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Research Article

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A generalized fixed-point theorem for set-valued mappings in b-metric spaces

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Abstract: The aim of this article is to present a local generalized fixed-point theorem for set-valued mappings in b-metric spaces, which brings together the framework of different (such as Nadler's and Kannan's) fixed-point theorems. For general set-valued graph contractions, a global fixed-point theorem and qualitative results concerning the fixed-point sets are obtained. We also provide an answer to the open question regarding Ulam-Hyers stability of the fixed-point inclusion proposed in [Petruşel et al., *Multi-valued graph contraction principle with applications*, Optimization **69** (2020), 1541–1556].

Keywords: fixed-point, general graph contraction, Ulam-Hyers stability, b-metric space

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1 Introduction

The fixed-point theory is an important branch of nonlinear analysis, which provides essential tools for solving various problems appeared in the existence theory of differential, integral and functional equations, game theory, fractal geometry, etc. For single-valued mappings, the well-known Banach contraction principle [1] is the most significant fundamental fixed-point result. Since this principle has a lot of applications in different branches of mathematics, many authors have extended, generalized, and improved it in many directions by considering different forms of mappings or various types of spaces [2–8]. Another type of contraction principle, known as the Kannan fixed-point theorem, is independent from the Banach contraction principle, and for more details and discussions, see [9–12].

The Banach contraction principle is a forceful tool in nonlinear analysis since it is easy to check the contraction condition under appropriate Lipschitzian type assumptions. On the other hand, the Kannan contraction condition is not convenient to be verified, especially for concrete nonlinear equations. However, it is very important from a theoretical point of view. Subrahmanyam [13] proved that Kannan's theorem characterizes the metric completeness. That is, a metric space X is complete if and only if every mapping satisfying Kannan's contraction has a fixed-point, while Banach contractions do not guarantee such results. Connell [14] provide an example of metric space X, which is not complete yet every contraction on X has a fixed-point.

In view of the fact that metric spaces have certain limitations in the application of fixed-point theorems, Czerwik [15] proposed the notion of b-metric space. The research on b-metric spaces has significantly promoted the development of fixed-point theory. Jovanović et al. [16] has successfully generalized the classic

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Banach fixed-point theorems to b-metric spaces. Other fixed-point theorems in b-metric spaces and their variations can be found in [10,15,17–19].

Set-valued fixed-point theorems exhibit broad applicability in practice. For set-valued mappings, as an extension of the Banach contraction principe, Nadler [20] proved the most important set-valued fixed-point theorem in 1969. Later on, Covitz and Nadler [21] gave a slight extension of the Nadler's fixed-point principle in 1970. Several other extensions of this principle were considered by many authors, such as the ones in different types of generalized metric spaces as well as the ones involving various set-valued generalized contractions [5,17,22–24]. Some set-valued generalized contraction principles have been successfully established in b-metric spaces [15,18]. This article aims to establish generalized set-valued fixed-point theorems for both local and global settings in b-metric spaces, thereby constructing a unified theoretical framework that encompasses diverse existing generalized set-valued contraction principles such as Nadler's and Kannan's fixed-point theorems.

The rest of the article is structured as follows. Section 2 contains definitions of the basic properties under consideration and some preliminary results used throughout the following sections. Section 3 is devoted to the study of a local generalized contraction principle for set-valued mappings in b-metric space. In Section 4, we establish a global fixed-point theorem for general set-valued graph contractions and obtain qualitative results concerning the fixed-point sets in b-metric space. Besides, we provide an answer to the open question regarding Ulam-Hyers stability of the fixed-point inclusion proposed in [8]. In Section 5, we give an application of the theoretical results to the altering point problem.

2 Notations and preliminary results

Unless otherwise specified, the framework of our study will be in complete b-metric spaces. For convenience of the readers, we first recall the definition of a b-metric space.

Definition 2.1. Let X be a nonempty set and $s \geq 1$. If a function $d : X \times X \to \mathbb{R}^+$ satisfies the following axioms:

(i) $d(u, v) = 0$ if and only if $u = v$;

(ii) $d(u, v) = d(v, u)$;

(iii) $d(u, v) \leq s[d(u, w) + d(w, v)]$ (for all $u, v, w \in X$),

then we say that d is a b-metric and the pair (X, d) is a b-metric space.

