{a,b,c}: l^{\varphi}(\omega) \to l^{\varphi}(\omega)$.

First, we are going to establish the upper bound for $\|T_{a,b,c}\|_e^{l^{\varphi}(\omega)\to l^{\varphi}(\omega)}$. We consider the "truncated" tridiagonal operator defined as follows: for $H \in \mathbb{N}$ fixed, we consider the function $T^{(H)}_{a,b,c}: l^{\varphi}(\omega) \to l^{\varphi}(\omega)$ such that $T^{(H)}_{a,b,c}(\mathbf{x}) = \mathbf{y}^{(H)} = (y_k^{(H)})$ with

$$y_k^{(H)} = b_{k-1}x_{k-1} + a_k x_k + c_k x_{k+1} \tag{18}$$

for $k = 1, 2, \dots, H$ and $y_k^{(H)} = 0$ otherwise. Clearly, $T^{(H)}_{a,b,c}: l^{\varphi}(\omega) \to l^{\varphi}(\omega)$ is a compact operator since $\dim(T^{(H)}_{a,b,c}(l^{\varphi}(\omega))) \le H$, that is, $T^{(H)}_{a,b,c}$ has finite rank. Hence, by the definition of the essential norm, we have

$$\|T_{a,b,c}\|_e^{l^{\varphi}(\omega)\to l^{\varphi}(\omega)} \le \|T_{a,b,c} - T^{(H)}_{a,b,c}\|,$$

for all $H \in \mathbb{N}$. Next, for $H \in \mathbb{N}$, we set

$$M_H = 3\left(\sup_{n\ge H}|b_n|\frac{\omega_{n+1}}{\omega_n} + \sup_{n\ge H}|a_n| + \sup_{n\ge H}|c_n|\frac{\omega_n}{\omega_{n+1}}\right) = 3B_H + 3A_H + 3C_H < \infty.$$

since $\mathbf{a} \in l^{\infty}$, $\mathbf{b} \in l^{\infty}(\beta)$ and $\mathbf{c} \in l^{\infty}(\gamma)$, where $A_H = \sup_{n\ge H}|a_n|$, $B_H = \sup_{n\ge H}|b_n|\frac{\omega_{n+1}}{\omega_n}$, and $C_H = \sup_{n\ge H}|c_n|\frac{\omega_n}{\omega_{n+1}}$. Then, for any nonnull sequence $\mathbf{x} \in l^{\varphi}(\omega)$, we obtain

$$\begin{aligned} N_{\varphi}\left(\frac{(T_{a,b,c} - T^{(H)}_{a,b,c})(\mathbf{x})}{M_H\,\|\mathbf{x}\|_{\varphi,\omega}}\right) &= \sum_{k=H+1}^{\infty}\varphi\left(\frac{|b_{k-1}x_{k-1} + a_k x_k + c_k x_{k+1}|}{M_H\,\|\mathbf{x}\|_{\varphi,\omega}}\omega_k\right) \\ &\le \sum_{k=H+1}^{\infty}\left[\frac{1}{3}\varphi\left(\frac{|b_{k-1}|}{B_H}\frac{|x_{k-1}|}{\|\mathbf{x}\|_{\varphi,\omega}}\omega_k\right) + \frac{1}{3}\varphi\left(\frac{|x_k|}{\|\mathbf{x}\|_{\varphi,\omega}}\omega_k\right) + \frac{1}{3}\varphi\left(\frac{|c_k|}{C_H}\frac{|x_{k+1}|}{\|\mathbf{x}\|_{\varphi,\omega}}\omega_k\right)\right], \end{aligned}$$

where we have used that φ is an increasing and convex function and the definition of M_H. Thus, arguing as in the proof of the continuity, we obtain that

$$N_{\varphi}\left(\frac{(T_{a,b,c} - T^{(H)}_{a,b,c})(\mathbf{x})}{M_H\,\|\mathbf{x}\|_{\varphi,\omega}}\right) \le N^{\varphi}_{\omega}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_{\varphi,\omega}}\right) \le 1,$$

and

$$\|(T_{a,b,c} - T^{(H)}_{a,b,c})(\mathbf{x})\|_{\varphi,\omega} \le M_H\,\|\mathbf{x}\|_{\varphi,\omega}$$

for all $\mathbf{x} \in l^{\varphi}(\omega)$ and for all $H \in \mathbb{N}$ in virtue of (6). This last relation tells us that $\|T_{a,b,c} - T^{(H)}_{a,b,c}\| \le M_H$ for all $H \in \mathbb{N}$, and therefore,

$$\begin{aligned} \|T_{a,b,c}\|_e^{l^{\varphi}(\omega)\to l^{\varphi}(\omega)} &\le \|T_{a,b,c} - T^{(H)}_{a,b,c}\| \le M_H \\ &\le 3\left(\sup_{n\ge H}|b_n|\frac{\omega_{n+1}}{\omega_n} + \sup_{n\ge H}|a_n| + \sup_{n\ge H}|c_n|\frac{\omega_n}{\omega_{n+1}}\right). \end{aligned}$$

Thus, by taking the limit as H tends to infinity, we obtain

$$\|T_{a,b,c}\|_e^{l^{\varphi}(\omega)\to l^{\varphi}(\omega)} \le 3\left(\limsup_{n\to\infty}|b_n|\frac{\omega_{n+1}}{\omega_n} + \limsup_{n\to\infty}|a_n| + \limsup_{n\to\infty}|c_n|\frac{\omega_n}{\omega_{n+1}}\right) < \infty,$$

which gives us a desired upper bound for $\|T_{a,b,c}\|_e^{l^{\varphi}(\omega)\to l^{\varphi}(\omega)}$.

