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Research Article

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# Carmichael numbers composed of Piatetski-Shapiro primes in Beatty sequences

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**Abstract:** The Piatetski-Shapiro sequences are sequences of the form $(\lfloor n^c \rfloor)_{n=1}^{\infty}$ and the Beatty sequence is the sequence of integers $(\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty}$. We prove that there are infinitely many Carmichael numbers composed of entirely the primes from the intersection of a Piatetski-Shapiro sequence and a Beatty sequence for $c \in \left(1, \frac{19137}{18746}\right)$, $\alpha > 1$ irrational and of finite type by investigating the Piatetski-Shapiro primes in arithmetic progressions in a Beatty sequence. Moreover, we also discuss the intersection of a Piatetski-Shapiro sequence and multiple Beatty sequences in arithmetic progressions.

**Keywords:** Beatty sequence, Piatetski-Shapiro prime, Carmichael number

**MSC 2020:** 11N05, 11L07, 11N80, 11B83

## 1 Introduction

The Piatetski-Shapiro sequences are sequences of the form

$$\mathcal{N}^{(c)} := (\lfloor n^c \rfloor)_{n=1}^{\infty} \quad (c > 1, c \notin \mathbb{N}),$$

where $\lfloor t \rfloor$ denotes the integer part of any $t \in \mathbb{R}$. Such sequences have been named in honor of Piatetski-Shapiro [1] who, in 1953, proved that $\mathcal{N}^{(c)}$ contains infinitely many primes provided that $c \in (1, \frac{12}{11})$. The range for $c$ in which it is known that $\mathcal{N}^{(c)}$ contains infinitely many primes has been enlarged many times over the years and is currently known to hold for all $c \in (1, \frac{243}{205})$, thanks to Rivat and Wu [2].

For fixed real numbers $\alpha$ and $\beta$, the associated non-homogeneous Beatty sequence is the sequence of integers defined by

$$\mathcal{B}_{\alpha,\beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty},$$

which are also called generalized arithmetic progressions. If $\alpha$ is irrational, it follows from a classical exponential sum estimate of Vinogradov [3] that $\mathcal{B}_{\alpha,\beta}$ contains infinitely many prime numbers.

Carmichael numbers are the composite natural numbers $N$ with the property that $N|(a^N - a)$ for every integer $a$. In 1994, Alford et al. [4] proved that there exist infinitely many Carmichael numbers. Baker et al. [5] showed that for every $c \in (1, \frac{147}{145})$, there are infinitely many Carmichael numbers composed entirely of Piatetski-Shapiro primes. Banks and Yeager [6] showed that there are infinitely many Carmichael numbers composed solely of primes from the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ for $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$ and $\alpha$ is irrational and of finite type.

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Since both Piatetski-Shapiro sequences and Beatty sequences produce infinitely many primes, Guo [7] investigated the intersection between a Piatetski-Shapiro sequence and a Beatty sequence by defining

$$\pi^{(c)}_{\alpha,\beta}(x) := \#\{p \leqslant x : p \in \mathscr{N}^{(c)} \cap \mathcal{B}_{\alpha,\beta}\}$$

and derived that

$$\pi^{(c)}_{\alpha,\beta}(x) = \frac{x^{\frac{1}{c}}}{\alpha \log x} + O\left(\frac{x^{\frac{1}{c}}}{\log^2 x}\right),$$

for $c \in (1, \frac{14}{13})$. Later, Guo et al. [8] extend the range of $c$ in this theorem to $(1, \frac{12}{11})$.

Guo and Qi [9] considered the following generalized Piatetski-Shapiro sequences:

$$\mathcal{N}^{(c)}_{\alpha,\beta} := (\lfloor \alpha n^c + \beta \rfloor)^{\infty}_{n=1}$$

and proved that there are infinitely many Carmichael numbers composed solely of primes from the numbers of the set $\mathcal{N}^{(c)}_{\alpha,\beta}$ for $c \in (1, \frac{64}{63})$.

In this article, we are interested in the relation between Carmichael numbers and the Piatetski-Shapiro primes in a Beatty sequence. For $(a, d) = 1$, let

$$\pi^{(c)}_{\alpha,\beta}(x;\, d, a) := \#\{p \leqslant x : p \in \mathscr{N}^{(c)} \cap \mathcal{B}_{\alpha,\beta} \quad \text{and} \quad p \equiv a \bmod d\}.$$

We prove the following theorem:

**Theorem 1.1.** *Let $\alpha \geqslant 1$ and $\beta$ be real numbers. Let $\alpha$ be irrational and of finite type. Let $c \in \left(1, \frac{12}{11}\right)$ and $\gamma := c^{-1}$.*

$$\pi^{(c)}_{\alpha,\beta}(x;\, d, a) = \alpha^{-1}\gamma x^{\gamma-1}\pi(x;\, d, a) + \alpha^{-1}\gamma(1-\gamma)\int\limits_2^x u^{\gamma-2}\pi(u;\, d, a)du + O\left(x^{\frac{7}{13}\gamma+\frac{11}{26}+\varepsilon}\right),$$

*where $\pi(x;\, d, a) := \#\{p \leqslant x : p \equiv a \bmod d\}$.*

**Theorem 1.2.** *Let $c \in (1, \frac{19137}{18746})$, $\alpha$ be irrational and of finite type. There are infinitely many Carmichael numbers composed of entirely the primes from the set $\mathscr{N}^{(c)} \cap \mathcal{B}_{\alpha,\beta}$.*

# 2 Preliminaries

## 2.1 Notation

We denote by $\lfloor t \rfloor$ and $\{t\}$ the integral part and the fractional part of $t$, respectively. As is customary, we put

$$\mathbf{e}(t) := e^{2\pi i t} \quad \text{and} \quad \{t\} := t - \lfloor t \rfloor.$$

Throughout the article, we make considerable use of the sawtooth function defined by

$$\psi(t) := t - \lfloor t \rfloor - \frac{1}{2} = \{t\} - \frac{1}{2}.$$

The notation $||t||$ is used to denote the distance from the real number $t$ to the nearest integer; that is,

$$||t|| := \min_{n\in\mathbb{Z}} |t - n|.$$

Let $\mathbb{P}$ denote the set of primes in $\mathbb{N}$. The letter $p$ always denotes a prime. For a Beatty sequence $(\lfloor \alpha n + \beta \rfloor)^{\infty}_{n=1}$, we denote $\omega := \alpha^{-1}$. We represent $\gamma := c^{-1}$ for the Piatetski-Shapiro sequence $(\lfloor n^c \rfloor)^{\infty}_{n=1}$. We use notation of the form $m \sim M$ as an abbreviation for $M < m \leqslant 2M$.