In the case of $s = 1$, a b-metric space is reduced to a metric space, which implies that every metric space is a b-metric space. However, in general, the converse is not true. It is known that every metric is a continuous functional while a b-metric does not necessarily possess this property, i.e., a b-metric does not have to be continuous. Consider the well-known space $L_p[0, 1]$ $(0 < p < 1)$. It is easy to see that $L_p[0, 1]$ $(0 < p < 1)$ is not a metric space, but it happens to be a b-metric space with $s = 2^{\frac{1}{p}}$.

Let $\{a_n\}$ be a sequence in b-metric space (X, d) and $a \in X$. We say that $\{a_n\}$ converges to a iff $d(a_n, a) \to 0$ as $n \to \infty$. We call $\{a_n\}$ a Cauchy sequence iff $d(a_n, a_m) \to 0$ as $n, m \to \infty$. A b-metric space (X, d) is said to be complete, if each Cauchy sequence in (X, d) is convergent in X. Let (X, d) be a b-metric space with constant $s \geq 1$. The symbol $B_a(x)$ denotes the closed ball of radius $a > 0$ centered at $x \in X$. Given subsets $C, D \subset X$, define the distance from $x \in X$ to C and the excess from C to D by

$$d(x, C) \coloneqq \inf\{d(x, c) : c \in C\} \quad \text{and} \quad e(C, D) \coloneqq \sup\{d(c, D) : c \in C\},$$

respectively, with the convention that $d(x, \emptyset) \coloneqq \infty$ and $e(\emptyset, D) \coloneqq 0$, for any $D \subset X$. The Pompeiu-Hausdorff distance between the sets C and D is defined as follows:

$$h(C, D) \coloneqq \max\{e(C, D), e(D, C)\}.$$

Let Y be another b-metric space and $F : X \rightrightarrows Y$ be a set-valued mapping. The domain and the graph of F are denoted, respectively, by

$$\mathrm{dom}(F) \coloneqq \{x \in X : F(x) \neq \emptyset\} \quad \text{and} \quad \mathrm{gph}(F) \coloneqq \{(x, y) \in X \times Y : \ y \in F(x)\}.$$

The symbol $F^{-1} : Y \rightrightarrows X$ stands for the inverse mapping of F with $F^{-1}(y) \coloneqq \{x \in X : y \in F(x)\}$, for any $y \in Y$. We say that F is closed, if $\mathrm{gph}(F)$ is a closed set in $X \times Y$. For a set-valued mapping $\Phi : X \rightrightarrows X$, we denote by $\mathrm{Fix}(F) \coloneqq \{x \in X : x \in F(x)\}$ the fixed-point set of F.

3 Local fixed-point theorem

We begin with establishing a local fixed-point theorem for set-valued mappings in b-metric spaces.

Theorem 3.1. *For any given $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu \in [0, +\infty)$ with $\lambda_3 < 1$, let s' be a positive root of the equation $\lambda_4 t^2 + (\lambda_1 + \lambda_2 + \lambda_4)t + \lambda_3 - 1 = 0$. Suppose that $1 \leq s < s'$, (X, d) be a complete b-metric space with constant s, $\Phi : X \rightrightarrows X$ be a set-valued mapping, $\bar{x} \in X$, and $r \in (0, +\infty)$. Assume the following assumptions hold*:

(*i*) $d(\bar{x}, \Phi(\bar{x})) < r\left(\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}\right)$,

(*ii*) $e(\Phi(u) \cap B_r(\bar{x}), \Phi(v)) \leq \lambda_1 d(u, v) + \lambda_2 d(u, \Phi(u)) + \lambda_3 d(v, \Phi(v)) + \lambda_4 d(u, \Phi(v)) + \mu d(v, \Phi(u)) \ \forall u, v \in B_r(\bar{x})$.

