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Research Article

Ibtesam Bajunaid, Madhu Venkataraman*, and Varadha Raj Manivannan

# Eigenfunctions on an infinite Schrödinger network

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**Abstract:** In this article, we show that there is a one-to-one correspondence between the eigenfunctions associated with the perturbed Laplacian operator $\Delta_q$ on a Schrödinger infinite network $\{X, t, q\}$ with weight function $q(a)$ and the eigenfunctions associated with classical Laplacian operator $\Delta$ on the infinite network $\{X, t\}$.

**Keywords:** Schrödinger infinite network, Laplace operator on an infinite network, eigenfunctions

**MSC 2020:** 15A18, 31C20, 35P05, 35R02

## 1 Introduction

Eigenfunctions of the Laplacian operators and eigenfunctions of the Schrödinger operator are load-bearing pillar of several area of physics and fluidodynamics problems (see, e.g., [1–3], and reference therein). In this article, we involve both topics. If $\lambda$ is a constant and there exists a positive function $\eta$ on $X$ such that $\lambda \geq \frac{\Delta\eta(a)}{\eta(a)}$, we show that a function $\sigma$ is an eigenfunction of the Laplacian operator $\Delta$ with eigenvalue $\lambda$ if and only if $\sigma$ can be written as the product of a positive function $\eta$ and a modified harmonic function $h$, i.e., $\sum_{b\sim a} t(a, b)\eta(b)[h(b) - h(a)] = 0$.

A function $u$ is called $\alpha$-harmonic on a Schrödinger infinite network $\{X, t, q\}$ if $\Delta_q u(a) = \alpha u(a)$. We show that $u$ is $\alpha$-harmonic, if and only if $\alpha$ is an eigenvalue of $\Delta^{**}$ associated with an eigenfunction $v$ on the network $\{X, t^{**}\}$, where $t^{**}(a, b) = \frac{\eta(b)}{\eta(a)} t(a, b)$, and $u$ can be expressed as the product of $v$ and the positive function $\eta$.

A similar relationship holds for $\alpha$-superharmonic functions.

## 2 Preliminaries

An infinite network $\{X, t\}$ is an infinite graph $X$ with a countable number of vertices where every vertex has a finite number of neighbors (locally finite), any two vertices can be connected by a path, and there are no edges that connect a vertex to itself.

Transition function $t(a, b)$ assigns a probability to each possible transition between vertices. It is only positive if there is an edge between vertices $a$ and $b$. By a function on a graph $X$, we mean a function on its set of vertices. The Laplacian $\Delta$ of a function $u$ on a network $\{X, t\}$ at $a$ is defined as $\Delta u(a) = \sum_{b\sim a} t(a, b)[u(b) - u(a)]$. A function

*** Corresponding author: Madhu Venkataraman,** Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore - 632 014, Tamil Nadu, India, e-mail: madhu.v@vit.ac.in
**Ibtesam Bajunaid:** Department of Mathematics, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia, e-mail: bajunaid@ksu.edu.sa
**Varadha Raj Manivannan:** Department of Mathematics, Sri Chandrasekharendra Saraswathi Viswa Mahavidyalaya, Kanchipuram - 631561, Tamil Nadu, India, e-mail: varadharaj.m219@gmail.com

$s(a)$ on $\{X, t\}$ is said to be $\Delta$-harmonic, $\Delta$-superharmonic, or $\Delta$-subharmonic at $a$ if $\Delta s(a)$ is equal to, at most, or at least zero, respectively. The function is said to be $\Delta$-harmonic, $\Delta$-superharmonic, or $\Delta$-subharmonic on $\{X, t\}$ if it is $\Delta$-harmonic, $\Delta$-superharmonic, or $\Delta$-subharmonic at each $a$ of the graph $X$.

A Schrödinger infinite network $\{X, t, q\}$ is essentially an infinite network where each vertex or edge is assigned a weight [4]. This weight, denoted by $q$, represents a real-valued function on $X$ such that $q \geq \frac{\Delta\xi}{\xi}$ for some function $\xi > 0$. Note that the function $q(a)$ can take some negative values also. The operator $\Delta_q$ is defined by $\Delta_q u(a) = \Delta u(a) - q(a)u(a)$. Note that $\xi(a)$ mentioned ealier is a positive $\Delta_q$-superharmonic function on $\{X, t, q\}$. If $q = \frac{\Delta\xi(a)}{\xi(a)}$, then $\xi(a)$ is a positive $\Delta_q$-harmonic function and the network $\{X, t, q\}$ is called hyperbolic. In hyperbolic networks, various potential-theoretic concepts, such as the minimum principle, domination principle, balayage, Dirichlet solution, and the existence of a Green kernel, are established. When $q$ is non-negative, the constant function 1 is $\Delta_q$-superharmonic.