Throughout the article, $\varepsilon$ always denotes an arbitrarily small positive constant, which may not be the same at different occurrences; the implied constants in symbols $O$, $\ll$ and $\gg$ may depend (where obvious) on the parameters $\alpha, \beta, c, \varepsilon$ but are absolute otherwise. For given functions $F$ and $G$, the notations $F \ll G, G \gg F$ and $F = O(G)$ are all equivalent to the statement that the inequality $|F| \leqslant C|G|$ holds with some constant $C > 0$.

## 2.2 Type of an irrational number

For any irrational number $\alpha$, we define its type $\tau = \tau(\alpha)$ by the following definition:

$$\tau \coloneqq \sup\left\{t \in \mathbb{R} : \liminf_{n\to\infty} n^t \|\alpha n\| = 0\right\}.$$

Using Dirichlet's approximation theorem, one can see that $\tau \geqslant 1$ for every irrational number $\alpha$. Thanks to the work of Khintchine [10] and Roth [11,12], it is known that $\tau = 1$ for almost all real numbers, in the sense of the Lebesgue measure, and for all irrational algebraic numbers, respectively. Moreover, if $\alpha$ is an irrational number of type $\tau < \infty$, then so are $\alpha^{-1}$ and $n\alpha^{-1}$ for all integer $n \geqslant 1$ [13].

## 2.3 Technical lemmas

We need the following well-known approximation of Vaaler [14].

**Lemma 2.1.** *For any $H \geqslant 1$, there exist numbers $a_h, b_h$ such that*

$$\left|\psi(t) - \sum_{0<|h|\leqslant H} a_h \, \mathbf{e}(th)\right| \leqslant \sum_{|h|\leqslant H} b_h \, \mathbf{e}(th), \quad a_h \ll \frac{1}{|h|}, b_h \ll \frac{1}{H}.$$

**Lemma 2.2.** *For an arithmetic function $g$ and $N' \sim N$, we have*

$$\sum_{N<p\leqslant N'} g(p) \ll \frac{1}{\log N} \max_{N<N_1\leqslant 2N} \left|\sum_{N<n\leqslant N_1} \Lambda(n) g(n)\right| + N^{1/2}.$$

**Proof.** See the argument on page 48 of [15]. □

**Lemma 2.3.** *Suppose that*

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2},$$

*with $(a, q) = 1, q \geqslant 1, |\theta| \leqslant 1$. Then there holds*

$$\sum_{\substack{m\leqslant N \\ m\equiv a \bmod d}} \Lambda(m)\mathbf{e}(m\alpha) \ll \left(Nqd^{-\frac{1}{2}} + N^{4/5} + N^{1/2}q^{1/2}\right)(\log N)^3 .$$

**Proof.** It is a simplified and weakened version of a theorem of Balog and Perelli [16]. □

**Lemma 2.4.** *Suppose that $\alpha$ is a fixed irrational number of finite type $\tau < \infty$ and $h \geqslant 1, m$ are integers. Then we have*

$$\sum_{\substack{m\leqslant M \\ m\equiv a \bmod d}} \Lambda(m)\mathbf{e}(\alpha hm) \ll h^{1/2} M^{1-1/(2\tau)+\varepsilon} + M^{1-\varepsilon}.$$

**Proof.** For any sufficiently small $\varepsilon > 0$, we set $\varrho = \tau + \varepsilon$. Since $\alpha$ is of type $\tau$, there exists some constant $\mathfrak{c} > 0$ such that

$$\|\alpha n\| > \mathfrak{c} n^{-\varrho}, \quad n \geqslant 1. \tag{2.1}$$

For given $h$ with $0 < h \leqslant H$, let $b/d$ be the convergent in the continued fraction expansion of $\alpha h$, which has the largest denominator $d$ not exceeding $M^{1-\eta}$ for a sufficiently small positive number $\eta$. Then we derive that

$$\left|\alpha h - \frac{b}{d}\right| \leqslant \frac{1}{dM^{1-\eta}} \leqslant \frac{1}{d^2}, \tag{2.2}$$

which combined with (2.1) yields

$$M^{-1+\eta} \geqslant |\alpha h d - b| \geqslant \|\alpha h d\| > \mathfrak{c}(hd)^{-\varrho}.$$

Taking $C_0 \coloneqq \mathfrak{c}^{1/\varrho}$, we obtain

$$d > C_0 h^{-1} M^{1/\varrho - \eta/\varrho}. \tag{2.3}$$

Combining (2.2) and (2.3), applying Lemma 2.3 and the fact that $d \leqslant M^{1-\eta}$, we deduce that

$$\begin{aligned}\sum_{\substack{m \leqslant M \\ m \equiv a \bmod d}} \Lambda(m)\mathbf{e}(\alpha h m) &\ll (Md^{-1/2} + M^{4/5} + M^{1/2}d^{1/2})(\log M)^3 \\ &\ll (h^{1/2}M^{1-1/(2\varrho)+\eta/(2\varrho)} + M^{4/5} + M^{1-\eta/2})(\log M)^3 \\ &\ll h^{1/2}M^{1-1/(2\tau)+\varepsilon} + M^{1-\varepsilon}.\end{aligned}$$

This completes the proof of Lemma 2.4. □

The following lemma gives a characterization of the numbers in the Beatty sequence $\mathcal{B}_{\alpha,\beta}$.

**Lemma 2.5.** *A natural number $m$ has the form $\lfloor \alpha n + \beta \rfloor$ if and only if $\mathcal{X}_{\alpha,\beta}(m) = 1$, where $\mathcal{X}_{\alpha,\beta}(m) \coloneqq \lfloor -\alpha^{-1}(m - \beta)\rfloor - \lfloor -\alpha^{-1}(m + 1 - \beta)\rfloor$.*

**Proof.** Note that an integer $m$ has the form $m = \lfloor \alpha n + \beta \rfloor$ for some integer $n$ if and only if

$$\frac{m - \beta}{\alpha} \leqslant n < \frac{m - \beta + 1}{\alpha}.$$ □

Finally, we use the following lemma, which provides a characterization of the numbers that occur in the Piatetski-Shapiro sequence $\mathcal{N}^{(c)}$.