Then we have that

(*a*) *if the set $\mathrm{gph}(\Phi) \cap B_r(\bar{x}) \times B_r(\bar{x})$ is closed in $X \times X$, then Φ has a fixed-point in $B_r(\bar{x})$, i.e., there exists $x \in B_r(\bar{x})$ such that $x \in \Phi(x)$;*

(*b*) *if Φ is single valued and $s\lambda_3 + s^2\lambda_4 < 1$, then Φ has a fixed-point in $B_r(\bar{x})$. Furthermore, Φ has a unique fixed-point in $B_r(\bar{x})$ provided that $\lambda_2 + \lambda_4 + \mu < 1$.*

Proof. By the assumption, we have $s(\lambda_1 + \lambda_2) + \lambda_3 + (s + s^2)\lambda_4 < 1$. For convenience, set

$$\alpha \coloneqq \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}.$$

Then $s\alpha \in [0, 1)$. Let $x_0 = \bar{x}$, by assumption (i) there exists $x_1 \in \Phi(\bar{x})$ such that $d(x_0, x_1) = d(\bar{x}, x_1) < \frac{r}{s}(1 - s\alpha)$. Proceeding by induction, suppose that there exists $x_{k+1} \in \Phi(x_k) \cap B_r(\bar{x})$ for $k = 0, 1, \ldots, j - 1$ with

$$d(x_k, x_{k+1}) < \frac{r}{s}(1 - s\alpha)\alpha^k. \tag{3.1}$$

By assumption (ii), we have

$$\begin{aligned} d(x_j, \Phi(x_j)) &\leq e(\Phi(x_{j-1}) \cap B_r(\bar{x}), \Phi(x_j)) \\ &\leq \lambda_1 d(x_{j-1}, x_j) + \lambda_2 d(x_{j-1}, \Phi(x_{j-1})) + \lambda_3 d(x_j, \Phi(x_j)) + \lambda_4 d(x_{j-1}, \Phi(x_j)) + \mu d(x_j, \Phi(x_{j-1})) \\ &\leq (\lambda_1 + \lambda_2 + s\lambda_4) d(x_{j-1}, x_j) + (\lambda_3 + s\lambda_4) d(x_j, \Phi(x_j)). \end{aligned}$$

And then

$$d(x_j, \Phi(x_j)) \leq \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4} d(x_{j-1}, x_j) = \alpha d(x_{j-1}, x_j).$$

According to assumption (3.1), one has $d(x_j, \Phi(x_j)) < \frac{r}{s}(1 - s\alpha)\alpha^j$. This implies that there exists an $x_{j+1} \in \Phi(x_j)$ such that

$$d(x_j, x_{j+1}) < \frac{r}{s}(1 - s\alpha)\alpha^j.$$

Employing the triangle inequality, we obtain that

$$d(\bar{x}, x_{j+1}) \leq \sum_{i=0}^{j} s^{i+1} d(x_i, x_{i+1}) < r(1 - s\alpha) \sum_{i=0}^{j} (s\alpha)^i < r.$$

Hence, $x_{j+1} \in \Phi(x_j) \cap B_r(\bar{x})$ and the induction step is complete. Then for any $k > m > 1$, we have

$$d(x_m, x_k) < \sum_{i=m}^{k-1} s^{i+1} d(x_i, x_{i+1}) < r(1 - s\alpha) \sum_{i=m}^{k-1} (s\alpha)^i < r(s\alpha)^m. \tag{3.2}$$

Thus, $\{x_k\}$ is a Cauchy sequence and consequently converges to some $x \in B_r(\bar{x})$.

Since $(x_{k-1}, x_k) \in \mathrm{gph}(\Phi) \cap B_r(\bar{x}) \times B_r(\bar{x})$, which is a closed set, we conclude that $x \in \Phi(x)$ by passing to the limit $k \to \infty$, which establishes (a).