It is proved that ([5], Theorem 4.1.9), there always exists a positive function $h$ such that $\Delta_q h(a) = 0$.

A real-valued function $u$ is $\alpha$-superharmonic if $\Delta_q u(a) \leq \alpha u(a)$.

For any $b \in X$, there exists a positive function ($q$-Green's function) $G_b(a) = G(a, b)$ that satisfies $\Delta_q G_b(a) = -\delta_b(a)$ and $G_b(a) \leq G_b(b)$ for all $a$ in $X$.

Note that if the transition function is non-symmetry (meaning the probability of moving from $a$ to $b$ may differ from the probability of moving from $b$ to $a$), a scalar product is not defined and potential-theoretic methods are used instead [6].

When $q \geq 0$, $q \not\equiv 0$, the transition functions are symmetric, and the network is locally finite, Yamasaki [7] provided a comprehensive analysis of $\Delta_q$-superharmonic functions.

Keller et al. [8,9] have introduced a new definition of subcritical networks that does not require a limit on the number of neighbors each vertex can have. Using Hilbert space methods, they have developed a theory for these networks. A network is considered uniformly subcritical if its Green function is bounded. They also impose a condition on the function $q$ to ensure that the network remains subcritical as follows: for any $\varphi$ vanishing outside a finite set, $\frac{1}{2}\sum_{b\sim a} t(a, b)[\varphi(a) - \varphi(b)]^2 + \sum_a q(a)[\varphi(a)]^2 \geq 0$.

## 3 Eigenfunctions of the Laplacian

Consider an infinite network $\{X, t\}$. If there is a real number $\lambda$ such that $\lambda \geq \frac{\Delta\xi(a)}{\xi(a)}$ for some $\xi > 0$, then there must be another positive function $h$ that satisfies $\Delta h(a) = \lambda h(a)$ ([5], Theorem 4.1.9, replacing the function $q(a)$ by $\lambda$).

**Lemma 3.1.** *In a Schrödinger network $\{X, t, q\}$, assume that $q \geq 0$ and $q \not\equiv 0$. If there is a non-negative bounded function $v$ such that $v \not\equiv 0$ and $\Delta v(a) \geq q(a)v(a)$, then there exists a bounded function $h > 0$ such that $\Delta h(a) = q(a)h(a)$.*

**Proof.** Assume that $v$ is bounded by $k$. Since $q \geq 0$, the constant function 1 is a $\Delta_q$-superharmonic. Hence, $v$ is $\Delta_q$-subharmonic function majorized by the $\Delta_q$-superharmonic function $k$. Therefore, there exists a $\Delta_q$-harmonic function $h$, $0 \leq v \leq h \leq k$ [5, Theorem 4.1.1]. If $h$ is zero at any vertex, it must be zero everywhere, which contradicts the assumption that $v$ is not identically zero. Therefore, $h$ is positive on the entire network. □

**Definition 3.2.** A $\lambda$-eigenfunction of the Laplacian operator $\Delta(\Delta_q)$ defined in the infinite network $\{X, t\}(\{X, t, q\})$ is a nonzero function $f$ on the graph $X$ such that $\Delta f = \lambda f (\Delta_q f = \lambda f)$. The constant $\lambda$ is called the eigenvalue of the eigenfunction $f$.

**Proposition 3.3.** *If $v$ is a non-negative bounded function such that $v \not\equiv 0$ and $\Delta v \geq \alpha v$, for $\alpha > 0$, then any positive number $\beta \leq \alpha$ is an eigenvalue of the Laplacian $\Delta$ with a corresponding positive, bounded eigenfunction $v$.*

**Proof.** We can use Lemma 3.1, which guarantees the existence of a positive, bounded function $h$ satisfying $\Delta h(a) = \beta h(a)$ on $X$. □

**Theorem 3.4.** *A function $\sigma$ is considered an eigenfunction associated with $\lambda = \frac{\Delta\eta(a)}{\eta(a)}$ for some $\eta > 0$ if and only if $\sigma$ can be expressed as the product of $\eta$ and a harmonic function $h$ on the network $\{X, t^*\}$, where $t^*(a,b) = \eta(b)t(a,b)$.*

