Research Article

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The evaluation of a definite integral by the method of brackets illustrating its flexibility

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Abstract: The method of brackets is a procedure to evaluate definite integrals over a half-line. It consists of a small number of rules. This article illustrates the method by evaluating an integral by several variations of the method. The integrand is the product of a Bessel function and an exponential integral function.

Keywords: integrals, Gradshteyn and Ryzhik, method of brackets

MSC 2020: 33C10, 33B15

1 Introduction

The goal of the present work is to describe a flexible method of integration, the so-called `method of brackets`, by discussing the evaluation of the identity

$$\begin{aligned}\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\,\mathrm{Ei}(-x^2y)K_0\left(\frac{x}{y}\right)\mathrm{d}x\mathrm{d}y &= -\frac{1}{12}\frac{\Gamma^2\left(\frac{\alpha+\beta}{3}\right)\Gamma^2\left(\frac{\alpha-2\beta}{6}\right)}{4^{(2\beta-\alpha)/6}\Gamma\left(\frac{\alpha+\beta}{3}+1\right)},\\ &= -\frac{1}{4}\frac{2^{(\alpha-2\beta)/3}}{\alpha+\beta}\Gamma\left(\frac{\alpha+\beta}{3}\right)\Gamma^2\left(\frac{\alpha-2\beta}{6}\right),\end{aligned} \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$, the Γ-function is the classical eulerian integral and the functions appearing in the integrand are the Bessel K_0 function and the exponential integral `Ei`. The definition of these functions is given in the following. The complexity of the integrand is chosen just to illustrate the power of the method of brackets. We present a variety of ways to use this method in the proof of (1.1). Throughout this article, we have chosen to continue using the expression on the first line earlier, instead of the simpler one given in the second line. The integral (1.1) was given to us, with the first answer, as a challenge to test the method of brackets described in the following. The authors have not found in the literature an evaluation of this example by classical methods.

The evaluation of definite integrals is one of the basic techniques found in the elementary basic calculus courses. One of the central problems is to create a class of functions $\mathcal{F}$ and decide if the functions in this class may be integrated. At the elementary level, one starts with simple powers x^n, $n \in \mathbb{N}$ in the class $\mathcal{F}$ and requires some algebraic properties on this class. For instance, it is often assumed that the class $\mathcal{F}$ should be closed under elementary operations, i.e., $\mathcal{F}$ should be closed under addition and products. It follows that $\mathcal{F}$ contains all polynomials in x.

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The evaluation of

$$I(n;\, a, b) = \int_a^b x^n \mathrm{d}x, \quad \text{for } n \in \mathbb{N} \text{ and } a, b \in \mathbb{R}, \tag{1.2}$$

is elementary since $\mathcal{F}$ contains the primitive of the integrand:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{n+1}x^{n+1}\right) = x^n, \quad \text{when } n \neq -1. \tag{1.3}$$

The formula

$$I(n;\, a, b) = \frac{1}{n+1}(b^{n+1} - a^{n+1}) \tag{1.4}$$

is now immediate. The linearity of integration now shows that the integration of any polynomial can be completed within the class $\mathcal{F}$.

The extension of (1.4) to a wider range of parameters n is now a question of elementary analysis. Once x^α has been defined for $\alpha \in \mathbb{R}$, (1.4) is valid with n replaced by α. Naturally, the exception $\alpha = -1$ remains. The integration of this final case requires the introduction of a new function:

$$\ln x := \int_1^x t^{-1} \mathrm{d}t, \tag{1.5}$$

the classical (natural) logarithm. This function is now added to the class $\mathcal{F}$ and the process may be continued.

Throughout history, the evaluated integrals have been collected in tables such as those created by Bierens de Haan [1] and extended in the current table by Gradshteyn and Ryzhik [2]. An effort has been made by the community to make the entries in these tables be free of errors. This is a continuing task. The techniques developed to prove these evaluations have generated a large number of articles and books. These include olders volumes [3], some elementary treatises [4,5] and [6] at a more advanced level. A very complete list of integrals is provided by the five volumes by Prudnikov et al. [7].

During the last few years, the authors have developed a method of integration, named `the method of brackets`. It consists of a small number of rules described in Section 2 and is based on the expansion of the integrand in a series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1}, \quad \text{with } a_n, \alpha, \beta \in \mathbb{R}, \tag{1.6}$$

(the extra −1 in the exponent simplifies the appearance of some formulas in the following).

Remark 1.1. In this work, the sum $\sum_{n=0}^{\infty} u_n$ will be written simply as $\sum_{n\geq 0} u_n$.

Remark 1.2. The original ideas for this method of integration came from the analysis of the so-called `method of negative dimensional integration` [8–10]. A detailed discussion of the power of this method is presented in [11], with a manuscript in preparation.

The goal of this note is to describe the flexibility of the method of brackets by presenting several ways to evaluate the integral (1.1). It is remarkable that the method of brackets evaluates this integral, in view of the fact that both functions in the integrand have logarithmic singularities at the origin.

The functions appearing in the integrand of (1.1) are now defined.

The Bessel function $K_0(x)$. The traditional manner to introduce Bessel functions is as solutions of the differential equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - \nu^2)y = 0. \tag{1.7}$$

With appropriate initial conditions, this gives the function J_ν, with power series expansion

$$J_\nu(x) = \sum_{k\geq 0} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}. \tag{1.8}$$

A second (linearly independent) solution is given by the function

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}, \quad \nu \notin \mathbb{Z}. \tag{1.9}$$

(The case when ν is an integer is treated by a limiting procedure.) References for Bessel functions include [12,13] and the encyclopedic treatise [14].

The modified Bessel function $K_\nu(x)$ is another family of special functions, closely related to J_ν and Y_ν. This function has the expansion

$$K_0(x) = -\ln\left(\frac{x}{2}\right)\left(\sum_{k\geq 0} \frac{1}{(k!)^2}\left(\frac{x}{2}\right)^{2k}\right) + \sum_{k\geq 0} \psi(k+1)\frac{1}{(k!)^2}\left(\frac{x}{2}\right)^{2k}, \tag{1.10}$$

showing a logarithmic singularity at the origin. This appears in [2], entry 8.447.3 as well as [14], formula WA 95(14). Here,

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{1.11}$$

is the logarithmic derivative of the gamma function.