To prove assertion (b), we assume that Φ is single valued. Then $x_k = \Phi(x_{k-1})$ (for all $k \in \mathbb{N}$) and $x_k \to x \in B_r(\bar{x})$. By assumption (ii), we have

$$\begin{aligned} d(x, \Phi(x)) &\le sd(x, x_k) + sd(x_k, \Phi(x)) \\ &= sd(x, x_k) + sd(\Phi(x_{k-1}), \Phi(x)) \\ &\le sd(x, x_k) + s\lambda_1 d(x_{k-1}, x) + s\lambda_2 d(x_{k-1}, \Phi(x_{k-1})) + s\lambda_3 d(x, \Phi(x)) + s\lambda_4 d(x_{k-1}, \Phi(x)) + s\mu d(x, \Phi(x_{k-1})) \\ &\le s(1 + \mu) d(x, x_k) + s(\lambda_1 + s\lambda_4) d(x_{k-1}, x) + s\lambda_2 d(x_{k-1}, x_k) + s(\lambda_3 + s\lambda_4) d(x, \Phi(x)). \end{aligned}$$

And then

$$d(x, \Phi(x)) \le \frac{s + s\mu}{1 - s\lambda_3 - s^2\lambda_4} d(x, x_k) + \frac{s\lambda_1 + s^2\lambda_4}{1 - s\lambda_3 - s^2\lambda_4} d(x_{k-1}, x) + \frac{s\lambda_2}{1 - s\lambda_3 - s^2\lambda_4} d(x_{k-1}, x_k).$$

Note that $x_k \to x$ and $d(x_{k-1}, x_k) \to 0$, we conclude that $d(x, \Phi(x)) = 0$, and then $x = \Phi(x)$. Hence, Φ has a fixed-point in $B_r(\bar{x})$. Furthermore, if $\lambda_1 + \lambda_4 + \mu < 1$, then Φ has a unique fixed-point in $B_r(\bar{x})$. We show this claim by contradiction. Assume to the contrary that Φ has two fixed-points in $B_r(\bar{x})$, i.e., there are $x, x' \in B_r(\bar{x})$, $x \neq x'$, such that $x = \Phi(x)$ and $x' = \Phi(x')$. Then we have

$$\begin{aligned} d(x, x') &= d(\Phi(x), \Phi(x')) \\ &\le \lambda_1 d(x, x') + \lambda_2 d(x, \Phi(x)) + \lambda_3 d(x', \Phi(x')) + \lambda_4 d(x, \Phi(x')) + \mu d(x', \Phi(x)) \\ &= (\lambda_1 + \lambda_4 + \mu) d(x, x'), \end{aligned}$$

which is a contradiction. □

Remark 3.2. The inequality $1 \le s < s'$ in Theorem 3.1 imposes a quantitative relationship between the constants. It is worth to note that there is no restriction on the nonegative constant μ. Indeed, following the proof of Theorem 3.1 (i), we can see that condition (ii) can be reduced to

$$\begin{aligned} &e(\Phi(u) \cap B_r(\bar{x}), \Phi(v)) \le \lambda_1 d(u, v) + \lambda_2 d(u, \Phi(u)) + \lambda_3 d(v, \Phi(v)) + \lambda_4 d(u, \Phi(v)) \\ &\forall (u, v) \in \mathrm{gph}(\Phi) \cap B_r(\bar{x}) \times B_r(\bar{x}). \end{aligned}$$

However, the following example shows that the closeness assumption on the set $\mathrm{gph}(\Phi) \cap B_r(\bar{x}) \times B_r(\bar{x})$ cannot be dropped.

Example 3.3. Let $X = \mathbb{R}$ be equipped with the metric $d(x, y) = |x - y|^2$, for all $x, y \in X$. It is clear that X is a complete b-metric space with constant $s = 2$. Assume that

$$\Phi(x) = \begin{cases} \left(\dfrac{x}{3}, \dfrac{1}{3}\right], & \text{if } x \in (-\infty, 0] \cup \left(\dfrac{1}{3}, 1\right]; \\ \left(\dfrac{x}{3}, \dfrac{1}{3}\right] \setminus \{x\}, & \text{if } x \in \left(0, \dfrac{1}{3}\right]; \\ \left\{\dfrac{x}{3}\right\}, & \text{if } x \in (1, +\infty). \end{cases}$$

Pick $\bar{x} = 0$ and $r = 1$, then $d(\bar{x}, \Phi(\bar{x})) = 0$ and $e(\Phi(u), \Phi(v)) \le \frac{1}{3} d(u, v)$, for all $u, v \in B_r(\bar{x})$. It is easy to verify that $\mathrm{gph}(\Phi) \cap B_r(\bar{x}) \times B_r(\bar{x})$ is not closed and Φ has no fixed-point in $B_r(\bar{x})$.