**Proof.** If $\Delta\sigma(x) = \lambda\sigma(a)$, then $\Delta\sigma(a) = \frac{\Delta\eta(a)}{\eta(a)}\sigma(a)$, which implies $\eta(a)\Delta\sigma(a) - \sigma(a)\Delta\eta(a) = 0$. Hence,

$$\begin{aligned}\sum_{b\sim a} t(a,b)[\eta(a)\sigma(b) - \eta(b)\sigma(a)] &= \sum_{b\sim a} t(a,b)\eta(b)\eta(a)\left[\frac{\sigma(b)}{\eta(b)} - \frac{\sigma(a)}{\eta(a)}\right] \\ &= \eta(a)\Delta^*\left[\frac{\sigma(a)}{\eta(a)}\right] \\ &= 0.\end{aligned}$$

Since $\eta > 0$, $\Delta^*\left[\frac{\sigma(a)}{\eta(a)}\right] = 0$. Thus, the function $\sigma$ can be written as the product of two functions: $h$ and $\eta$. The function $h = \frac{\sigma(a)}{\eta(a)}$ is a $\Delta^*$- harmonic function. Therefore, the dimension of the eigenspace associated with a particular eigenvalue $\lambda$ with respect to $\Delta$ is equal to the dimension of the space of harmonic functions on the network $\{X, t^*\}$.

Conversely, if $\sigma$ is the product of $h$ and $\eta$, where $h$ is a $\Delta^*$-harmonic function, then

$$\begin{aligned}\Delta\sigma(a) &= \Delta h(a)\eta(a) \\ &= \sum_{b\sim a} t(a,b)\{h(b)\eta(b) - h(a)\eta(a)\} \\ &= \sum_{b\sim a} t(a,b)\{\eta(b)[h(b) - h(a)] + h(a)[\eta(b) - \eta(a)]\} \\ &= \Delta^*h(a) + h(a)[\Delta\eta(a)] \\ &= 0 + h(a)[\lambda\eta(a)] \\ &= \lambda\sigma(a)\end{aligned}$$

□

**Remark 1.** With minor adjustments to the proof, we can show that the inequality $\Delta\sigma(a) \le \lambda\sigma(a)$ holds if and only if $\sigma(a)$ can be expressed as the product of two functions: $s$ and $\eta$, where $\Delta^* s \le 0$ on $\{X, t^*\}$.

# 4 Eigenfunctions of the Schrödinger operator

In this section, $\{X, t, q\}$ is a Schrödinger network, where $q \ge \frac{\Delta\xi}{\xi}$, $\xi > 0$ and $\eta > 0$ is a solution to the equation $\Delta_q(\eta) = 0$.

**Proposition 4.1.** *If $\alpha$ is an eigenvalue with a corresponding non-negative eigenfunction $\varphi$ that is non-identically zero, then $\varphi > 0$ and $\alpha > -[t(a) + q(a)]$ for all all points on the network $\{X, t, q\}$.*

**Proof.** To prove this, we can use the fact that $q(a) \ge \frac{\Delta\xi(a)}{\xi(a)}$ which implies, $[t(a) + q(a)] > 0$. Now, $\sum_b t(a,b)\varphi(b) = \alpha\varphi(a) + [t(a) + q(a)]\varphi(a)$. Hence, if the eigenfunction is zero at any point $c$, then $\varphi(b) = 0$ for all $b \sim c$ by the connectedness of $X$, it must be zero everywhere, which contradicts the assumption $\varphi \not\equiv 0$. Finally, since

$$\sum_b t(a,b)\varphi(b) - [t(a) + q(a)]\varphi(a) = \alpha\varphi(a),$$

$$\begin{aligned}\alpha + [t(a) + q(a)] &= \sum_b t(a,b)\frac{\varphi(b)}{\varphi(a)} \\ &> 0,\end{aligned}$$

for all $a$. □

We are ready to prove a necessary and sufficient condition to be eigenfunctions associated with the modified Laplacian operator.

**Theorem 4.2.** *A function u is an α-eigenfunction of the Schrödinger operator* $\Delta_q$ *if and only if α is an eigenvalue of the modified Laplacian* $\Delta^{**}$ *of the network* $\{X, t^{**}\}$, *where* $t^{**}(a,b) = \frac{\eta(b)}{\eta(a)} t(a,b)$. *The relationship between the eigenfunctions of these two operators is given by the equation* $u(a) = v(a)\eta(a)$, *where v is an eigenfunction of* $\Delta^{**}$ *associated with* $\alpha$.