Next we shall establish the lower bound for $\|T_{a,b,c}\|_e^{l^{\varphi}(\omega)\to l^{\varphi}(\omega)}$. Let us consider the normalized canonical sequences $\{\hat{\mathbf{e}}_n\}$ defined by

$$\hat{\mathbf{e}}_n = \frac{\mathbf{e_n}}{\|\mathbf{e_n}\|_{\varphi,\omega}} = \frac{\varphi^{-1}(1)}{\omega_n}\mathbf{e_n},$$

where $n \geq 1$. Observe that if G is any bounded linear functional on $l^{\varphi}(\omega)$, then by Riesz's representation theorem (10) there exists $\mathbf{y} \in l^{\psi}(\omega)$ such that

$$G(\mathbf{x}) = \sum_{k=1}^{\infty} x_k y_k \, \omega_k^2$$

for all $\mathbf{x} \in l^{\varphi}(\omega)$. In particular, for each $n \in \mathbb{N}$, we have

$$G(\hat{\mathbf{e}}_n) = \varphi^{-1}(1) y_n \omega_n \to 0$$

as $n \to \infty$ since $l^{\psi}(\omega) \subset c_0(\omega)$. This means that the sequence $\{\hat{\mathbf{e}}_n\}$ converges weakly to zero, and therefore, $\lim_{n\to\infty} \|K(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} = 0$ for all compact operator $K : l^{\varphi}(\omega) \to l^{\varphi}(\omega)$. Thus, it is enough to estimate $\|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega}$ for all $n \in \mathbb{N}$. If $|a_n| \neq 0$, the definition of $T_{a,b,c}(\hat{\mathbf{e}}_n)$ tells us that

$$N_{\omega}^{\varphi}\left(\frac{T_{a,b,c}(\hat{\mathbf{e}}_n)}{|a_n|}\right) \geq \varphi\left(\frac{|a_n|}{|a_n| \|\mathbf{e_n}\|_{\varphi,\omega}} \omega_n\right) = 1,$$

which implies that $\|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq |a_n|$ for all $n \in \mathbb{N}$ in virtue of (6). In similar way, if $|b_n| \neq 0$, then we can write

$$N_{\omega}^{\varphi}\left(\frac{T_{a,b,c}(\hat{\mathbf{e}}_n)}{|b_n|} \frac{\omega_n}{\omega_{n+1}}\right) \geq \varphi\left(\frac{|b_n|}{|b_n| \|\mathbf{e_n}\|_{\varphi,\omega}} \frac{\omega_n}{\omega_{n+1}} \omega_{n+1}\right) = 1,$$

and hence, $\|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq |b_n| \frac{\omega_{n+1}}{\omega_n}$ for all $n \in \mathbb{N}$. Furthermore, if $n \geq 2$ and $|c_{n-1}| \neq 0$, then we have

$$N_{\omega}^{\varphi}\left(\frac{T_{a,b,c}(\hat{\mathbf{e}}_n)}{|c_{n-1}|} \frac{\omega_n}{\omega_{n-1}}\right) \geq \varphi\left(\frac{|c_{n-1}|}{|c_{n-1}| \|\mathbf{e_n}\|_{\varphi,\omega}} \frac{\omega_n}{\omega_{n-1}} \omega_{n-1}\right) = 1,$$

and therefore,

$$\|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq |c_{n-1}| \frac{\omega_{n-1}}{\omega_n}$$

for all $n \geq 2$. Hence, for any $H \in \mathbb{N}$ fixed, we obtain

$$\sup_{n\geq H} \|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq \frac{1}{3}\left(\sup_{n\geq H} |a_n| + \sup_{n\geq H} |b_n| \frac{\omega_{n+1}}{\omega_n} + \sup_{n\geq H} |c_{n-1}| \frac{\omega_{n-1}}{\omega_n}\right).$$

Thus, since $\|\hat{\mathbf{e}}_n\|_{\varphi,\omega} = 1$ for all $n \in \mathbb{N}$, for any compact operator $K : l^{\varphi}(\omega) \to l^{\varphi}(\omega)$, we have

$$\|T_{a,b,c} - K\| \geq \|(T_{a,b,c} - K)(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq \|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} - \|K(\hat{\mathbf{e}}_n)\|_{\varphi,\omega}.$$

Hence, for any $H \in \mathbb{N}$ fixed, we obtain

$$\begin{aligned} &\frac{1}{3}\left(\sup_{n\geq H} |a_n| + \sup_{n\geq H} |b_n| \frac{\omega_{n+1}}{\omega_n} + \sup_{n\geq H} |c_{n-1}| \frac{\omega_{n-1}}{\omega_n}\right) \\ &\quad \leq \sup_{n\geq H} \|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \leq \|T_{a,b,c} - K\| + \sup_{n\geq H} \|K(\hat{\mathbf{e}}_n)\|_{\varphi,\omega}, \end{aligned}$$

and taking limit when H goes to infinity

$$\|T_{a,b,c}\|_e^{l^{\varphi}(\omega)\to l^{\varphi}(\omega)} \geq \frac{1}{3}\left(\limsup_{n\to\infty} |b_n| \frac{\omega_{n+1}}{\omega_n} + \limsup_{n\to\infty} |a_n| + \limsup_{n\to\infty} |c_n| \frac{\omega_n}{\omega_{n+1}}\right).$$

This proves the first estimation.