**Lemma 2.6.** *A natural number $m$ has the form $\lfloor n^c \rfloor$ if and only if $\mathcal{X}^{(c)}(m) = 1$, where $\mathcal{X}^{(c)}(m) \coloneqq \lfloor -m^\gamma \rfloor - \lfloor -(m+1)^\gamma \rfloor$. Moreover,*

$$\mathcal{X}^{(c)}(m) = \gamma m^{\gamma-1} + \psi(-(m+1)^\gamma) - \psi(-m^\gamma) + O(m^{\gamma-2}).$$

**Proof.** The proof of Lemma 2.6 is similar to that of Lemma 2.5, so we omit the details herein. □

**Lemma 2.7.** *For $1 < c < \frac{2817}{2426}$, there holds*

$$\pi^{(c)}(x) = \sum_{p \leqslant x} \mathcal{X}^{(c)}(p) = \frac{x^\gamma}{\log x} + O\left(\frac{x^\gamma}{\log^2 x}\right). \tag{2.4}$$

**Proof.** See Theorem 1 of Rivat and Sargos [17]. □

**Lemma 2.8.** *Suppose that*

$$L(H) = \sum_{i=1}^{m} A_i H^{a_i} + \sum_{j=1}^{n} B_j H^{-b_j},$$

*where $A_i$, $B_j$, $a_i$, and $b_j$ are positive. Assume further that $H_1 \leqslant H_2$. Then there exists some $H$ with $H_1 \leqslant H \leqslant H_2$ and*

$$L(H) \ll \sum_{i=1}^{m} A_i H_1^{a_i} + \sum_{j=1}^{n} B_j H_2^{-b_j} + \sum_{i=1}^{m}\sum_{j=1}^{n} (A_i^{b_j} B_j^{a_i})^{1/(a_i+b_j)}.$$

*The implied constant depends only on $m$ and $n$.*

**Proof.** See Lemma 3 of Srinivasan [18]. □

**Lemma 2.9.** *For real numbers $m_1$, $m_2$, and $N < t \leqslant N_1$, we have*

$$\left|\sum_{N<n\leqslant N_1} \Lambda(n)\mathbf{e}(hn^{\gamma} + m_1 n + m_2)\right| \ll N^{\varepsilon}\left(|h|^{\frac{1}{6}} N^{\frac{\gamma}{6}+\frac{3}{4}} + |h|^{-\frac{1}{3}} N^{1-\frac{\gamma}{3}} + |h|^{\frac{1}{4}} N^{\frac{\gamma}{4}+\frac{5}{8}} + |h|^{-\frac{1}{4}} N^{1-\frac{\gamma}{4}} + N^{\frac{22}{25}}\right).$$

**Proof.** See [8, Lemma 2.14]. □

# 3 Proof of Theorem 1.1

For a Beatty sequence

$$\mathcal{B}_{\alpha,\beta} \coloneqq \lfloor \alpha n + \beta \rfloor,$$

recall that $\omega \coloneqq \alpha^{-1}$. By the definition of $\pi_{\alpha,\beta}^{(c)}(x)$, we have that

$$\pi_{\alpha,\beta}^{(c)}(x;\, d, a) = \sum_{\substack{p\leqslant x\\ p\equiv a \bmod d}} \mathcal{X}_{\alpha,\beta}(p)\mathcal{X}^{(c)}(p) = S_1 + S_2 + S_3,$$

where

$$S_1 \coloneqq \sum_{\substack{p\leqslant x\\ p\equiv a \bmod d}} \omega\mathcal{X}^{(c)}(p);$$

$$S_2 \coloneqq \sum_{\substack{p\leqslant x\\ p\equiv a \bmod d}} (\gamma p^{\gamma-1} + O(p^{\gamma-2}))(\psi(-\omega(p+1-\beta)) - \psi(-\omega(p-\beta))),$$

$$S_3 \coloneqq \sum_{\substack{p\leqslant x\\ p\equiv a \bmod d}} (\psi(-(p+1)^{\gamma}) - \psi(-p^{\gamma}))(\psi(-\omega(p+1-\beta)) - \psi(-\omega(p-\beta))).$$

A partial summation gives

$$S_1 = \omega\gamma x^{\gamma-1}\pi(x;\, d, a) + \omega\gamma(1-\gamma)\int_2^x u^{\gamma-2}\pi(u;\, d, a)du + O(x^{\gamma-1} + 1).$$

By applying Lemma 2.1, we take $H_1 \coloneqq x^{\varepsilon}$ and let $H_2$ be chosen later. With a sufficiently small positive number $\varepsilon$, we have that

$$\begin{aligned}&\psi(-\omega(p+1-\beta)) - \psi(-\omega(p-\beta))\\ &= \sum_{0<|h_1|\leqslant H_1} a_{h_1}(\mathbf{e}(\omega h_1(p+1-\beta)) - \mathbf{e}(\omega h_1(p-\beta))) + O\left(\sum_{|h_1|\leqslant H_1} b_{h_1}(\mathbf{e}(\omega h_1(p+1-\beta)) + \mathbf{e}(\omega h_1(p-\beta)))\right)\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}&\psi(-(p+1)^{\gamma})-\psi(-p^{\gamma})\\&=\sum_{0<|h_2|\leqslant H_2}a_{h_2}(\mathbf{e}(h_2(p+1)^{\gamma})-\mathbf{e}(h_2p^{\gamma}))+O\left(\sum_{|h_2|\leqslant H_2}b_{h_2}(\mathbf{e}(h_2(p+1)^{\gamma})+\mathbf{e}(h_2p^{\gamma}))\right).\end{aligned}\tag{3.2}$$

We mention that for $j=1,2$ there holds

$$a_{h_j}\ll|h_j|^{-1}\quad\text{and}\quad b_{h_j}\ll H_j^{-1}.$$

## 3.1 Upper bounds of $S_2$

Let $N\leqslant x$ and $N_1\leqslant 2N$. We write $S_2=S_{21}+O(S_{22})$, where

$$S_{21}:=\sum_{\substack{p\leqslant N\\p\equiv a\bmod d}}\gamma p^{\gamma-1}(\psi(-\omega(p+1-\beta))-\psi(-\omega(p-\beta)))$$

and

$$S_{22}:=\sum_{\substack{p\leqslant N\\p\equiv a\bmod d}}\gamma p^{\gamma-2}(\psi(-\omega(p+1-\beta))-\psi(-\omega(p-\beta))).$$