The exponential integral $\mathrm{Ei}(x)$. This function is defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt, \tag{1.12}$$

for $x < 0$. In the case $x > 0$, one defines it using the Cauchy principal value

$$\mathrm{Ei}(x) = -\lim_{\varepsilon\to 0^+}\left[\int_{-x}^{-\varepsilon} \frac{e^{-t}}{t} dt + \int_{\varepsilon}^{\infty} \frac{e^{-t}}{t} dt\right], \tag{1.13}$$

appearing as entry **3.351.6** in [2]. The series expansion

$$\mathrm{Ei}(-x) = \gamma + \ln x + \sum_{k\geq 1} \frac{(-x)^k}{kk!}, \tag{1.14}$$

with γ the Euler's constant, shows that `Ei` has a logarithmic singularity at zero.

This article describes how to use several aspects of the methods to evaluate the single integral (1.1). The outline of this article is as follows: Section 2 summarizes the rules of the method. These are of two types: production and evaluation. Section 3 outlines the ideas behind the main analytic result: that the evaluation of a bracket series gives the value of the integral. Section 4 shows how to generate bracket series when the functions appearing in the integrand are given by an integral representation. Section 5 shows that the representation of the integrand in divergent series is also compatible with the method of brackets. Section 6 shows how to use this method when the Mellin transform of the integrand is known. Section 7 combines the method of brackets with contour integration. Finally, Section 8 presents how to combine several of the methods presented here.

2 Method of brackets

This section presents the main rules for the method of brackets, a procedure for the evaluation of definite integrals over the half line $[0, \infty)$. The application of the method consists of a small number of rules, deduced in a heuristic form. These rules have not been placed on solid ground [15] and [16].

Consider the integral

$$\mathcal{I}(f) = \int_0^\infty f(x)\mathrm{d}x, \tag{2.1}$$

and define the formal symbol

$$\langle a \rangle = \int_0^\infty x^{a-1}\mathrm{d}x, \tag{2.2}$$

called the *bracket* of a. Observe that the integral on (2.2) is divergent for any choice of the parameter a. To simplify some computations while operating with brackets, introduce the symbol

$$\phi_n = \frac{(-1)^n}{\Gamma(n+1)}, \tag{2.3}$$

called the `indicator` of n, and let $\phi_{i_1,i_2,\ldots,i_r}$, denote the product $\phi_{i_1}\phi_{i_2}\cdots\phi_{i_r}$.

Using this notation, the basic rules for the production and evaluation of a so-called `bracket series` associated with $\mathcal{I}(f)$ are described in the following (see [17] for details).

Rule P_1. Assume f has the expansion

$$f(x) = \sum_{n\geq 0} \phi_n a_n x^{\alpha n+\beta-1}. \tag{2.4}$$

Then, $\mathcal{I}(f)$ is assigned the *bracket series*

$$\mathcal{B}(f) = \sum_{n\geq 0} \phi_n a_n \langle \alpha n + \beta \rangle. \tag{2.5}$$

In the situation where the integrand f contains a multinomial power, it is convenient to assign a bracket series to the integrand using Rule P_2 `before` using Rule P_1. This produces multi-dimensional bracket series.

Rule P_2. For $\alpha \in \mathbb{R}$, the multinomial power $(a_1 + a_2 + \ldots + a_r)^\alpha$ is assigned the r-dimensional bracket series

$$\sum_{n_1\geq 0}\sum_{n_2\geq 0}\cdots\sum_{n_r\geq 0} \phi_{n_1,n_2,\ldots,n_r} a_1^{n_1}\cdots a_r^{n_r} \frac{\langle n_1+\ldots+n_r-\alpha\rangle}{\Gamma(-\alpha)}. \tag{2.6}$$

Definition. Each representation of an integral by a bracket series has an associated *complexity index of the representation* via

$$\text{complexity index} = \text{number of sums} - \text{number of brackets}. \tag{2.7}$$

It is important to note that the complexity index is attached to each specific representation of the integral and not just to integral itself. The level of difficulty in the analysis of the resulting bracket series increases with the complexity index; however, it has been shown that the value of integral is independent of the bracket series representation.

Rule E_1. Let $\alpha, \beta \in \mathbb{R}$. The one-dimensional bracket series is assigned the value

$$\sum_{n\geq 0} \phi_n f(n)\langle \alpha n + \beta\rangle = \frac{1}{|\alpha|} f(n^*)\Gamma(-n^*), \tag{2.8}$$

where n^* is obtained from the vanishing of the bracket, i.e., n^* solves $\alpha n + \beta = 0$. This is precisely the Ramanujan's master theorem [15].

The following rule provides a value for multi-dimensional bracket series of index 0, i.e., the number of sums is equal to the number of brackets.

Rule E_2. Let $a_{ij} \in \mathbb{R}$. Assuming the matrix $A = (a_{ij})$ is non-singular, then the assignment is

$$\sum_{n_1 \geq 0} \sum_{n_2 \geq 0} \dots \sum_{n_r \geq 0} \phi_{n_1 \dots n_r} f(n_1, \dots, n_r) \langle a_{11} n_1 + \dots + a_{1r} n_r + c_1 \rangle \\ \dots \langle a_{r1} n_1 + \dots + a_{rr} n_r + c_r \rangle = \frac{1}{|\det A|} f(n_1^*, \dots, n_r^*) \Gamma(-n_1^*) \dots \Gamma(-n_r^*), \tag{2.9}$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if A is singular.

Rule E_3. The value of a multi-dimensional bracket series of non-negative index is obtained by computing all the contributions of maximal rank using Rule E_2. These contributions to the integral appear as series in the free parameters. There is no assignment to a bracket series of negative index.

The next result will be used in later sections.