Case (2) $T_{a,b,c} : l^{\varphi}(\omega) \to l^{\infty}(\omega)$.

This estimations is similar to the above one, but we have to change the norms. For example, for the estimation of $\|T_{a,b,c}\|_e^{l^{\varphi}(\omega)\to l^{\infty}(\omega)}$, we have

$$\|(T_{a,b,c} - T_{a,b,c}^{(H)})(\mathbf{x})\|_{\infty,\omega} = \sup_{k\geq H+1} |b_{k-1}x_{k-1} + a_k x_k + c_k x_{k+1}| \omega_k$$

for all $\mathbf{x} \in l^{\varphi}(\omega)$ and for all $H \in \mathbb{N}$, and we obtain

$$\|(T_{a,b,c} - T_{a,b,c}^{(H)})(\mathbf{x})\|_{\infty,\omega} \le \varphi^{-1}(1)\left(\sup_{k\ge H}|a_k| + \sup_{k\ge H+1}|b_k|\beta_k + \sup_{k\ge H+1}|c_k|\gamma_k\right)\|\mathbf{x}\|_{\varphi,\omega},$$

where we use the relation (7). While for any $n \in \mathbb{N}$, we have that the sequence $\{\hat{\mathbf{e}}_n\}$ converges weakly to zero and

$$\|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\infty,\omega} = \varphi^{-1}(1)\max\{|c_{n-1}|\gamma_{n-1}, |a_n|, |b_n|\beta_n\}.$$

The estimation of the essential norm of $T_{a,b,c} : l^{\varphi}(\omega) \to l^{\infty}(\omega)$ follows as in the previous case.
Case (3) $T_{a,b,c} : l^{\infty}(\omega) \to l^{\infty}(\omega)$.

For simplicity, in this subsection, we work with real sequences. We will estimate the essential norm of $T_{a,b,c} : l^{\infty}(\omega) \to l^{\infty}(\omega)$. The argument in the previous cases tells us that for any $\mathbf{x} \in l^{\infty}(\omega)$ and $H \in \mathbb{N}$ fixed, we have

$$\|(T_{a,b,c} - T_{a,b,c}^{(H)})(\mathbf{x})\|_{\infty.\omega} \le \left(\sup_{k\ge H+1}|a_k| + \sup_{k\ge H+1}|b_{k-1}|\beta_{k-1} + \sup_{k\ge H+1}|c_k|\gamma_k\right)\|\mathbf{x}\|_{\infty,\omega},$$

and hence, taking limit when H goes to infinity, we obtain the upper bound

$$\|T_{a,b,c}\|_e^{l^{\infty}(\omega)\to l^{\infty}(\omega)} \le \limsup_{n\to\infty}|b_n|\beta_n + \limsup_{n\to\infty}|a_n| + \limsup_{n\to\infty}|c_n|\gamma_n.$$

While Lemma 1 tells us that if $K : l^{\infty}(\omega) \to l^{\infty}(\omega)$ is any compact operator, then $\|K(\hat{\mathbf{e}}_n)\|_{\infty,\omega} \to 0$ as $n \to \infty$. In addition, we also have

$$\|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\infty,\omega} = \max\{|c_{n-1}|\gamma_{n-1}, |a_n|, |b_n|\beta_n\},$$

and therefore,

$$\|T_{a,b,c}\|_e^{l^{\infty}(\omega)\to l^{\infty}(\omega)} \ge \frac{1}{3}\left(\limsup_{n\to\infty}|b_n|\frac{\omega_{n+1}}{\omega_n} + \limsup_{n\to\infty}|a_n| + \limsup_{n\to\infty}|c_n|\frac{\omega_n}{\omega_{n+1}}\right).$$

This conclude the proof of Theorem 4. □

4.1 Case $T_{a,b,c} : l^{\infty}(\omega) \to l^{\varphi}(\omega)$

Next we analyze the remain case of a continuous tridiagonal operator acting from $l^{\infty}(\omega)$ into $l^{\varphi}(\omega)$. Here, continuity and compactness are equivalent such as we can see in the following result.

Theorem 5. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a Young function and satisfying a global Δ_2-condition, and let ω be a weight. For fixed sequences $\mathbf{a} = (a_k)$, $\mathbf{b} = (b_k)$, and $\mathbf{c} = (c_k)$, the following are equivalent:*
(1) *the tridiagonal operator $T_{a,b,c}$ is continuous from $l^{\infty}(\omega)$ into $l^{\varphi}(\omega)$,*
(2) *the tridiagonal operator $T_{a,b,c}$ is compact from $l^{\infty}(\omega)$ into $l^{\varphi}(\omega)$,*
(3) $\mathbf{a} \in l^{\varphi}$, $\mathbf{b} \in l^{\varphi}(\beta)$, *and* $\mathbf{c} \in l^{\varphi}(\gamma)$, *where* $\beta_k = \frac{\omega_{k+1}}{\omega_k}$ *and* $\gamma_k = \frac{\omega_k}{\omega_{k+1}}$ *for all* $k \in \mathbb{N}$.