By (3.1), Lemma 2.2 and a splitting argument, we obtain that $S_{21}=S_{23}+O(S_{24})$, where

$$S_{23}:=\sum_{\substack{N<n\leqslant N_1\\n\equiv a\bmod d}}\sum_{0<|h_1|\leqslant H_1}a_{h_1}n^{\gamma-1}\Lambda(n)(\mathbf{e}(\omega h_1(n+1-\beta))-\mathbf{e}(\omega h_1(n-\beta)))\tag{3.3}$$

and

$$S_{24}:=\sum_{\substack{N<n\leqslant N_1\\n\equiv a\bmod d}}\sum_{0<|h_1|\leqslant H_1}b_{h_1}n^{\gamma-1}\Lambda(n)(\mathbf{e}(\omega h_1(n+1-\beta))-\mathbf{e}(\omega h_1(n-\beta))).\tag{3.4}$$

First, we estimate $S_{23}$. Let

$$\theta_{h_1}:=\mathbf{e}(\omega h_1)-1.\tag{3.5}$$

It follows from partial summation and the trivial estimate $\theta_{h_1}\ll 1$ that

$$\begin{aligned}S_{23}&\ll\sum_{0<h_1\leqslant H_1}a_{h_1}\sum_{\substack{N<n\leqslant N_1\\n\equiv a\bmod d}}n^{\gamma-1}\Lambda(n)\theta_{h_1}\mathbf{e}(\omega h_1(n-\beta))\\&\ll N^{\gamma-1}\sum_{0<h_1\leqslant H_1}h_1^{-1}\max_{N_1\leqslant 2N}\left|\sum_{\substack{N<n\leqslant N_1\\n\equiv a\bmod d}}\Lambda(n)\mathbf{e}(\omega h_1 n)\right|.\end{aligned}\tag{3.6}$$

Hence, we need to bound

$$T:=\sum_{\substack{N<n\leqslant N_1\\n\equiv a\bmod d}}\Lambda(n)\mathbf{e}(\omega h_1 n).\tag{3.7}$$

By Lemma 2.4, we obtain

$$T\ll h_1^{\frac{1}{2}}N^{1-\frac{1}{2\tau}+\varepsilon}+N^{1-\varepsilon},\tag{3.8}$$

for $\varepsilon$ being a small positive number.

Now we work on the bound of $S_{32}$. The contribution of $S_{24}$ from $h_1=0$ is

$$2b_0\sum_{\substack{n\leqslant N\\n\equiv a\bmod d}}\Lambda(n)n^{\gamma-1}\ll\frac{b_0N^{\gamma}}{\phi(d)}\ll H_1^{-1}N^{\gamma},\tag{3.9}$$

where the function $\phi(d)$ is the Euler function and $b_0 \ll H_1^{-1}$. The contribution from $h_1 \neq 0$ is

$$\ll N^{\gamma-1}H_1^{-1}\max_{N_1\leqslant 2N}\sum_{0<h_1\leqslant H_1}\left|\sum_{\substack{N<n\leqslant N_1\\ n\equiv a \bmod d}}\Lambda(n)\mathbf{e}(\omega h_1 n)\right|. \tag{3.10}$$

The right-hand side of (3.11) can be estimated by the same method of (3.7). Therefore, by inserting (3.8) into (3.6) and (3.11) and combining with (3.9), it follows that

$$S_{21} \ll S_{23} + S_{24} \ll H_1^{\frac{1}{2}}N^{\gamma-\frac{1}{2\tau}+\varepsilon} + N^{\gamma+\varepsilon} + H_1^{-1}N^{\gamma} \ll N^{\gamma+\varepsilon},$$

where we use $H_1 = N^{\varepsilon}$. Moreover, the bound of $S_{22}$ can be estimated similarly. Hence, we obtain

$$S_2 \ll S_{21} + S_{22} \ll N^{\gamma+\varepsilon}. \tag{3.11}$$

## 3.2 Upper bounds of $S_3$

We only give the details of the estimation of $S_3$, By (3.1) and (3.2), it is easy to see that

$$S_3 = S_{31} + O(S_{32} + S_{33} + S_{34}), \tag{3.12}$$

where

$$\begin{aligned}
S_{31} &:= \sum_{\substack{p\leqslant x\\ p\equiv a \bmod d}}\sum_{0<|h_2|\leqslant H_2} a_{h_2}(\mathbf{e}(h_2(p+1)^{\gamma}) - \mathbf{e}(h_2p^{\gamma}))\\
&\quad\times \sum_{0<|h_1|\leqslant H_1} a_{h_1}(\mathbf{e}(\omega h_1(p+1-\beta)) - \mathbf{e}(\omega h_1(p-\beta))),\\
S_{32} &:= \sum_{\substack{p\leqslant x\\ p\equiv a \bmod d}}\sum_{0<|h_2|\leqslant H_2} a_{h_2}(\mathbf{e}(h_2(p+1)^{\gamma}) - \mathbf{e}(h_2p^{\gamma}))\\
&\quad\times \sum_{|h_1|\leqslant H_1} b_{h_1}(\mathbf{e}(\omega h_1(p+1-\beta)) + \mathbf{e}(\omega h_1(p-\beta))),\\
S_{33} &:= \sum_{\substack{p\leqslant x\\ p\equiv a \bmod d}}\sum_{|h_2|\leqslant H_2} b_{h_2}(\mathbf{e}(h_2(p+1)^{\gamma}) + \mathbf{e}(h_2p^{\gamma}))\\
&\quad\times \sum_{0<|h_1|\leqslant H_1} a_{h_1}(\mathbf{e}(\omega h_1(p+1-\beta)) - \mathbf{e}(\omega h_1(p-\beta))),\\
S_{34} &:= \sum_{\substack{p\leqslant x\\ p\equiv a \bmod d}}\sum_{|h_2|\leqslant H_2} b_{h_2}(\mathbf{e}(h_2(p+1)^{\gamma}) + \mathbf{e}(h_2p^{\gamma}))\\
&\quad\times \sum_{|h_1|\leqslant H_1} b_{h_1}(\mathbf{e}(\omega h_1(p+1-\beta)) + \mathbf{e}(\omega h_1(p-\beta))).
\end{aligned}$$

### 3.2.1 Estimation of $S_{31}$

By Lemma 2.2 and a splitting argument, we estimate $S_{31}$ by considering

$$\begin{aligned}
&\sum_{\substack{N<n\leqslant N_1\\ n\equiv a \bmod d}}\Lambda(n)\sum_{0<|h_2|\leqslant H_2} a_{h_2}(\mathbf{e}(h_2(n+1)^{\gamma}) - \mathbf{e}(h_2n^{\gamma}))\\
&\quad\times \sum_{0<|h_1|\leqslant H_1} a_{h_1}(\mathbf{e}(\omega h_1(n+1-\beta)) - \mathbf{e}(\omega h_1(n-\beta))).
\end{aligned} \tag{3.13}$$