Lemma 2.1. *For any* $\alpha, \beta \in \mathbb{R}$, *with* $\alpha \neq 0$, *the bracket satisfies*

$$\langle \alpha\gamma + \beta \rangle = \frac{1}{|\alpha|} \left\langle \gamma + \frac{\beta}{\alpha} \right\rangle. \tag{2.10}$$

The goal of this article is to illustrate the power of the method of brackets by evaluating the complicated example (1.1). As a preliminary example, we present an evaluation of the classical `Wallis integral`

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{\pi}{2^{2m+1}} \binom{2m}{m}. \tag{2.11}$$

Many elementary evaluations of this integral appear in Chapter 9 of [4]. Two proofs using the method of brackets are discussed here:

Proof. The binomial theorem

$$(1-t)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} t^k \tag{2.12}$$

gives the expansion

$$\frac{1}{(x^2+1)^{m+1}} = \sum_{k=0}^{\infty} \frac{(m+1)_k}{k!} (-x^2)^k. \tag{2.13}$$

This is an expansion of the form (2.4), converging for $|x| < 1$. Integrating produces

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \sum_{k=0}^{\infty} \phi_k (m+1)_k \langle 2k+1 \rangle. \tag{2.14}$$

Rule E_1 now requires to solve $2k + 1 = 0$, to produce $k^* = -1/2$ and then

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \frac{1}{2} (m+1)_{-1/2} \Gamma\left(\frac{1}{2}\right). \tag{2.15}$$

Now, simplify using

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \quad \text{and} \quad \Gamma\left(m + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m}} \cdot \frac{(2m)!}{m!} \tag{2.16}$$

to confirm the evaluation (2.11). □

Proof. Rule P_2 gives

$$(x^2+1)^{-m-1} \mapsto \sum_{n_1} \sum_{n_2} \phi_{12} x^{2n_1} \frac{\langle n_1 + n_2 + m + 1 \rangle}{\Gamma(m+1)}, \tag{2.17}$$

and integrating yields

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{\langle 2n_1+1\rangle\langle n_1+n_2+m+1\rangle}{m!}. \tag{2.18}$$

This is a bracket series of index 0, and Rule E_2 gives its value by solving the linear system of equations

$$2n_1 + 1 = 0,$$
$$n_1 + n_2 + m + 1 = 0.$$

The solutions are $n_1 = -\frac{1}{2}$, $n_2 = -m - \frac{1}{2}$, and the formula in Rule E_2 confirms (2.11). □

These two proofs confirm the flexibility of the method of brackets. Many other examples appear in [17,18].

3 Justification of the method of brackets

The rules of the method of brackets presented in Section 2 have the advantage of being easy to operate. The present work illustrates a variety of forms to use the method to evaluate the single example (1.1). This section outlines the proofs of the rules mentioned in the previous section. Complete details will appear in [16].

The method of brackets associates with the integral of a function f

$$\mathcal{I}(f) = \int_0^\infty f(x)dx, \tag{3.1}$$

a formal object, called a bracket series, of the form

$$\mathcal{B}(f) = \sum_{n_1,\dots,n_r\geq 0} \phi_{n_1,\dots,n_r} F(n_1,\dots,n_r) \prod_{j=1}^{r} \langle \alpha_j n_j + \beta_j\rangle. \tag{3.2}$$

(A similar procedure exists for a multi-dimensional integral over $[0,\infty)\times\dots[0,\infty)$.)

The main analytic statement is this:

Theorem 3.1. *The method of brackets consists in two parts: given a function f*
- *associate with $\mathcal{I}(f)$ in a bracket series $\mathcal{B}(f)$. For a given function f, this association is not necessarily unique*;
- *assign to a bracket series $\mathcal{B}(f)$ a unique complex number* $\text{Eval}(\mathcal{B}(f))$.

Then, the integral agrees with the evaluation of the associated bracket series, i.e.,

$$\mathcal{I}(f) = \text{Eval}(\mathcal{B}(f)), \tag{3.3}$$

where $\mathcal{B}(f)$ is a bracket series associated with $\mathcal{I}(f)$. The value of $\text{Eval}(\mathcal{B}(f))$, is `independent` *of the bracket series associated with $\mathcal{I}(f)$.*

The rules for this procedure are those given in Section 2. A complete detailed proof of this result will appear in [16]. The remaining of this section presents an outline of the proof.

`Production rules`. These are forms to generate brackets series, i.e., the result is a formal object. For instance, Rule P_1 states that if the integrand has an expansion

$$f(x) = \sum_{n\geq 0} \phi_n a_n x^{\alpha n+\beta-1}, \tag{3.4}$$

then one creates the bracket series

$$\mathcal{B}(f) = \sum_{n\geq 0} \phi_n a_n \langle \alpha n + \beta\rangle. \tag{3.5}$$

This follows directly by integration and the definition of brackets. Since this is a formal result, conditions on expansion (3.4) are not part of the analysis. These conditions play a role in the evaluation of the associated bracket series, stated in the following.

Rule P_2 is the second production rule. This appears from scaling the definition of the gamma function to obtain

$$U^a = \frac{1}{\Gamma(-a)} \int_0^\infty x^{-a-1} e^{-Ux} \mathrm{d}x. \tag{3.6}$$

Now, let $U = u_1 + \ldots + u_r$ to produce

$$(u_1 + \ldots + u_r)^a = \frac{1}{\Gamma(-a)} \int_0^\infty x^{-a-1} e^{-(u_1 + \ldots + u_r)x} \mathrm{d}x = \frac{1}{\Gamma(-a)} \int_0^\infty x^{-a-1} e^{-u_1 x} \ldots e^{-u_r x} \mathrm{d}x. \tag{3.7}$$

Now, expand the exponential terms in power series to produce the bracket series appearing in Rule 2.

`Evaluation rules`. These are rules that assign a (complex) number to a bracket series. Rule E_1 gives the assignment

$$\sum_{n \geq 0} \phi_n f(n) \langle \alpha n + \beta \rangle = \frac{1}{|\alpha|} f(n^*) \Gamma(-n^*), \tag{3.8}$$

where n^* solves $\alpha n + \beta = 0$. The proof of this rule is based on a classical interpolation of theorem of Ramanujan applied to the function $\lambda(u) = \Gamma(u + 1) f(u)$. Naturally, this requires an extension of f, first defined on $\mathbb{N}$, to the complex plane.