Proof. We know that every compact operator is a continuous operator, and hence, Theorem 2 tells us that it is enough to prove the implication [(3) ⇒ (2)]. We can assume that none of the sequences $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the null sequence. Also, we set

$$M = \max\{\|\mathbf{a}\|_{\varphi}, \|\mathbf{b}\|_{\varphi,\beta}, \|\mathbf{c}\|_{\varphi,\gamma}\}.$$

As mentioned earlier, for $H \in \mathbb{N}$ fixed, we consider the truncated tridiagonal operator $T_{a,b,c}^{(H)} : l^{\infty}(\omega) \to l^{\varphi}(\omega)$ defined as in equation (18), which is a compact operator since it has finite rank. We are going to prove that

$$\lim_{H\to\infty} \|T_{a,b,c} - T_{a,b,c}^{(H)}\| = 0.$$

To see this, we consider the sequence

$$\mathbf{a}^{(H)} = (a_k^{(H)}) = (0, \ldots, 0, a_{H+1}, a_{H+2}, \ldots).$$

That is, $a_k^{(H)} = a_k$ when $k \geq H+1$ and $a_k^{(H)} = 0$ otherwise. Similarly, we define $\mathbf{b}^{(H)} = (b_k^{(H)}) = (0, \ldots, 0, b_H, b_{H+1}, \ldots)$ and

$$\mathbf{c}^{(H)} = (c_k^{(H)}) = (0, \ldots, 0, c_{H+1}, c_{H+2}, \ldots).$$

Hypothesis (3) tells us that $\mathbf{a}^{(H)} \in l^{\varphi}$, $\mathbf{b}^{(H)} \in l^{\varphi}(\beta)$, and $\mathbf{c}^{(H)} \in l^{\varphi}(\gamma)$. We also consider

$$M_H = \max\{\|\mathbf{a}^{(H)}\|_{\varphi}, \|\mathbf{b}^{(H)}\|_{\varphi,\beta}, \|\mathbf{c}^{(H)}\|_{\varphi,\gamma}\},$$

and we can suppose that $M_H > 0$. Then for any non-null sequence $\mathbf{x} = (x_k) \in l^{\infty}(\omega)$, we have

$$N_{\omega}^{\varphi}\left(\frac{(T_{a,b,c} - T_{a,b,c}^{(H)})(\mathbf{x})}{3M_H\|\mathbf{x}\|_{\infty,\omega}}\right) \leq \frac{1}{3}\sum_{k=H+1}^{\infty}\varphi\left(\frac{|b_{k-1}|}{\|\mathbf{b}^{(H)}\|_{\varphi,\beta}}\beta_{k-1}\right) + \frac{1}{3}\sum_{k=H+1}^{\infty}\varphi\left(\frac{|a_k|}{\|\mathbf{a}^{(H)}\|_{\varphi}}\right) + \frac{1}{3}\sum_{k=H+1}^{\infty}\varphi\left(\frac{|c_k|}{\|\mathbf{c}^{(H)}\|_{\varphi,\gamma}}\gamma_k\right) \leq 1$$

and therefore, the equivalence (6) gives us $\|T_{a,b,c} - T_{a,b,c}^{(H)}\| \leq 3M_H$ for all $H \in \mathbb{N}$. Furthermore, for $\varepsilon > 0$ arbitrary, the global Δ_2-condition allows us to find a constant $K > 0$ such that

$$\varphi\left(\frac{M}{\varepsilon}t\right) \leq K\varphi(t)$$

for all $t \geq 0$, and then the fact that $\mathbf{b}^{(H)} \in l^{\varphi}(\beta)$ implies that there exists $H_0 \in \mathbb{N}$ such that

$$K\sum_{k=H+1}^{\infty}\varphi\left(\frac{|b_k|}{\|\mathbf{b}\|_{\varphi,\beta}}\beta_k\right) \leq 1$$

for all $H \geq H_0$. Hence,

$$N_{\beta}^{\varphi}\left(\frac{\mathbf{b}^{(H)}}{\varepsilon}\right) \leq \sum_{k=H+1}^{\infty}\varphi\left(\frac{M}{\varepsilon}\frac{|b_k|}{\|\mathbf{b}\|_{\varphi,\beta}}\beta_k\right) \leq K\sum_{k=H+1}^{\infty}\varphi\left(\frac{|b_k|}{\|\mathbf{b}\|_{\varphi,\beta}}\beta_k\right) \leq 1$$

for all $H \geq H_0$. Therefore, $\lim_{H\to\infty}\|\mathbf{b}^{(H)}\|_{\varphi,\beta} = 0$. In a similar way, we also have $\|\mathbf{a}^{(H)}\|_{\varphi} \to 0$ and $\|\mathbf{c}^{(H)}\|_{\varphi,\gamma} \to 0$ as $H \to \infty$, and we conclude that

$$\lim_{H\to\infty} \|T_{a,b,c} - T_{a,b,c}^{(H)}\| = 0,$$

which means that $T_{a,b,c} : l^{\infty}(\omega) \to l^{\varphi}(\omega)$ is a compact operator and the proof is complete. □