Define

$$\phi_{h_2}(t) := \mathbf{e}(h_2((t+1)^\gamma - t^\gamma)) - 1. \tag{3.14}$$

Then we have

$$\phi_{h_2}(t) \ll |h_2| t^{\gamma-1} \quad \text{and} \quad \frac{\partial \phi_{h_2}(t)}{\partial t} \ll |h_2| t^{\gamma-2}.$$

It follows from the aforementioned estimate, (3.5) and partial summation that the formula (3.13) is

$$\begin{aligned}
&\ll \sum_{0<|h_2|\leqslant H_2} \frac{1}{|h_2|} \left| \sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n)\phi_{h_2}(n)\mathbf{e}(h_2 n^\gamma) \sum_{0<|h_1|\leqslant H_1} a_{h_1}\theta_{h_1}\mathbf{e}(\omega h_1(n-\beta)) \right| \\
&\ll \sum_{0<|h_2|\leqslant H_2} \frac{1}{|h_2|} \left| \int_N^{N_1} \phi_{h_2}(t) \mathrm{d} \sum_{\substack{N<n\leqslant t \\ n\equiv a \bmod d}} \Lambda(n)\mathbf{e}(h_2 n^\gamma) \sum_{0<|h_1|\leqslant H_1} a_{h_1}\theta_{h_1}\mathbf{e}(\omega h_1(n-\beta)) \right| \\
&\ll \sum_{0<|h_2|\leqslant H_2} \frac{1}{|h_2|} |\phi_{h_2}(N)| \left| \sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n)\mathbf{e}(h_2 n^\gamma) \sum_{0<|h_1|\leqslant H_1} a_{h_1}\theta_{h_1}\mathbf{e}(\omega h_1(n-\beta)) \right| \\
&\quad + \int_N^{N_1} \sum_{0<|h_2|\leqslant H_2} \frac{1}{|h_2|} \left| \frac{\partial \phi_{h_2}(t)}{\partial t} \right| \left| \sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n)\mathbf{e}(h_2 n^\gamma) \sum_{0<|h_1|\leqslant H_1} a_{h_1}\theta_{h_1}\mathbf{e}(\omega h_1(n-\beta)) \right| \mathrm{d}t \qquad (3.15) \\
&\ll N^{\gamma-1} \max_{N_1\leqslant 2N} \sum_{0<|h_2|\leqslant H_2} \left| \sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n)\mathbf{e}(h_2 n^\gamma) \sum_{0<|h_1|\leqslant H_1} a_{h_1}\theta_{h_1}\mathbf{e}(\omega h_1(n-\beta)) \right| \\
&= N^{\gamma-1} \max_{N_1\leqslant 2N} \sum_{0<|h_2|\leqslant H_2} \left| \sum_{0<|h_1|\leqslant H_1} a_{h_1}\theta_{h_1} \sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n)\mathbf{e}(h_2 n^\gamma + \omega h_1 n - \omega h_1 \beta) \right| \\
&\ll N^{\gamma-1} \sum_{0<|h_1|\leqslant H_1} \frac{1}{|h_1|} \max_{N_1\leqslant 2N} \sum_{0<|h_2|\leqslant H_2} \left| \sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n)\mathbf{e}(h_2 n^\gamma + \omega h_1 n - \omega h_1 \beta) \right|.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n)\mathbf{e}(h_2 n^\gamma + \omega h_1 n - \omega h_1 \beta) \\
&\quad = \frac{1}{d} \sum_{m=1}^{d} \sum_{N<n\leqslant N_1} \Lambda(n)\mathbf{e}\left(h_2 n^\gamma + \omega h_1 n - \omega h_1 \beta + \frac{(n-a)m}{d}\right).
\end{aligned}$$

Hence, we need to bound

$$T_1 := \sum_{N<n\leqslant N_1} \Lambda(n)\mathbf{e}\left(h_2 n^\gamma + \left(\omega h_1 + \frac{m}{d}\right)n + \frac{am}{d} - \omega h_1 \beta\right)$$

By Lemma 2.9, we have

$$T_1 N^{-\varepsilon} \ll |h_2|^{\frac{1}{6}} N^{\frac{\gamma}{6}+\frac{3}{4}} + |h_2|^{-\frac{1}{3}} N^{1-\frac{\gamma}{3}} + |h_2|^{\frac{1}{4}} N^{\frac{\gamma}{4}+\frac{5}{8}} + |h_2|^{-\frac{1}{4}} N^{1-\frac{\gamma}{4}} + N^{\frac{22}{25}}. \tag{3.16}$$

Recalling $H_1 = N^\varepsilon$ and inserting (3.16) to (3.15), we have

$$S_{31} N^{-\varepsilon} \ll H_2^{\frac{7}{6}} N^{\frac{7\gamma}{6}-\frac{1}{4}} + H_2^{\frac{2}{3}} N^{\frac{2\gamma}{3}} + H_2^{\frac{5}{4}} N^{\frac{5\gamma}{4}-\frac{3}{8}} + H_2^{\frac{3}{4}} N^{\frac{3\gamma}{4}} + H N^{\gamma-\frac{3}{25}}. \tag{3.17}$$

### 3.2.2 Estimations of $S_{32}$ and $S_{33}$

We only give the proof of $S_{32}$ since the bound of $S_{33}$ can be obtained similarly. Let $N \leqslant x$ and $N_1 \leqslant 2N$. By Lemma 2.2 and a splitting argument, we can see

$$S_{32} = \sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n) \sum_{0<|h_2|\leqslant H_2} a_{h_2}(\mathbf{e}(h_2(n+1)^\gamma) - \mathbf{e}(h_2 n^\gamma)) \times \sum_{0<|h_1|\leqslant H_1} b_{h_1}(\mathbf{e}(\omega h_1(n+1-\beta)) - \mathbf{e}(\omega h_1(n-\beta))).$$