Theorem 3.2. (Ramanujan's master theorem). *Let $\varphi(z)$ be an analytic (single-valued) function, defined on a half-plane*

$$H(\delta) = \{z \in \mathbb{C} : \mathrm{Re}_z \geq -\delta\}, \tag{3.9}$$

for some $0 < \delta < 1$. Suppose that, for some $A < \pi$ and $P \in \mathbb{R}$, the function φ satisfies the growth condition

$$|\varphi(v + iw)| < Ce^{Pv + A|w|}, \tag{3.10}$$

for all $z = v + iw \in H(\delta)$. Then,

$$\int_0^\infty x^{s-1} \{\varphi(0) - x\varphi(1) + x^2 \varphi(2) - \ldots\} \mathrm{d}x = \frac{\pi}{\sin \pi s} \varphi(-s). \tag{3.11}$$

Rule E_2 is proved by iterating Rule E_1. The appearance of the determinant is natural in the process. Finally, Rule E_3 is proved by making a choice of free parameters and observing that the non-chosen parameters generate a bracket series of rank 0 (equal number of sums as brackets). These are then evaluated using Rule E_3.

4 Integral representations

The brackets series of an integral may be obtained if the components of the integrand have proper integral representations, i.e., representations in terms of functions with series expansions. In the case considered here the analysis begins with

$$K_0(\xi) = \int_0^\infty \frac{\cos(\xi t)}{(t^2 + 1)^{1/2}} \mathrm{d}t \tag{4.1}$$

and

$$\mathrm{Ei}(-\xi) = -\int_{\xi}^{\infty} \frac{\exp(-t)}{t}\mathrm{d}t, \tag{4.2}$$

appearing in [19] and [2], respectively. The classical Taylor series for cosine is written as

$$\cos(\xi t) = \sum_{n\geq 0} \phi_n \frac{\Gamma(\frac{1}{2})\xi^{2n}}{\Gamma(\frac{1}{2}+n)4^n} t^{2n}, \tag{4.3}$$

and the binomial expansion of $(t^2+1)^{-1/2}$, written as

$$\frac{1}{(t^2+1)^{1/2}} = \sum_{m\geq 0}\sum_{\ell\geq 0} \phi_{m,\ell} \frac{\langle \frac{1}{2}+m+\ell\rangle}{\Gamma(\frac{1}{2})} t^{2m}, \tag{4.4}$$

is obtained from (2.6). Then, it follows from (4.1) that

$$\begin{aligned} K_0(\xi) &= \sum_{n\geq 0}\sum_{m\geq 0}\sum_{\ell\geq 0} \phi_{n,m,\ell} \frac{\langle \frac{1}{2}+m+\ell\rangle}{4^n\Gamma(\frac{1}{2}+n)} \xi^{2n} \int_0^{\infty} t^{2n+2m}\mathrm{d}t \\ &= \sum_{n\geq 0}\sum_{m\geq 0}\sum_{\ell\geq 0} \phi_{n,m,\ell} \frac{\langle \frac{1}{2}+m+\ell\rangle\langle 2n+2m+1\rangle}{4^n\Gamma(\frac{1}{2}+n)} \xi^{2n} \\ &= \frac{1}{2}\sum_{n\geq 0}\sum_{m\geq 0}\sum_{\ell\geq 0} \phi_{n,m,\ell} \frac{\langle \frac{1}{2}+m+\ell\rangle\langle n+m+\frac{1}{2}\rangle}{4^n\Gamma(\frac{1}{2}+n)} \xi^{2n}, \end{aligned} \tag{4.5}$$

where the last equality comes from Lemma 2.1. This is a bracket series of $K_0(\xi)$. Recall from (1.10) that K_0 has a logarithmic singularity at the origin. The discussion presented previously shows that it is possible to assign a bracket series to functions without a power series around zero. It is the convergence of the original integral that makes the process work.

In order to obtain a bracket series representation of the exponential integral, start with

$$\mathrm{Ei}(-\xi) = -\int_{\xi}^{\infty} \frac{\exp(-t)}{t}\mathrm{d}t = -\int_0^{\infty} \frac{\exp[-(z+\xi)]}{z+\xi}\mathrm{d}z. \tag{4.6}$$

Expanding the exponential and using the binomial theorem yield

$$\begin{aligned} \mathrm{Ei}(-\xi) &= -\sum_{i\geq 0} \phi_i \int_0^{\infty} (z+\xi)^{i-1}\mathrm{d}z \\ &= -\sum_{i\geq 0}\sum_{j\geq 0}\sum_{k\geq 0} \phi_{i,j,k}\xi^k \frac{\langle 1-i+j+k\rangle}{\Gamma(1-i)} \int_0^{\infty} z^j\mathrm{d}z \\ &= -\sum_{i\geq 0}\sum_{j\geq 0}\sum_{k\geq 0} \phi_{i,j,k}\xi^k \frac{\langle 1-i+j+k\rangle}{\Gamma(1-i)} \langle j+1\rangle. \end{aligned} \tag{4.7}$$

This is a bracket series with three three-dimensional sums and two brackets, so it is of complexity index 1. Rule 1 is now used to reduce this to two-dimensional sums by eliminating a bracket with one of the sums. The simplest choice is to eliminate the index j and this leads to the representation

$$\mathrm{Ei}(-\xi) = -\sum_{i\geq 0}\sum_{k\geq 0} \phi_{i,k}\xi^k \frac{\langle k-i\rangle}{\Gamma(1-i)}. \tag{4.8}$$

Let $I(\alpha,\beta)$ denote the integral in (1.1). Replacing the representations (4.5) and (4.8) in (1.1) give

$$I(\alpha,\beta) = -\frac{1}{2}\sum_{i\geq 0}\sum_{k\geq 0}\sum_{n\geq 0}\sum_{m\geq 0}\sum_{\ell\geq 0} \phi_{i,k,n,m,\ell} \frac{\langle k-i\rangle\langle m+\ell+\frac{1}{2}\rangle\langle n+m+\frac{1}{2}\rangle}{4^n\Gamma(1-i)\Gamma(\frac{1}{2}+n)} \langle \alpha+2n+2k\rangle\langle \beta-2n+k\rangle. \tag{4.9}$$