5 Another kind of weighted Orlicz sequence spaces

There is another way to define weighted Orlicz sequence spaces as we have done in Section 2. In this case, the weight ω appears outside of the argument of the function φ. That is, we consider the Orlicz space obtained when we use the σ-additive weighted counting measure on $2^{\mathbb{N}}$ defined by $\mu(\{n\}) = \omega_n$ for all $n \in \mathbb{N}$. More precisely, using the same notations as in Section 2, given a Young function φ, a weight $\boldsymbol{\omega}$, and a complex sequence $\mathbf{x} = (x_k)$, we set

$$N_{\omega}^{\varphi}(\mathbf{x}) = \sum_{k=1}^{\infty}\varphi(|x_k|)\omega_k = \lim_{m\to\infty}\sum_{k=1}^{m}\varphi(|x_k|)\omega_k, \tag{19}$$

and then we said that $\mathbf{x}$ belongs to the weighted Orlicz sequence space $l_{\varphi}(\omega)$ if there exists a $\lambda > 0$ such that $N_{\omega}^{\varphi}\left(\frac{\mathbf{x}}{\lambda}\right) \leq 1$. This kind of weighted Orlicz sequence spaces has been considered by some researchers such as

Fuentes and Hernández [19], Hudzik et al. [20], and Zlatanov [21]. As mentioned earlier, it can be seen that the relation

$$\|\mathbf{x}\|_{\varphi,\omega} = \inf\left\{\lambda > 0 : N_\omega^\varphi\left(\frac{\mathbf{x}}{\lambda}\right) \le 1\right\} \tag{20}$$

is a norm for $l_\varphi(\omega)$ and the pair $(l_\varphi(\omega), \|\cdot\|_{\varphi,\omega})$ is a Banach space. In fact, we can see that

$$N_\omega^\varphi\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_{\varphi,\omega}}\right) \le 1.$$

for all non-null sequence $\mathbf{x} \in l_\varphi(\omega)$, which allows us to conclude that

$$\|\mathbf{x}\|_{\varphi,\omega} \le 1 \;\Leftrightarrow\; N_\omega^\varphi(\mathbf{x}) \le 1$$

for all $\mathbf{x} \in l_\varphi(\omega)$. In particular, for any $\mathbf{x} = (x_k) \in l_\varphi(\omega)$, we have

$$\lim_{k\to\infty} \frac{|x_k|}{\varphi^{-1}\left(\frac{1}{\omega_k}\right)} = 0,$$

which means that, in this case, $l_\varphi(\omega)$ is continuously embedded in the weighted space $c_0(\delta)$ with

$$\delta_k = \frac{1}{\varphi^{-1}\left(\frac{1}{\omega_k}\right)} = \|\mathbf{e_k}\|_{\varphi,\omega} \tag{21}$$

for all $k \in \mathbb{N}$. Hence, $\|\mathbf{e_k}\|_{\varphi,\omega} \to 0$ if and only if $\omega_k \to 0$. Also, in this context, it is common to impose conditions on the weight ω and on the Young function φ to get genuine extensions or generalizations of the properties of the classical Orlicz sequence space l_φ (without weight). For instance, Fuentes and Hernández in [19] consider Young functions satisfying a global Δ_2-condition and weights ω such that

$$\liminf_{n\to\infty} \frac{1}{\omega_n} \sum_{k=n+1}^{\infty} \omega_k > 0,$$

and they characterize when $l_\varphi(\omega)$ contain isomorphic copies of c_0 and l^∞. In our case, we consider seminormalized weights, and we have the following result:

Theorem 6. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a Young function and let ω be a weight. Suppose that $\boldsymbol{\omega} = (\omega_k)$ is seminormalized in the sense that*

$$1 \le \frac{\omega_{k-1}}{\omega_k} \le 2$$

for all $k \in \mathbb{N}$. For fixed sequences $\mathbf{a} = (a_k)$, $\mathbf{b} = (b_k)$, and $\mathbf{c} = (c_k)$, the tridiagonal operator $T_{a,b,c}$ is continuous from $l_\varphi(\omega)$ into itself if and only if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in l^\infty$.

Proof. Let us suppose first that the tridiagonal operator $T_{a,b,c} : l_\varphi(\omega) \to l_\varphi(\omega)$ is continuous. There exists a constant $M > 0$ such that

$$\left\|\frac{T_{a,b,c}(\mathbf{e_n})}{M\,\|\mathbf{e_n}\|_{\varphi,\omega}}\right\|_{\varphi,\omega} \le 1$$

for all $n \in \mathbb{N}$. Thus, by relation (20), we have

$$N_\omega^\varphi\left(\frac{T_{a,b,c}(\mathbf{e_n})}{M\,\|\mathbf{e_n}\|_{\varphi.\omega}}\right) \le 1.$$

Hence, the definition of $T_{a,b,c}(\mathbf{e_n})$ (expression (1)) and the definition of N_ω^φ in (19) imply that

$$\varphi\left(\frac{|c_{n-1}|}{M\,\|\mathbf{e_n}\|_{\varphi,\omega}}\right)\omega_{n-1} \le 1, \quad \varphi\left(\frac{|a_n|}{M\,\|\mathbf{e_n}\|_{\varphi,\omega}}\right)\omega_n \le 1 \quad \text{and} \quad \varphi\left(\frac{|b_n|}{M\,\|\mathbf{e_n}\|_{\varphi,\omega}}\right)\omega_{n+1} \le 1.$$

Now, since φ is an increasing function, from the first inequality, we obtain

$$|c_{n-1}| \le M\,\|\mathbf{e_n}\|_{\varphi,\omega}\varphi^{-1}\left(\frac{1}{\omega_{n-1}}\right) = M\frac{\varphi^{-1}\left(\frac{1}{\omega_{n-1}}\right)}{\varphi^{-1}\left(\frac{1}{\omega_n}\right)} \le M,$$

for all $n \ge 2$, where we have used the relation (21), the fact that φ^{-1} is an increasing function and the hypothesis that the weight ω is a decreasing sequence. This last inequality means that $\mathbf{c} \in l^\infty$.