By (3.14) and Lemma 2.9, the contribution of $S_{32}$ from $h_1 = 0$ is

$$\begin{aligned} &\ll H_1^{-1}N^{\gamma-1} \sum_{0<|h_2|\leqslant H_2} \max_{N_1\leqslant 2N} \left| \sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n)(\mathbf{e}(\theta h_2 n^\gamma)) \right| \\ &\ll H_1^{-1}N^{\gamma-1} \sum_{0<|h_2|\leqslant H_2} \max_{N_1\leqslant 2N} \left| \frac{1}{d} \sum_{m=1}^{d} \sum_{N<n\leqslant N_1} \Lambda(n)\mathbf{e}\left(h_2 n^\gamma + \frac{m(n-a)}{d}\right) \right| \\ &\ll N^\varepsilon \left( H_2^{\frac{7}{6}} N^{\frac{7\gamma}{6}-\frac{1}{4}} + H_2^{\frac{2}{3}} N^{\frac{2\gamma}{3}} + H_2^{\frac{5}{4}} N^{\frac{5\gamma}{4}-\frac{3}{8}} + H_2^{\frac{3}{4}} N^{\frac{3\gamma}{4}} + HN^{\gamma-\frac{3}{25}} \right). \end{aligned} \tag{3.18}$$

The contribution of $S_{32}$ from $h_1 \neq 0$ is

$$\sum_{\substack{p\leqslant x \\ p\equiv a \bmod d}} \sum_{0<|h_2|\leqslant H_2} a_{h_2}(\mathbf{e}(h_2(p+1)^\gamma) - \mathbf{e}(h_2 p^\gamma)) \times \sum_{0<|h_1|\leqslant H_1} b_{h_1}(\mathbf{e}(\omega h_1(p+1-\beta)) + \mathbf{e}(\omega h_1(p-\beta))),$$

which can be get the upper bound (3.18) by the same method of $S_{31}$. So we have

$$(S_{32} + S_{33})N^{-\varepsilon} \ll H_2^{\frac{7}{6}} N^{\frac{7\gamma}{6}-\frac{1}{4}} + H_2^{\frac{2}{3}} N^{\frac{2\gamma}{3}} + H_2^{\frac{5}{4}} N^{\frac{5\gamma}{4}-\frac{3}{8}} + H_2^{\frac{3}{4}} N^{\frac{3\gamma}{4}} + HN^{\gamma-\frac{3}{25}}. \tag{3.19}$$

### 3.2.3 Estimation of $S_{34}$ and conclusions

The contribution of $S_{34}$ from $h_1 = h_2 = 0$ is

$$\sum_{\substack{p\leqslant N \\ p\equiv a \bmod d}} H_2^{-1}H_1^{-1} \ll H_2^{-1}N^{1+\varepsilon}. \tag{3.20}$$

By (3.14) and Lemma 2.9, the contribution of $S_{34}$ from $h_1 = 0$ and $h_2 \neq 0$ is

$$\begin{aligned} &2b_0 \sum_{\substack{p\leqslant x \\ p\equiv a \bmod d}} \sum_{0<|h_2|\leqslant H_2} b_{h_2}(\mathbf{e}(h_2(p+1)^\gamma) + \mathbf{e}(h_2 p^\gamma)) \\ &\ll H_1^{-1}H_2^{-1} \sum_{0<|h_2|\leqslant H_2} \max_{N_1\leqslant 2N} \left| \sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n)(\mathbf{e}(h_2 n^\gamma)) \right| \\ &\ll N^\varepsilon \left( H_2^{\frac{1}{6}} N^{\frac{\gamma}{6}+\frac{3}{4}} + H_2^{-\frac{1}{3}} N^{1-\frac{\gamma}{3}} + H_2^{\frac{1}{4}} N^{\frac{\gamma}{4}+\frac{5}{8}} + H_2^{-\frac{1}{4}} N^{1-\frac{\gamma}{4}} + N^{\frac{22}{25}} \right), \end{aligned} \tag{3.21}$$

where $H_1 = N^{\varepsilon}$ and $b_{h_j} \ll \frac{1}{H}$. Similarly, by (3.5) and Lemma 2.4, the contribution of $S_{34}$ from $h_1 \neq 0$ and $h_2 = 0$ is

$$\begin{aligned}
&2b_0 \sum_{\substack{p \leqslant x \\ p \equiv a \bmod d}} \sum_{0<|h_1|\leqslant H_1} b_{h_1}(\mathbf{e}(\omega h_1(p+1-\beta)) + \mathbf{e}(\omega h_1(p-\beta))) \\
&\ll H_1^{-1}H_2^{-1} \sum_{0<|h_1|\leqslant H_1} \max_{N_1 \leqslant 2N} \left| \sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n)(\mathbf{e}(\omega h_1 n)) \right| \\
&\ll H_2^{-1}(H_1^{\frac{1}{2}} N^{1-\frac{1}{2\tau}+\varepsilon} + N^{1-\varepsilon}) \ll H_2^{-1}N^{1+\varepsilon}.
\end{aligned} \tag{3.22}$$

The contribution of $S_{34}$ from $h_1 \neq 0$ and $h_2 \neq 0$ is

$$\begin{aligned}
&\sum_{\substack{N<n\leqslant N_1 \\ n\equiv a \bmod d}} \Lambda(n) \sum_{0<|h_2|\leqslant H_2} b_{h_2}(\mathbf{e}(h_2(n+1)^{\gamma}) + \mathbf{e}(h_2 n^{\gamma})) \\
&\times \sum_{0<|h_1|\leqslant H_1} b_{h_1}(\mathbf{e}(\omega h_1(n+1-\beta)) + \mathbf{e}(\omega h_1(n-\beta))),
\end{aligned}$$

which can be estimated similarly. Now the estimation

$$S_{34}N^{-\varepsilon} \ll H_2^{\frac{1}{6}} N^{\frac{\gamma}{6}+\frac{3}{4}} + H_2^{-\frac{1}{3}} N^{1-\frac{\gamma}{3}} + H_2^{\frac{1}{4}} N^{\frac{\gamma}{4}+\frac{5}{8}} + H_2^{-\frac{1}{4}} N^{1-\frac{\gamma}{4}} + N^{\frac{22}{25}} + H_2^{-1}N. \tag{3.23}$$

follows from (3.20), (3.21), and (3.22). In the end, by combining (3.17), (3.19), (3.23), (3.12), and (3.11), one has

$$\begin{aligned}
(S_2+S_3)N^{-\varepsilon} \ll{}& H_2^{\frac{1}{6}} N^{\frac{\gamma}{6}+\frac{3}{4}} + H_2^{-\frac{1}{3}} N^{1-\frac{\gamma}{3}} + H_2^{\frac{1}{4}} N^{\frac{\gamma}{4}+\frac{5}{8}} + H_2^{-\frac{1}{4}} N^{1-\frac{\gamma}{4}} + N^{\frac{22}{25}} + H_2^{\frac{7}{6}} N^{\frac{7\gamma}{6}-\frac{1}{4}} \\
&+ H_2^{\frac{2}{3}} N^{\frac{2\gamma}{3}} + H_2^{\frac{5}{4}} N^{\frac{5\gamma}{4}-\frac{3}{8}} + H_2^{\frac{3}{4}} N^{\frac{3\gamma}{4}} + HN^{\gamma-\frac{3}{25}} + H_2^{-1}N.
\end{aligned}$$