The number of sums equals the number of brackets; therefore, this is a representation of index zero. According to (2.9), the value of $I(\alpha,\beta)$ is then given by

$$I(\alpha,\beta) = -\frac{1}{2|\det A|}\frac{\Gamma(-i^*)\Gamma(-k^*)\Gamma(-n^*)\Gamma(-m^*)\Gamma(-\ell^*)}{4^{n^*}\Gamma(1-i^*)\Gamma(\frac{1}{2}+n^*)}, \tag{4.10}$$

where the set $(i^*, k^*, n^*, m^*, \ell^*)$ is the unique solution of

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{pmatrix}\begin{pmatrix} i \\ k \\ n \\ m \\ \ell \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\alpha \\ -\beta \end{pmatrix}. \tag{4.11}$$

The matrix of coefficients of the aforementioned system is denoted by A. The solutions are given by

$$i^* = k^* = -\frac{1}{3}(\alpha+\beta), \quad n^* = \ell^* = -\frac{1}{6}(\alpha-2\beta), \quad m^* = \frac{1}{6}(\alpha-2\beta) - \frac{1}{2},$$

and, since $|\det A| = 6$, this gives

$$I(\alpha,\beta) = -\frac{1}{12}\frac{\Gamma^2\left(\frac{\alpha+\beta}{3}\right)\Gamma^2\left(\frac{\alpha-2\beta}{6}\right)}{4^{(2\beta-\alpha)/6}\Gamma\left(\frac{\alpha+\beta}{3}+1\right)}. \tag{4.12}$$

This is the first proof of (1.1).

Remark 4.1. An alternative evaluation of $I(\alpha,\beta)$ is obtained by replacing the integral representations of K_0 and `Ei` and write

$$I(\alpha,\beta) = -\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\frac{\exp[-(z+x^2y)]}{(z+x^2y)}\frac{\cos(xt/y)}{(t^2+1)^{1/2}}\,dt\,dx\,dy\,dz.$$

Then, evaluate this integral by the method of brackets. The reader is invited to complete the details.

5 Use of null and divergent series for the integrand

This section presents the evaluation of the integral (1.1) by a technique derived from the method of brackets, which allows us to assign a (non-classical) series to functions without a regular power series at zero. These new type of series are classified as divergent and null. These terms are illustrated next.

For the functions in the integrand of (1.1), the assigned series are:

$$K_0(\xi) = \begin{cases} \frac{1}{2}\sum_{n\geq 0}\phi_n\frac{\Gamma(-n)}{4^n}\xi^{2n}, & \text{Divergent representation}, \\ \frac{1}{\xi}\sum_{n\geq 0}\phi_n\frac{4^n\Gamma^2\left(n+\frac{1}{2}\right)}{\Gamma(-n)}\xi^{-2n}, & \text{Null representation}, \end{cases} \tag{5.1}$$

$$\mathrm{Ei}(-\xi) = \sum_{\ell\geq 0}\phi_l\frac{\xi^l}{l}, \qquad \text{Divergent representation.} \tag{5.2}$$

Remark 5.1. The first representation of $K_0(\xi)$ is called divergent since all the coefficients are infinite, since they come from evaluating the gamma function at its poles. The second representation is called null, since all the coefficients vanish. It is clear that these are not classical power series representations. Nevertheless, it will be seen that these expansions can be used in the method of brackets to evaluate the integral given in (1.1).

This gives a new approach for the evaluation of the integral (1.1). First, considering the divergent versions for $\mathrm{Ei}(-x^2y)$ and $K_0(\frac{x}{y})$,

$$\begin{aligned} I(\alpha,\beta) &= \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\left(\sum_{\ell\geq 0}\phi_l\frac{(x^2y)^l}{l}\right)\left(\frac{1}{2}\sum_{n\geq 0}\phi_n\frac{\Gamma(-n)}{4^n}\left(\frac{x}{y}\right)^{2n}\right)\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2}\sum_{\ell\geq 0}\sum_{n\geq 0}\phi_{l,n}\frac{\Gamma(-n)}{4^n l}\int_0^\infty\int_0^\infty x^{\alpha+2l+2n-1}y^{\beta+l-2n-1}\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{5.3}$$

which gives the bracket series

$$I(\alpha,\beta) = \frac{1}{2}\sum_{\ell\geq 0}\sum_{n\geq 0}\phi_{l,n}\frac{\Gamma(-n)}{4^n l}\langle\alpha+2l+2n\rangle\langle\beta+l-2n\rangle. \tag{5.4}$$

Evaluating this bracket series according to Rule 2.9, it can be shown that this approach gives the same result as found in (4.12).

On the other hand, using the divergent representation for $\mathrm{Ei}(-x^2y)$ and the null representation for $K_0(\frac{x}{y})$,

$$\begin{aligned} I(\alpha,\beta) &= \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\left[\sum_{\ell\geq 0}\phi_l\frac{(x^2y)^\ell}{\ell}\right]\left[\sum_{n\geq 0}\phi_n\frac{4^n\Gamma^2\left(n+\frac{1}{2}\right)}{\Gamma(-n)}\left(\frac{x}{y}\right)^{-2n-1}\right]\mathrm{d}x\mathrm{d}y \\ &= \sum_{\ell\geq 0}\sum_{n\geq 0}\phi_{\ell,n}\frac{4^n\Gamma^2\left(n+\frac{1}{2}\right)}{\ell\Gamma(-n)}\int_0^\infty\int_0^\infty x^{\alpha+2\ell-2n-2}y^{\beta+\ell+2n}\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{5.5}$$

which leads to the bracket series

$$I(\alpha,\beta) = \sum_{\ell\geq 0}\sum_{n\geq 0}\phi_{\ell,n}\frac{4^n\Gamma^2\left(n+\frac{1}{2}\right)}{\ell\Gamma(-n)}\langle\alpha+2\ell-2n-1\rangle\langle\beta+\ell+2n+1\rangle. \tag{5.6}$$

Now, it can be shown that the evaluation of this bracket series gives the same result for (4.12) as earlier.