Theorem 3.1 provides a generalized unified framework for several different set-valued contraction principles. When $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and $\mu = 0$ in Theorem 3.1, we have the following Nadler's contraction principle in b-metric spaces.

Corollary 3.4. *Let (X, d) be a complete b-metric space with constant $s \geq 1$, $\Phi : X \rightrightarrows X$ be a set-valued mapping, $\bar{x} \in X$ and $r \in (0, +\infty)$. Given $\lambda \in [0, 1/s)$, impose the following assumptions*:

(*i*) $d(\bar{x}, \Phi(\bar{x})) < r\left(\frac{1}{s} - \lambda\right)$,

(*ii*) $e(\Phi(u) \cap B_r(\bar{x}), \Phi(v)) \leq \lambda d(u, v)\ \forall u, v \in B_r(\bar{x})$.

Then we have that

(*a*) *if the set* $\mathrm{gph}(\Phi) \cap B_r(\bar{x}) \times B_r(\bar{x})$ *is closed in $X \times X$, then Φ has a fixed-point in $B_r(\bar{x})$,*

(*b*) *if Φ is single valued, then Φ has a unique fixed-point in $B_r(\bar{x})$.*

When $\lambda_1 = \lambda_4 = 0$ and $\mu = 0$ in Theorem 3.1, we have the following Kannan contraction principle in b-metric space, which goes back to Kannan's fixed-point theorem if Φ is single valued.

Corollary 3.5. *Let (X, d) be a complete b-metric space with constant $s \geq 1$, $\Phi : X \rightrightarrows X$ be a set-valued mapping, $\bar{x} \in X$ and $r \in (0, +\infty)$. Given $\lambda_2, \lambda_3 \in [0, +\infty)$ with $s\lambda_2 + \lambda_3 < 1$, impose the following assumptions*:

(*i*) $d(\bar{x}, \Phi(\bar{x})) < r\left(\frac{1}{s} - \frac{\lambda_2}{1 - \lambda_3}\right)$,

(*ii*) $e(\Phi(u) \cap B_r(\bar{x}), \Phi(v)) \leq \lambda_2 d(u, \Phi(u)) + \lambda_3 d(v, \Phi(v))\ \forall u, v \in B_r(\bar{x})$.

Then we have that

(*a*) *if the set* $\mathrm{gph}(\Phi) \cap B_r(\bar{x}) \times B_r(\bar{x})$ *is closed in $X \times X$, then Φ has a fixed-point in $B_r(\bar{x})$,*

(*b*) *if Φ is single valued and $s\lambda_3 < 1$, then Φ has a fixed-point in $B_r(\bar{x})$ and Φ has a unique fixed-point in $B_r(\bar{x})$.*

When $\lambda_1 = \lambda_2 = \lambda_3 = 0$ in Theorem 3.1, we obtain another contraction principle for set-valued mappings.

Corollary 3.6. *Let (X, d) be a complete b-metric space with constant $s \geq 1$, $\Phi : X \rightrightarrows X$ be a set-valued mapping, $\bar{x} \in X$ and $r \in (0, +\infty)$. Given $\lambda, \mu \in [0, +\infty)$ with $(s + s^2)\lambda < 1$, impose the following assumptions*:

(*i*) $d(\bar{x}, \Phi(\bar{x})) < r\left(\frac{1}{s} - \frac{s\lambda}{1 - s\lambda}\right)$,

(*ii*) $e(\Phi(u) \cap B_r(\bar{x}), \Phi(v)) \leq \lambda d(u, \Phi(v)) + \mu d(v, \Phi(u))\ \forall u, v \in B_r(\bar{x})$.