By using Lemma 2.8, we obtain

$$\begin{aligned}
(S_2+S_3)N^{-\varepsilon} \ll{}& N^{\frac{3\gamma}{4}} + N^{\gamma-\frac{3}{25}} + N^{\frac{5\gamma}{4}-\frac{3}{8}} + N^{\frac{7\gamma}{6}-\frac{1}{4}} + N^{\frac{\gamma}{6}+\frac{3}{4}} + N^{\frac{\gamma}{4}+\frac{5}{8}} + N^{\frac{22}{25}} \\
&+ N^{\frac{7\gamma}{13}+\frac{11}{26}} + N^{\frac{5\gamma}{9}+\frac{7}{18}} + N^{\frac{3\gamma}{7}+\frac{3}{7}} + N^{\frac{\gamma}{2}+\frac{11}{25}} + N^{\frac{\gamma}{7}+\frac{11}{14}} + N^{\frac{\gamma}{5}+\frac{7}{10}}.
\end{aligned}$$

Note that $S_1 \ll x^{\gamma}$, so we need that $S_2 + S_3 \ll x^{\gamma-\varepsilon}$. Hence,

$$\gamma > \max\left(\frac{9}{10}, \frac{5}{6}, \frac{22}{25}, \frac{11}{12}, \frac{7}{8}, \frac{3}{4}\right) = \frac{11}{12}$$

and

$$S_2 + S_3 \ll x^{\frac{7\gamma}{13}+\frac{11}{26}+\varepsilon}.$$

## 4 Sketch of proof of Theorem 1.2

We sketch the proof of Theorem 1.2 because the idea of the proof is close to the proof in [9, Section 4]. We only give the changes that are necessary for our Theorem 1.2.

We set

$$\vartheta(x;\, d, a) \coloneqq \sum_{\substack{p \leqslant x \\ p\equiv a \bmod d}} \log p$$

and consider a weighted counting function

$$\vartheta_{\alpha,\beta}^{(c)}(x;\, d, a) \coloneqq \sum_{\substack{p \leqslant x \\ p \in \pi_{\alpha,\beta}^{(c)}(x) \\ p\equiv a \bmod d}} \log p = \sum_{\substack{p \leqslant x \\ p\equiv a \bmod d}} \mathcal{X}_{\alpha,\beta}(p)\mathcal{X}^{(c)}(p) \log p.$$

By a similar argument as in the proof of Theorem 1.1, we conclude the following.

**Theorem 4.1.** *Let $\alpha \geqslant 1$ and $\beta$ be real numbers. Let $c \in \left(1, \frac{12}{11}\right)$. Then*

$$\vartheta^{(c)}_{\alpha,\beta}(x;\, d, a) = \alpha^{-1}\gamma x^{\gamma-1}\vartheta(x;\, d, a) + \alpha^{-1}\gamma(1-\gamma)\int_2^x u^{\gamma-2}\vartheta(u;\, d, a)\mathrm{d}u + O\left(x^{\frac{7\gamma}{13}+\frac{11}{26}+\varepsilon}\right).$$

The proof of our Theorem 1.2 is similar to [9, Section 4] by switching the conditions

$$1 < c < \frac{14}{13} \quad \text{into} \quad 1 < c < \frac{12}{11},$$
$$-\frac{13}{35} + \frac{2\gamma}{5} \quad \text{into} \quad -\frac{11}{26} + \frac{6\gamma}{13},$$
$$\mathcal{N}^{(c)}_{\alpha,\beta} \quad \text{into} \quad \pi^{(c)}_{\alpha,\beta}$$

and

$$\theta \quad \text{into} \quad \omega.$$

Let $\pi(x, y)$ be the number of those primes for which $p-1$ is free of prime factors exceeding $y$. Let $\mathcal{E}$ be the set of numbers $E$ in the range $0 < E < 1$ for which

$$\pi(x, x^{1-E}) \geqslant x^{1+o(1)} \quad (x \to \infty),$$

where the function implied by $o(1)$ depends only on $E$. By a similar argument as in [5, Page 64–66], we conclude the following statement.

**Lemma 4.2.** *Let $\alpha \geqslant 1$ and $\beta$ be real numbers. Let $c \in \left(1, \frac{38}{37}\right)$. Let $B$ and $B_1$ be positive real numbers such that $B_1 < B < -\frac{11}{26} + \frac{6\gamma}{13}$. For any $E \in \mathcal{E}$, there is a number $x_3$ depending on $c, B, B_1, E$, and $\varepsilon$, such that for any $x \geqslant x_1$, there are at least $x^{EB+(1-B+B_1)(\gamma-1)-\varepsilon}$ Carmichael numbers up to $x$ composed solely of primes from $\pi^{(c)}_{\alpha,\beta}$.*

Taking $B$ and $B_1$ arbitrarily close to $-\frac{11}{26} + \frac{6\gamma}{13}$, Lemma 4.2 implies that there are infinitely many Carmichael numbers composed entirely of the primes from $\pi^{(c)}_{\alpha,\beta}$ with

$$\left(-\frac{11}{26} + \frac{6\gamma}{13}\right)E + \gamma - 1 > 0.$$

Taking $E = 0.7039$ from [19], we eventually have $\gamma > \frac{18746}{19137}$.

# 5 More Beatty sequences

Guo et al. [8] proved that there are infinitely many primes in the intersection of a Piatetski-Shapiro sequence and multiple Beatty sequences with some restrictions; see [8, Theorem 1.3] for more details. We mention that by the similar techniques in the proof of Theorem 1.1 and the proof of [8, Theorem 1.3], Piatetski-Shapiro primes in arithmetic progressions and the intersection of multiple Beatty sequences can also be detected. Therefore, we state the following theorem without proofs.