In similar way, we obtain $\mathbf{a} \in l^\infty$, and by relation (21), we can write

$$|b_n| \le M\,\|\mathbf{e_n}\|_{\varphi,\omega}\varphi^{-1}\left(\frac{1}{\omega_{n+1}}\right) = M\frac{\varphi^{-1}\left(\frac{1}{\omega_{n+1}}\right)}{\varphi^{-1}\left(\frac{1}{\omega_n}\right)},$$

for all $n \in \mathbb{N}$. But φ is a convex function such that $\varphi(0) = 0$, and then for $0 < t_1 < t_2$, we set $\lambda = \frac{t_1}{t_2} \in (0,1)$, and we have

$$\frac{\varphi(t_1)}{t_1} = \frac{\varphi(\lambda t_2)}{t_1} \le \lambda\frac{\varphi(t_2)}{t_1} = \frac{\varphi(t_2)}{t_2},$$

that is, for all $0 < t_1 < t_2$, we have

$$\frac{t_2}{t_1} \le \frac{\varphi(t_2)}{\varphi(t_1)}.$$

This last in turn implies that for any $n \in \mathbb{N}$ we have

$$|b_n| \le M\frac{\varphi^{-1}\left(\frac{1}{\omega_{n+1}}\right)}{\varphi^{-1}\left(\frac{1}{\omega_n}\right)} \le \frac{\omega_{k-1}}{\omega_k} \le 2$$

in virtue of the normalized condition of ω and $\mathbf{b} \in l^\infty$. This conclude the proof of the implication.

Let us suppose now that $\mathbf{a} \in l^\infty$, $\mathbf{b} \in l^\infty$ and $\mathbf{c} \in l^\infty$. We set

$$M = 6(\|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty + \|\mathbf{c}\|_\infty).$$

Then any non-null $\mathbf{x} \in l^\varphi(\omega)$, we can write

$$\begin{aligned}
N_\omega^\varphi\left(\frac{|T_{a,b,c}(\mathbf{x})|}{M\,\|\mathbf{x}\|_{\varphi,\omega}}\right) &\le \sum_{k=1}^\infty \varphi\left(\frac{|b_{k-1}x_{k-1}| + |a_k x_k| + |c_k x_{k+1}|}{M\,\|\mathbf{x}\|_{\varphi,\omega}}\right)\omega_k \\
&\le \frac{1}{6}\sum_{k=1}^\infty \varphi\left(\frac{|x_{k-1}|}{\|\mathbf{x}\|_{\varphi,\omega}}\right)\omega_k + \frac{1}{6}\sum_{k=1}^\infty \varphi\left(\frac{|x_k|}{\|\mathbf{x}\|_{\varphi,\omega}}\right)\omega_k + \frac{1}{6}\sum_{k=1}^\infty \varphi\left(\frac{|x_{k+1}|}{\|\mathbf{x}\|_{\varphi,\omega}}\right)\omega_k \\
&\le \frac{1}{6}\sum_{k=2}^\infty \varphi\left(\frac{|x_{k-1}|}{\|\mathbf{x}\|_{\varphi,\omega}}\right)\omega_{k-1} + \frac{1}{6}\sum_{k=1}^\infty \varphi\left(\frac{|x_k|}{\|\mathbf{x}\|_{\varphi,\omega}}\right)\omega_k + \frac{2}{6}\sum_{k=1}^\infty \varphi\left(\frac{|x_{k+1}|}{\|\mathbf{x}\|_{\varphi,\omega}}\right)\omega_{k+1} \\
&\le \frac{4}{6}N_\omega^\varphi\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_{\varphi,\omega}}\right) \le 1,
\end{aligned}$$

where we have used that φ is an increasing function in the first inequality. In the second inequality, we have used that φ is a convex function and $\varphi(0) = 0$. In the third inequality, we have used the seminormalization of the weight ω. We conclude that $\|T_{a,b,c}(\mathbf{x})\|_{\varphi,\omega} \le M\,\|\mathbf{x}\|_{\varphi,\omega}$ for all $\mathbf{x} \in l_\varphi(\omega)$ and $T_{a,b,c}$ is continuous from $l_\varphi(\omega)$ into itself. □

Remark 2. It is important to note that the seminormalized condition imposed on the weight ω implies that the associated weights $\beta = (\beta_k)$ and $\gamma = (\gamma_k)$ defined by $\beta_k = \frac{\omega_{k+1}}{\omega_k}$ and $\gamma_k = \frac{\omega_k}{\omega_{k+1}}$ for all $k \in \mathbb{N}$, are also seminormalized, and therefore, the conclusion in the aforementioned theorem is also $\mathbf{a} \in l^\infty$, $\mathbf{b} \in l^\infty(\beta)$ and $\mathbf{c} \in l^\infty(\gamma)$ such as occurs in Theorem 1.