Then we have that

(*a*) *if the set* $\mathrm{gph}(\Phi) \cap B_r(\bar{x}) \times B_r(\bar{x})$ *is closed in $X \times X$, then Φ has a fixed-point in $B_r(\bar{x})$,*

(*b*) *if Φ is single valued, then Φ has a fixed-point in $B_r(\bar{x})$ and Φ has a unique fixed-point in $B_r(\bar{x})$, provided $\lambda + \mu < 1$.*

4 Global fixed-point theorem

In this section, we dedicate our efforts to establish a global version of the fixed-point theorem for set-valued mappings. Let us start with the definition of a general graph contraction.

Definition 4.1. Let (X, d) be a b-metric space with constant $s \geq 1$, $\Phi : X \rightrightarrows X$ be a set-valued mapping and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu \in [0, +\infty)$.

(i) The mapping Φ is said to be a general $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu)$-contraction, if for all $u, v \in X$, we have

$$h(\Phi(u), \Phi(v)) \leq \lambda_1 d(u, v) + \lambda_2 d(u, \Phi(u)) + \lambda_3 d(v, \Phi(v)) + \lambda_4 d(u, \Phi(v)) + \mu d(v, \Phi(u)).$$

(ii) The mapping Φ is said to be a general graph $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-contraction, if for all $u, v \in \mathrm{gph}(\Phi)$, we have

$$e(\Phi(u), \Phi(v)) \leq \lambda_1 d(u, v) + \lambda_2 d(u, \Phi(u)) + \lambda_3 d(v, \Phi(v)) + \lambda_4 d(u, \Phi(v)).$$

Note that when $u, v \in \mathrm{gph}(\Phi)$, we have $d(v, \Phi(u)) = 0$, which indicates immediately that a general contraction readily implies a general graph contraction. It is clear that a general graph $(\lambda_1, 0, 0, 0)$-contraction implies the usual graph contraction, for more details, see [4,8,18,25,26].

Theorem 4.2. *For any given $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, +\infty)$ with $\lambda_3 < 1$, let s' be a positive root of the equation $\lambda_4 t^2 + (\lambda_1 + \lambda_2 + \lambda_4)t + \lambda_3 - 1 = 0$. Suppose that $1 \leq s < s'$, (X, d) be a complete b-metric space with constant s and $\Phi : X \rightrightarrows X$ be a closed set-valued mapping. If Φ is a general graph $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-contraction, then, for any $x \in \mathrm{dom}(\Phi)$, there exists a sequence $\{x_n\}$ of successive approximations for Φ starting from x which converges to a point $x_* \in \mathrm{Fix}(\Phi)$ and*

$$d(x, \mathrm{Fix}(\Phi)) \leq d(x, x_*) \leq \frac{1}{\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}} d(x, \Phi(x)).$$

Proof. By the assumption, one has $s(\lambda_1 + \lambda_2) + \lambda_3 + (s + s^2)\lambda_4 < 1$. For convenience, we set

$$\alpha := \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}.$$

Then $s\alpha \in [0, 1)$. Pick any $x \in \mathrm{dom}(\Phi)$, $q \in (1, \frac{1}{s\alpha})$ and let $x_0 = x$. If $x_0 \in \Phi(x_0)$, we set $x_1 = x_0$. Otherwise, there exists $x_1 \in \Phi(x_0)$ such that $d(x_0, x_1) \leq q d(x_0, \Phi(x_0))$. By the assumption that Φ is general graph $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-contraction, we have

$$\begin{aligned} d(x_1, \Phi(x_1)) &\leq e(\Phi(x_0), \Phi(x_1)) \\ &\leq \lambda_1 d(x_0, x_1) + \lambda_2 d(x_0, \Phi(x_0)) + \lambda_3 d(x_1, \Phi(x_1)) + \lambda_4 d(x_0, \Phi(x_1)) \\ &\leq (\lambda_1 + \lambda_2 + s\lambda_4) d(x_0, x_1) + (\lambda_3 + s\lambda_4) d(x_1, \Phi(x_1)). \end{aligned}$$

And then

$$d(x_1, \Phi(x_1)) \leq \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4} d(x_0, x_1) = \alpha d(x_0, x_1).$$