**Theorem 5.1.** *Suppose that $\xi$ is a positive integer, and $\alpha_1, \dots, \alpha_\xi, \beta_1, \dots, \beta_\xi \in \mathbb{R}$. Let $\alpha_1, \dots, \alpha_\xi > 1$ be irrational and of finite type such that*

$$1, \alpha_1^{-1}, \dots, \alpha_\xi^{-1} \quad \textit{are linearly independent over } \mathbb{Q}.$$

*For* $c \in (1, \frac{12}{11})$, *the counting function*

$$\pi^{(c)}_{\alpha_1,\beta_1;\ldots;\alpha_\xi,\beta_\xi}(x;\ d,a) := \#\{\text{prime}\, p \leqslant x : p \equiv a \bmod d, p \in \mathcal{B}_{\alpha_1,\beta_1} \cap \ldots \cap \mathcal{B}_{\alpha_\xi,\beta_\xi} \cap \mathcal{N}^{(c)}\}$$

*satisfies*

$$\pi^{(c)}_{\alpha_1,\beta_1;\ldots;\alpha_\xi,\beta_\xi}(x;\ d,a) = \alpha_1^{-1}\ldots\alpha_\xi^{-1}\gamma x^{\gamma-1}\pi(x;\ d,a) + \alpha_1^{-1}\ldots\alpha_\xi^{-1}\gamma(1-\gamma)\int_2^x u^{\gamma-2}\pi(u;\ d,a)du + O\left(x^{\frac{7}{13}\gamma+\frac{11}{26}+\varepsilon}\right),$$

*where the implied constant depends only on* $\alpha_1,\ldots,\alpha_\xi$ *and* $c$.

Then by the same technique in the proof of Theorem 1.2, we state the following theorem without proofs.

**Theorem 5.2.** *Suppose that* $\xi$ *is a positive integer, and* $\alpha_1,\ldots,\alpha_\xi,\beta_1,\ldots,\beta_\xi \in \mathbb{R}$. *Let* $\alpha_1,\ldots,\alpha_\xi > 1$ *be irrational and of finite type such that*

$$1, \alpha_1^{-1},\ldots,\alpha_\xi^{-1} \text{ are linearly independent over } \mathbb{Q}.$$

*For* $c \in (1, \frac{19137}{18746})$, *there are infinitely many Carmichael numbers composed entirely of the primes from the set*

$$\mathcal{B}_{\alpha_1,\beta_1} \cap \ldots \cap \mathcal{B}_{\alpha_\xi,\beta_\xi} \cap \mathcal{N}^{(c)}.$$

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# References


[1] I. I. Piatetski-Shapiro, *On the distribution of prime numbers in the sequence of the form* $\lfloor f(n) \rfloor$, Mat. Sb. **33** (1953), 559–566.

[2] J. Rivat and J. Wu, *Prime numbers of the form* $[n^c]$, Glasg. Math. J. **43** (2001), no. 2, 237–254, DOI: https://doi.org/10.1017/S0017089501020080.

[3] I. M. Vinogradov, *A new estimate of a certain sum containing primes (Russian)*, Rec. Math. Moscou, n. Ser. **2(44)** (1937), no. 5, 783–792, English translation: *New estimations of trigonometrical sums containing primes*, C. R. (Dokl.) Acad. Sci. URSS, n. Ser. **17** (1937), 165–166.

[4] W. R. Alford, A. Granville, and C. Pomerance, *There are infinitely many Carmichael numbers*, Ann. of Math. (2) **139** (1994), no. 3, 703–722, DOI: https://doi.org/10.2307/2118576.

[5] R. C. Baker, W. D. Banks, J. Brüdern, I. E. Shparlinski, and A. J. Weingartner, *Piatetski-Shapiro sequences*, Acta Arith. **157** (2013), no. 1, 37–68, DOI: https://doi.org/10.4064/aa157-1-3.

[6] W. C. Banks and A. M. Yeager, *Carmichael numbers composed of primes from a Beatty sequence*, Colloq. Math. **125** (2011), no. 1, 129–137, DOI: https://doi.org/10.4064/cm125-1-9.

[7] V. Z. Guo, *Piatetski-Shapiro primes in a Beatty sequence*, J. Number Theory **156** (2015), 317–330, DOI: https://doi.org/10.1016/j.jnt.2015.05.001.

[8] V. Z. Guo, J. Li, and M. Zhang, *Piatetski-Shapiro primes in the intersection of multiple Beatty sequences*, Rocky Mountain J. Math. **52** (2022), no. 4, 1375–1394. DOI: https://doi.org/10.1216/rmj.2022.52.1375.

[9] V. Z. Guo and J. Qi, *A generalization of Piatetski-Shapiro sequences*, Taiwanese J. Math. **26** (2022), no. 1, 33–47, DOI: https://doi.org/10.11650/tjm/210901.

[10] A. Khintchine, *Zur metrischen Theorie der diophantischen Approximationen*, Math. Z. **24** (1926), no. 1, 706–714.

[11] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20, DOI: https://doi.org/10.1112/S0025579300000644.

[12] K. F. Roth, *Corrigendum to Rational approximations to algebraic numbers*, Mathematika **2** (1955), 168.

[13] J. Qi and V. Z. Guo, *The* $k$*-fold divisor function over Beatty sequences*, Ramanujan J. **66** (2025), no. 3, 54, DOI: https://doi.org/10.1007/s11139-024-00875-w.

[14] J. D. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc. **12** (1985), no. 2, 183–216, DOI: https://doi.org/10.1090/S0273-0979-1985-15349-2.

[15] S. W. Graham and G. Kolesnik, *Van der Corputas Method of Exponential Sums*, Cambridge University Press, Cambridge, 1991.

[16] A. Balog and A. Perelli, *Exponential sums over primes in an arithmetic progression*, Proc. Amer. Math. Soc. **93** (1985), no. 4, 578–582, DOI: https://doi.org/10.1090/S0002-9939-1985-0776182-0.

[17] J. Rivat and S. Sargos, *Nombres premiers de la forme* $\lfloor n^c \rfloor$, Canad. J. Math. **53** (2001), no. 2, 414–433, DOI: https://doi.org/10.4153/CJM-2001-017-0.

[18] B. R. Srinivasan, *The lattice point problem of many-dimensional hyperboloids* $II$, Acta Arith. **8** (1963), 173–204.

[19] R. C. Baker and G. Harman, *Shifted primes without large prime factors*, Acta Arith. **83** (1998), no. 4, 331–361, DOI: https://doi.org/10.4064/aa-83-4-331-361.