6 Use of Mellin transforms

This section illustrates how to compute an integral by using the method of brackets, combined with the fact that one may obtain a power series representation for a function whose Mellin transform is known [20].

The main idea is explained next. Let $f(\xi)$ be a function with an explicit expression for its Mellin transform

$$M(s) = \int_0^\infty \xi^{s-1}f(\xi)\mathrm{d}\xi. \tag{6.1}$$

To determine a (perhaps non-classical) power series for $f(\xi)$, suppose

$$f(\xi) = \sum_{n\geq 0}\phi_n F(n)\xi^{an+b}, \tag{6.2}$$

where a and b are the real parameters. The coefficients $F(n)$ are determined using the method of brackets:

$$\begin{aligned} M(s) &= \int_0^\infty \xi^{s-1} f(\xi) d\xi \\ &= \sum_{n\geq 0} \phi_n F(n) \int_0^\infty \xi^{s+an+b-1} d\xi \\ &= \sum_{n\geq 0} \phi_n F(n) \langle s + an + b \rangle \\ &= \frac{1}{|a|} \Gamma(-n) F(n) \bigg|_{n=-\frac{s+b}{a}}. \end{aligned} \tag{6.3}$$

This is equivalent to

$$M(s)|_{s=-an-b} = \frac{1}{|a|} \Gamma(-n) F(n), \tag{6.4}$$

which provides an expression for the coefficients $F(n)$ in terms of the Mellin transform of f as

$$F(n) = |a| \frac{M(-an-b)}{\Gamma(-n)}. \tag{6.5}$$

Therefore, the series assigned to $f(\xi)$ is

$$f(\xi) = |a| \sum_{n\geq 0} \phi_n \frac{M(-an-b)}{\Gamma(-n)} \xi^{an+b}, \tag{6.6}$$

where the parameters a and b are, up to now, arbitrary.

These ideas are now used to calculate the integral (1.1). Start with the Mellin transforms

$$\int_0^\infty \xi^{s-1} \operatorname{Ei}(-\xi) d\xi = -\frac{\Gamma(s)}{s} \tag{6.7}$$

and

$$\int_0^\infty \xi^{s-1} K_0(\xi) d\xi = \frac{2^s}{4} \Gamma^2\left(\frac{s}{2}\right), \tag{6.8}$$

appearing as entries 6.223 and 6.561.16 in [2].

The computation of the series assigned to the integrand of (1.1), starts by writing

$$\operatorname{Ei}(-\xi) = \sum_{\ell\geq 0} \phi_\ell F(\ell) \xi^{a\ell+b}, \tag{6.9}$$

$$K_0(\xi) = \sum_{n\geq 0} \phi_n G(n) \xi^{An+B}, \tag{6.10}$$

where a, b, A, and B are real. Using the procedure described earlier, the coefficients of $F(\ell)$ and $G(n)$ are given by

$$F(\ell) = |a| \frac{\Gamma(-a\ell-b)}{(a\ell+b)\Gamma(-\ell)}, \tag{6.11}$$

$$G(n) = 2^{-An-B} \frac{|A|}{4} \frac{\Gamma^2\left(-\frac{An+B}{2}\right)}{\Gamma(-n)}. \tag{6.12}$$

Thus, the series representations obtained by this method are

$$\operatorname{Ei}(-\xi) = |a| \sum_{\ell\geq 0} \phi_\ell \frac{\Gamma(-a\ell-b)}{(a\ell+b)\Gamma(-\ell)} \xi^{a\ell+b} \tag{6.13}$$

and

$$K_0(\xi) = \frac{|A|}{2^{2+B}} \sum_{n\geq 0} \phi_n 2^{-An} \frac{\Gamma^2\left(-\frac{An+B}{2}\right)}{\Gamma(-n)} \xi^{An+B}. \tag{6.14}$$

Finally, this yields

$$\begin{aligned} I(\alpha,\beta) &= \int_0^\infty\int_0^\infty x^{\alpha-1} y^{\beta-1} \operatorname{Ei}(-x^2 y) K_0\left(\frac{x}{y}\right) dx dy \\ &= \frac{|A||a|}{2^{2+B}} \sum_{\ell\geq 0}\sum_{n\geq 0} \phi_{\ell,n} \frac{\Gamma(-a\ell-b)\Gamma^2\left(-\frac{An+B}{2}\right)}{2^{An}(a\ell+b)\Gamma(-\ell)\Gamma(-n)} \\ &\quad \times \int_0^\infty\int_0^\infty x^{\alpha+2a\ell+2b+An+B-1} y^{\beta+a\ell+b-An-B-1} dx dy \\ &= \frac{|A||a|}{2^{2+B}} \sum_{\ell\geq 0}\sum_{n\geq 0} \phi_{\ell,n} \frac{\Gamma(-a\ell-b)\Gamma^2\left(-\frac{An+B}{2}\right)}{2^{An}(a\ell+b)\Gamma(-\ell)\Gamma(-n)} \\ &\quad \times \langle \alpha+2a\ell+2b+An+B\rangle\langle \beta+a\ell+b-An-B\rangle, \end{aligned} \tag{6.15}$$

and then use Rule E_2, formula (2.9), to evaluate the bracket series and provide a new proof of (4.12).

7 Use of contour integrals

This section presents how the method of brackets extends naturally to Mellin-Barnes integrals [21]. This technique is based in the following rule for the evaluation of a *bracket integral* over a complex contour.

Rule E_4. Assume F is a function defined on $\mathbb{C}$. Then,

$$\int_{-i\infty}^{i\infty} F(s)\langle \alpha+\beta s\rangle ds = \frac{2\pi i}{|\beta|} F\left(-\frac{\alpha}{\beta}\right). \tag{7.1}$$

This rule has a multi-dimensional version described in the following.