**Proof.** Let $\Delta_q u(a) = \alpha u(a)$. Then,

$$\Delta u(a) - q(a)u(a) = \alpha u(a),$$
$$\eta(a)\Delta u(a) - \Delta\eta(a)u(a) = \alpha u(a)\eta(a),$$
$$\sum_b t(a,b)\eta(a)\eta(b)\left[\frac{u(b)}{\eta(b)} - \frac{u(a)}{\eta(a)}\right] = \alpha u(a)\eta(a).$$

If we divide both sides by $[\eta(a)]^2$, we obtain

$$\sum_b t(a,b)\frac{\eta(b)}{\eta(a)}\left[\frac{u(b)}{\eta(b)} - \frac{u(a)}{\eta(a)}\right] = \alpha\left[\frac{u(a)}{\eta(a)}\right],$$
$$\Delta^{**}\left[\frac{u(a)}{\eta(a)}\right] = \alpha\left[\frac{u(a)}{\eta(a)}\right].$$

Hence, $v(a) = \frac{u(a)}{\eta(a)}$ is an eigenfunction of $\Delta^{**}$ associated with $\alpha$.

On the other hand, if $u(a) = v(a)\eta(a)$, where $\Delta^{**}v = \alpha v$, we want to prove that $\Delta_q u(a) = \alpha u(a)$:

$$\begin{aligned}
\Delta_q u(a) &= \Delta_q[v(a)\eta(a)] \\
&= \Delta[v(a)\eta(a)] - q(a)[v(a)\eta(a)] \\
&= \sum_b t(a,b)[v(b)\eta(b) - v(a)\eta(a)] - q(a)[v(a)\eta(a)] \\
&= \sum_b \frac{\eta(a)}{\eta(b)} t^{**}(a,b)\{\eta(b)[v(b) - v(a)] + v(a)[\eta(b) - \eta(a)]\} - q(a)[v(a)\eta(a)] \\
&= \sum_b \eta(a)t^{**}(a,b)[v(b) - v(a)] + \sum_b v(a)\frac{\eta(a)}{\eta(b)} t^{**}(a,b)[\eta(b) - \eta(a)] - q(a)[v(a)\eta(a)] \\
&= \eta(a)\Delta^{**}v(a) + \sum_b v(a)t(a,b)[\eta(b) - \eta(a)] - q(a)[v(a)\eta(a)] \\
&= \eta(a)[\alpha v(a)] + v(a)[\Delta\eta(a)] - q(a)[v(a)\eta(a)] \\
&= \alpha u(a) + v(a)[q(a)\eta(a)] - q(a)[v(a)\eta(a)] \\
&= \alpha u(a).
\end{aligned}$$

□

**Remark 2.**

(1) By making minor adjustments to the proof, we can also show that the inequality $\Delta_q u(a) \leq \alpha u(a)$ holds on $\{X, t, q\}$ if and only if $u = v\eta$, where $\Delta^{**}v \leq \alpha v$. If $v$ satisfies the inequality $\Delta^{**}v \leq \alpha v$, then the product of $v$ and $\eta$ is an $\alpha$-superharmonic function on the original Schrödinger network. $\alpha$-superharmonic functions are used to describe the lowest possible eigenvalue, as seen in the Agmon-Allegretto theorem. For more information, you can refer to a related article, by Lennx and Stollmann [10].

(2) If $\alpha$ is a constant such that $\alpha \geq \frac{\Delta^{**}v}{v}$ for a function $v > 0$ on $\{X, t^{**}\}$, then there exist $\alpha$-superharmonic functions on the original Schrödinger network $\{X, t, q\}$. One example of such a function is $u = v\eta$, which is positive on $\{X, t, q\}$.

(3) There exists a positive eigenfunction of the Schrödinger operator $\Delta_q$ on $\{X, t, q\}$. As discussed earlier, there is a positive function $h$ such that $\alpha = \frac{\Delta^{**}v}{v}$ on $\{X, t^{**}\}$. Using Theorem 4.2, we proved earlier, $u(x) = h(x)\eta(x)$ is a positive eigenfunction of $\Delta_q$ on $\{X, t, q\}$.

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