Next, we are going to estimate the essential norm of a tridiagonal operator acting on $l_\varphi(\omega)$ spaces. We need some information about the dual of $l_\varphi(\omega)$. We recall that the space $l_\varphi(\omega)$ is obtained from the Orlicz function space $L^\varphi(\Omega, \Sigma, \mu)$ when we consider the set $\Omega = \mathbb{N}$, the σ-algebra $\Sigma = 2^{\mathbb{N}}$, and the σ-additive measure μ defined by $\mu(\{n\}) = \omega_n$ for all $n \in \mathbb{N}$. Thus, it is known that if φ satisfies the global Δ_2-condition, then the dual space of $l_\varphi(\omega)$ can be identified with $l_\psi(\omega)$ defined by the complementary function ψ of φ (defined in (9)). In fact, in this case, for each $T \in (l_\varphi(\omega))'$, there exists an unique $\mathbf{y} = (y_k) \in l_\psi(\omega)$ such that

$$T(\mathbf{x}) = \sum_{k=1}^{\infty} x_k y_k \omega_k \tag{22}$$

for all $\mathbf{x} = (x_k) \in l_\varphi(\omega)$. In this case $\|T\| \simeq \|\mathbf{y}\|_{\psi,\omega}$. In particular, we have the following result:

Lemma 2. *Let φ be a Young function satisfying a global Δ_2-condition, ψ its complementary function, and let ω be a weight. The normalized canonical sequences $\{\hat{\mathbf{e}}_n\}$ defined by*

$$\hat{\mathbf{e}}_n = \frac{\mathbf{e_n}}{\|\mathbf{e_n}\|_{\varphi,\omega}} \tag{23}$$

converge weakly to zero in $l_\varphi(\omega)$.

Proof. Indeed, since φ is a Young function satisfying a global Δ_2-condition, then for each $T \in (l_\varphi(\omega))'$, we can find a $\mathbf{y} \in l_\psi(\omega)$ such that the expression (22) hold for all $\mathbf{x} \in l_\varphi(\omega)$ and $\|T\| \simeq \|\mathbf{y}\|_{\psi,\omega}$. In particular, for all $n \in \mathbb{N}$, we have

$$|T(\hat{\mathbf{e}}_n)| = |y_n| \frac{\omega_n}{\|\mathbf{e_n}\|_{\varphi,\omega}}.$$

Furthermore, we note that the sequence $\{\mathbf{f_n}\}$ defined by $\mathbf{f_n} = \frac{\mathbf{e_n}}{\omega_n}$ belongs to $l_\psi(\omega)$, and hence, it defines a bounded functional linear T_n on $l_\varphi(\omega)$, which coincides with the evaluation functional at n (see (8)). Thus, we can write

$$\frac{1}{\omega_n} \|\mathbf{e_n}\|_{\psi,\omega} = \|\mathbf{f_n}\|_{\psi,\omega} \simeq \|T_n\| = \frac{1}{\|\mathbf{e_n}\|_{\varphi,\omega}}.$$

Hence, we obtain $|T(\hat{\mathbf{e}}_n)| \simeq |y_n| \|\mathbf{e_n}\|_{\psi,\omega}$, and the result follows since $l_\psi(\omega)$ is continuously embedded in the weighted space $c_0(\delta)$ with

$$\delta_k = \frac{1}{\psi^{-1}\left(\frac{1}{\omega_k}\right)} = \|\mathbf{e_k}\|_{\psi,\omega}$$

for all $k \in \mathbb{N}$. □

Next we will see that under certain reasonable conditions, Theorem 4 can be extended to the weighted Orlicz sequence spaces defined in this section

Theorem 7. *Let φ be a Young function satisfying a global Δ_2-condition, and let ω be a seminormalized weight in the sense that*

$$1 \leq \frac{\omega_{k-1}}{\omega_k} \leq 2 \tag{24}$$

for all $k \in \mathbb{N}$. *Suppose further that* $\mathbf{a} = (a_k) \in l^\infty$, $\mathbf{b} = (b_k) \in l^\infty$, *and* $\mathbf{c} = (c_k) \in l^\infty$. *Then the following relations hold*:

$$\|T_{a,b,c}\|_e^{l_\varphi(\omega)\to l_\varphi(\omega)} \simeq \limsup_{n\to\infty}|a_n| + \limsup_{n\to\infty}|b_n| + \limsup_{n\to\infty}|c_n|. \tag{25}$$

Proof. The proof is similar to the one given in Theorem 4. By Lemma 2, we know that the normalized canonical sequence $\{\hat{\mathbf{e}}_n\}$ defined in (23) converges weakly to zero, and hence, for any compact operator $K : l_\varphi(\omega) \to l_\varphi(\omega)$, we can write

$$\|T_{a,b,c} - K\| \geq \limsup_{n\to\infty} \|(T_{a,b,c} - K)(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq \limsup_{n\to\infty} \|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega}$$

since $\lim_{n\to\infty} \|K(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} = 0$. Next, we can see that for each $n \geq 2$ and $|c_{n-1}| \neq 0$, we have

$$N_\omega^\varphi\left(\frac{T_{a,b,c}(\mathbf{e_n})}{|c_{n-1}|\|\mathbf{e_n}\|_{\varphi,\omega}}\right) \geq \varphi\left(\frac{1}{\|\mathbf{e_n}\|_{\varphi,\omega}}\right)\omega_{n-1} = \frac{\omega_{n-1}}{\omega_n} \geq 1,$$

since the weight ω is a decreasing sequence. Hence, $\|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq |c_{n-1}|$ for all $n \geq 2$, and therefore,

$$\limsup_{n\to\infty} \|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq \limsup_{n\to\infty}|c_n|.$$