Rule E_5. Let $A = (a_{ij})$ be a non-singular matrix, the following expression

$$\begin{aligned} I &= \left(\frac{1}{2\pi i}\right)^N \int_{-i\infty}^{i\infty} \dots \int_{-i\infty}^{i\infty} F(s_1,\dots,s_N) \\ &\quad \times \langle a_{11}s_1 + \dots a_{1N}s_N + c_1\rangle \dots \langle a_{N1}s_1 + \dots + a_{NN}s_N + c_N\rangle \prod_{k=1}^N ds_k \end{aligned} \tag{7.2}$$

is evaluated as the value

$$I = \frac{1}{|\det A|} F(s_1^*,\dots,s_N^*), \tag{7.3}$$

where $\{s_k^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets.

The rules presented earlier are now used to evaluate the integral of interest (1.1). Recall that

$$\int_0^\infty \xi^{s-1} \operatorname{Ei}(-\xi) d\xi = -\frac{\Gamma(s)}{s} \tag{7.4}$$

and

$$\int_0^\infty \xi^{s-1} K_0(\xi)\,d\xi = \frac{2^s}{4}\Gamma^2\left(\frac{s}{2}\right). \tag{7.5}$$

This yields the Mellin-Barnes representations

$$\operatorname{Ei}(-\xi) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \xi^{-s}\frac{\Gamma(s)}{s}\,ds, \tag{7.6}$$

$$K_0(\xi) = \frac{1}{8\pi i}\int_{-i\infty}^{i\infty} \xi^{-z} 2^z \Gamma^2\left(\frac{z}{2}\right)dz. \tag{7.7}$$

Replacing this in (1.1) gives

$$\begin{aligned} I(\alpha,\beta) &= \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\operatorname{Ei}(-x^2y)K_0\left(\frac{x}{y}\right)dx\,dy \\ &= \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\left[-\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(x^2y)^{-s}\frac{\Gamma(s)}{s}ds\right]\times\left[\frac{1}{8\pi i}\int_{-i\infty}^{i\infty}\left(\frac{x}{y}\right)^{-z}2^z\Gamma^2\left(\frac{z}{2}\right)dz\right]dx\,dy \\ &= \frac{-1}{4(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}2^z\frac{\Gamma(s)}{s}\Gamma^2\left(\frac{z}{2}\right)\left[\int_0^\infty\int_0^\infty x^{\alpha-2s-z-1}y^{\beta-s+z-1}dx\,dy\right]ds\,dz, \end{aligned} \tag{7.8}$$

from which one obtains the bracket integral

$$I(\alpha,\beta) = -\frac{1}{4}\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}2^z\frac{\Gamma(s)}{s}\Gamma^2\left(\frac{z}{2}\right)\langle\alpha-2s-z\rangle\langle\beta-s+z\rangle\,ds\,dz. \tag{7.9}$$

This is evaluated by the repeated use of rule (7.1) on each variable, or via rule (7.3) with the result

$$I(\alpha,\beta) = -\frac{1}{4|-3|}2^z\frac{\Gamma(s)}{s}\Gamma^2\left(\frac{z}{2}\right)\Bigg|_{s^*,z^*}, \tag{7.10}$$

where $s^* = \frac{1}{3}\alpha + \frac{1}{3}\beta$ and $z^* = \frac{1}{3}\alpha - \frac{2}{3}\beta$. This can be written as

$$I(\alpha,\beta) = -\frac{1}{12}2^{(\alpha-2\beta)/3}\frac{\Gamma\left(\frac{\alpha+\beta}{3}\right)}{\frac{\alpha+\beta}{3}}\cdot\Gamma^2\left(\frac{\alpha-2\beta}{6}\right). \tag{7.11}$$

The fact that this expression gives another proof of (4.12) is a direct consequence of the basic identity $\Gamma(u+1) = u\Gamma(u)$, with $u = \frac{1}{3}(\alpha+\beta)$.

8 Mixed method

This section shows that the method of brackets can be applied by combining the techniques from previous sections. Three different approaches will be analyzed. The reader is encouraged to explore other possible combinations.

8.1 Divergent series and Mellin-Barnes, version 1

Combine the divergent series representation for $\mathrm{Ei}(-\xi)$ with the Mellin-Barnes representation for $K_0(\xi)$. Recall these representations are

$$\mathrm{Ei}(-\xi) = \sum_{\ell\geq 0}\phi_\ell \frac{\xi^\ell}{\ell}, \tag{8.1}$$

$$K_0(\xi) = \frac{1}{8\pi i}\int_{-i\infty}^{i\infty}\xi^{-z}2^z\Gamma^2\left(\frac{z}{2}\right)\mathrm{d}z. \tag{8.2}$$

Then,

$$\begin{aligned}
I(\alpha,\beta) &= \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\mathrm{Ei}(-x^2y)K_0\left(\frac{x}{y}\right)\mathrm{d}x\mathrm{d}y\\
&= \int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}\left[\sum_{\ell\geq 0}\phi_\ell\frac{(x^2y)^\ell}{\ell}\right]\left[\frac{1}{8\pi i}\int_{-i\infty}^{i\infty}\left(\frac{x}{y}\right)^{-z}2^z\Gamma^2\left(\frac{z}{2}\right)\mathrm{d}z\right]\mathrm{d}x\mathrm{d}y\\
&= \frac{1}{8\pi i}\int_{-i\infty}^{i\infty}\sum_{\ell\geq 0}\phi_\ell\frac{2^z\Gamma^2\left(\frac{z}{2}\right)}{\ell}\left[\int_0^\infty\int_0^\infty x^{\alpha+2\ell-z-1}y^{\beta+\ell+z-1}\mathrm{d}x\mathrm{d}y\right]\mathrm{d}z\\
&= \frac{1}{8\pi i}\int_{-i\infty}^{i\infty}\sum_{\ell\geq 0}\phi_\ell\frac{2^z\Gamma^2\left(\frac{z}{2}\right)}{\ell}\langle\alpha+2\ell-z\rangle\langle\beta+\ell+z\rangle\mathrm{d}z.
\end{aligned}\tag{8.3}$$

Now, use the rules of the brackets to evaluate the sum and the integral. First, use rule (7.1) to produce

$$\begin{aligned}
I(\alpha,\beta) &= \frac{1}{4}\sum_{\ell\geq 0}\phi_\ell\frac{2^z\Gamma^2\left(\frac{z}{2}\right)}{\ell}\langle\beta+\ell+z\rangle\Bigg|_{z=\alpha+2\ell}\\
&= \frac{1}{4}\sum_{\ell\geq 0}\phi_\ell\frac{2^{\alpha+2\ell}\Gamma^2\left(\frac{\alpha+2\ell}{2}\right)}{\ell}\langle\beta+\alpha+3\ell\rangle\\
&= \frac{1}{12}\sum_{\ell\geq 0}\phi_\ell\frac{2^{\alpha+2\ell}\Gamma^2\left(\frac{\alpha+2\ell}{2}\right)}{\ell}\left\langle\frac{\beta+\alpha}{3}+\ell\right\rangle,
\end{aligned}\tag{8.4}$$

using Lemma 2.1 in the last equality.