In similar way, we have $\|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq |a_n|$ for all $n \geq 1$, and

$$\limsup_{n\to\infty} \|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq \limsup_{n\to\infty}|a_n|.$$

While for $n \geq 1$ and $|b_n| \neq 0$, the convexity of φ allows us to write

$$N_\omega^\varphi\left(\frac{2T_{a,b,c}(\mathbf{e_n})}{|b_n|\|\mathbf{e_n}\|_{\varphi,\omega}}\right) \geq 2\varphi\left(\frac{1}{\|\mathbf{e_n}\|_{\varphi,\omega}}\right)\omega_{n+1} = 2\frac{\omega_{n+1}}{\omega_n} \geq 1,$$

since ω satisfies the normalized condition in (24). Thus, $\|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq \frac{1}{2}|b_n|$ for all $n \geq 1$ and

$$\limsup_{n\to\infty} \|T_{a,b,c}(\hat{\mathbf{e}}_n)\|_{\varphi,\omega} \geq \frac{1}{2}\limsup_{n\to\infty}|b_n|.$$

Therefore,

$$\|T_{a,b,c}\|_e^{l_\varphi(\omega)\to l_\varphi(\omega)} \geq \frac{1}{2}\left(\limsup_{n\to\infty}|a_n| + \limsup_{n\to\infty}|b_n| + \limsup_{n\to\infty}|c_n|\right),$$

which gives us the lower bound for the essential norm of $T_{a,b,c} : l_\varphi(\omega) \to l_\varphi(\omega)$.

Next we shall get the upper bound. Arguing as the proof of Theorem 5, we consider the truncated tridiagonal operator $T_{a,b,c}^{(H)} : l_\varphi(\omega) \to l_\varphi(\omega)$ defined as in equation (18), where $H \in \mathbb{N}$ is fixed. We consider the sequence $\mathbf{a}^{(H)} = (a_k^{(H)})$ defined by $a_k^{(H)} = a_k$ when $k \geq H+1$ and $a_k^{(H)} = 0$ otherwise. Similarly, we define $\mathbf{b}^{(H)} = (b_k^{(H)}) = (0, \dots, 0, b_H, b_{H+1}, \dots)$ and $\mathbf{c}^{(H)} = (c_k^{(H)}) = (0, \dots, 0, c_{H+1}, c_{H+2}, \dots)$, and we also consider

$$M_H = \max\{\|\mathbf{a}^{(H)}\|_\infty, \|\mathbf{b}^{(H)}\|_\infty, \|\mathbf{c}^{(H)}\|_\infty\}.$$

We can suppose that $M_H > 0$. Then for any nonnull sequence $\mathbf{x} = (x_k) \in l_\varphi(\omega)$, we have

$$\begin{aligned} N_\omega^\varphi\left(\frac{(T_{a,b,c} - T_{a,b,c}^{(H)})(\mathbf{x})}{6M_H\ \|\mathbf{x}\|_{\varphi,\omega}}\right) &\leq \frac{1}{3}\sum_{k=H+1}^\infty \varphi\left(\frac{|x_{k-1}|}{2\ \|\mathbf{x}\|_{\varphi,\omega}}\right)\omega_k + \frac{1}{3}\sum_{k=H+1}^\infty \varphi\left(\frac{|x_k|}{2\ \|\mathbf{x}\|_{\varphi,\omega}}\right)\omega_k + \frac{1}{3}\sum_{k=H+1}^\infty \varphi\left(\frac{|x_{k+1}|}{2\ \|\mathbf{x}\|_{\varphi,\omega}}\right)\omega_k \\ &\leq \frac{1}{6} + \frac{1}{6} + \frac{1}{3} \leq 1, \end{aligned}$$

and therefore,

$$\|T_{a,b,c}\|_e^{l_\varphi(\omega)\to l_\varphi(\omega)} \leq 6\left(\limsup_{n\to\infty}|a_n| + \limsup_{n\to\infty}|b_n| + \limsup_{n\to\infty}|c_n|\right).$$

This proves the result. □

6 Final comments

We finish this article by noting that Theorems 4 and 7 indeed represent genuine generalizations and extensions of Theorem 4.1 in [12] and Theorem 3.2 in [13]. Additionally, as an immediate and significant consequence of our results, we obtain new criteria for the compactness of the tridiagonal operator acting between weighted Orlicz and weighted l^∞ spaces, and *vice versa*. For instance, the operator $T_{a,b,c} : l^\infty(\omega) \to l^\infty(\omega)$ is compact if and only if $\mathbf{a} \in c_0$, $\mathbf{b} \in c_0(\beta)$ and $\mathbf{c} \in c_0(\gamma)$.

Acknowledgments: The authors would like to thank the referees for their valuable comments that helped improve the manuscript.

Funding information: The first author received partial financial support from the Oficina de Investigaciones of Universidad Distrital Francisco José de Caldas for the payment of the Article Processing Charge (APC).

Author contributions: The results presented in this article are the outcome of continuous collaboration and discussions among all authors. All authors contributed equally to the development of the main ideas, the writing of the manuscript, and the final approval of the version submitted.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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