Then, evaluate the bracket series with Rule (2.8) to obtain

$$\begin{aligned}
I(\alpha,\beta) &= \frac{1}{12}\frac{2^{\alpha+2\ell}\Gamma^2\left(\frac{\alpha+2\ell}{2}\right)\Gamma(-\ell)}{\ell}\Big|_{\ell=-\frac{\beta+\alpha}{3}}\\
&= \frac{1}{12}\frac{2^{\alpha+2\left(-\frac{\beta+\alpha}{3}\right)}\Gamma^2\left(\frac{\alpha+2\left(-\frac{\beta+\alpha}{3}\right)}{2}\right)\Gamma\left(\frac{\beta+\alpha}{3}\right)}{\left(-\frac{\beta+\alpha}{3}\right)}\\
&= -\frac{1}{12}\frac{\Gamma^2\left(\frac{\alpha+\beta}{3}\right)\Gamma^2\left(\frac{\alpha-2\beta}{6}\right)}{4^{(2\beta-\alpha)/6}\Gamma\left(\frac{\alpha+\beta}{3}+1\right)},
\end{aligned}\tag{8.5}$$

which simplifies to the usual result in (4.12)

8.2 Divergent series and Mellin-Barnes, version 2

Next, use the divergent series for $\text{Ei}(-\xi)$ and a bracket series for $K_0(\xi)$ obtained from its corresponding Mellin-Barnes representation. Start with the bracket series for the gamma function

$$\Gamma(\xi) = \sum_{n\geq 0} \phi_n \langle \xi + n \rangle.$$

Then, the Mellin-Barnes representation for $K_0(\xi)$ gives

$$\begin{aligned} K_0(\xi) &= \frac{1}{8\pi i} \int_{-i\infty}^{i\infty} \xi^{-z} 2^z \Gamma^2\left(\frac{z}{2}\right) \mathrm{d}z \\ &= \frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \xi^{-2t} 4^t \Gamma^2(t) \mathrm{d}t \\ &= \frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \xi^{-2t} 4^t \left(\sum_{n\geq 0} \phi_n \langle t + n \rangle \right) \left(\sum_{m\geq 0} \phi_m \langle t + m \rangle \right) \mathrm{d}t \\ &= \frac{1}{4\pi i} \sum_{n\geq 0, m\geq 0} \phi_{n,m} \int_{-i\infty}^{i\infty} \xi^{-2t} 4^t \langle t + n \rangle \langle t + m \rangle \mathrm{d}t. \end{aligned} \tag{8.6}$$

Using (7.1) on the bracket $\langle t + m \rangle$, and taking $F(t) = \xi^{-2t} 4^t \langle t + n \rangle$, eliminates the integral to obtain

$$K_0(\xi) = \frac{1}{2} \sum_{n\geq 0, m\geq 0} \phi_{n,m} \xi^{2m} 4^{-m} \langle n - m \rangle, \tag{8.7}$$

which is another bracket series for $K_0(\xi)$. Then, the integral of interest is given by

$$\begin{aligned} I(\alpha, \beta) &= \int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} \text{Ei}(-x^2 y) K_0\left(\frac{x}{y}\right) \mathrm{d}x \mathrm{d}y \\ &= \int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} \left(\sum_{\ell\geq 0} \phi_\ell \frac{(x^2 y)^\ell}{\ell} \right) \left(\frac{1}{2} \sum_{n\geq 0, m\geq 0} \phi_{n,m} \left(\frac{x}{y}\right)^{2m} 4^{-m} \langle n - m \rangle \right) \mathrm{d}x \mathrm{d}y \\ &= \sum_{\ell\geq 0, n\geq 0, m\geq 0} \phi_{\ell,n,m} \frac{4^{-m} \langle n - m \rangle}{\ell} \int_0^\infty \int_0^\infty x^{\alpha+2\ell+2m-1} y^{\beta+\ell-2m-1} \mathrm{d}x \mathrm{d}y, \end{aligned} \tag{8.8}$$

which yields the three-dimensional bracket series

$$I(\alpha, \beta) = \sum_{\ell\geq 0, n\geq 0, m\geq 0} \phi_{\ell,n,m} \frac{\langle n - m \rangle \langle \alpha + 2\ell + 2m \rangle \langle \beta + \ell - 2m \rangle}{4^m \ell}. \tag{8.9}$$

Rule (2.9) evaluates this bracket series and gives the usual result for (4.12).

Remark 8.1. The bracket series representation for $K_0(\xi)$ obtained in (8.7) can be evaluated using rule (2.8) to eliminate one of the two sums. Independently of the choice of the parameter to be eliminated, it gives

$$K_0(\xi) = \frac{1}{2} \sum_{k\geq 0} \phi_k \frac{\Gamma(-k)}{4^k} \xi^{2k}, \tag{8.10}$$

the previously known divergent series representation for $K_0(\xi)$. This is obtained now by a completely different approach to the one used in [20], illustrating the flexibility of the method of brackets.

9 Conclusions

The method of brackets is an algorithm designed to evaluate definite integrals over a half-line. The method is based on a small number of operational rules. Recently, this method has been extended to deal with functions that lack of a power series representations [20]. In the present work, we illustrate these new techniques by evaluating an integral where the integrand is a combination of a Bessel function and the exponential integral function.

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