If $x_1 \in \Phi(x_1)$, we set $x_2 = x_1$. Otherwise, there exists $x_2 \in \Phi(x_1)$ such that $d(x_1, x_2) \leq q\alpha d(x_0, x_1)$. Proceeding with induction, we have $x_{k+1} \in \Phi(x_k)$ such that

$$d(x_k, x_{k+1}) \leq (q\alpha)^k d(x_0, x_1) \quad \forall k = 0, 1, \ldots.$$

Employing the triangle inequality, we obtain that

$$d(x_0, x_{k+1}) \leq \sum_{i=0}^{k} s^{i+1} d(x_i, x_{i+1}) \leq s d(x_0, x_1) \sum_{i=0}^{k} (sq\alpha)^i \leq \frac{s d(x_0, x_1)}{1 - sq\alpha}. \tag{4.1}$$

Then for any $k > m \geq 0$, we have

$$d(x_m, x_k) \leq \sum_{i=m}^{k-1} s^{i+1} d(x_i, x_{i+1}) \leq s d(x_0, x_1) \sum_{i=m}^{k-1} (sq\alpha)^i \leq s d(x_0, x_1) \frac{(sq\alpha)^m}{1 - sq\alpha}.$$

Due to the fact that $sq\alpha < 1$, it follows that $\{x_k\}$ is a Cauchy sequence and consequently converges to some $x_* \in X$ (thanks to the completeness of X). Let $k \to \infty$ in (4.1), one has

$$d(x_0, x_*) \leq \frac{s d(x_0, x_1)}{1 - sq\alpha} \leq \frac{sq d(x_0, \Phi(x_0))}{1 - sq\alpha}.$$

Then by letting $q \to 1$, we obtain the conclusion. □

The following example serves as an illustration of Theorem 4.2.

Example 4.3. Let the b-metric on $X = \mathbb{R}$ be the same as the b-metric in Example 3.3. Then X is a b-metric space with constant $s = 2$. Assume that

$$\Phi(x) = \begin{cases} \left[\frac{x}{5}, \frac{x}{4}\right], & \text{if } x \in [0, \infty); \\ \varnothing, & \text{if } x \in (-\infty, 0). \end{cases}$$

Let $\lambda_1 = \lambda_4 = 0$ and $\lambda_2 = \lambda_3 = \frac{2}{9}$, for any $u \in [0, \infty)$ and $v \in \Phi(u) = \left[\frac{u}{5}, \frac{u}{4}\right]$, we have $e(\Phi(u), \Phi(v)) = \frac{1}{16}|u - v|^2$, $d(u, \Phi(u)) = \frac{9}{16}|u|^2$ and $d(v, \Phi(v)) = \frac{9}{16}|v|^2$. This shows that

$$e(\Phi(u), \Phi(v)) \leq \frac{2}{16}(|u|^2 + |v|^2) = \frac{2}{9}(d(u, \Phi(u)) + d(v, \Phi(v))) \quad \forall u, v \in \mathrm{gph}(\Phi).$$

It follows that Φ is a general graph $(0, 2/9, 2/9, 0)$-contraction. It is easy to verify that $\mathrm{Fix}(\Phi) = \{0\}$ and

$$d(x, \mathrm{Fix}(\Phi)) = |x|^2 \leq \frac{14}{3} \times \frac{9}{16}|x|^2 = \frac{1}{\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}} d(x, \Phi(x)) \quad \forall x \in \mathrm{dom}(\Phi) = [0, \infty).$$

Recall that for a set-valued mapping $\Phi : X \rightrightarrows X$, the fixed-point problem

$$x \in \Phi(x), \quad x \in X, \tag{4.2}$$

is said to be Ulam-Hyers stable if there exists $c > 0$ such that, for any $\varepsilon > 0$ and any ε-solution z of the fixed-point problem (4.2) (i.e. $d(z, \Phi(z)) \leq \varepsilon$), there exists an $x \in \mathrm{Fix}(\Phi)$ such that $d(z, x) \leq c\varepsilon$ (cf. [8,27]). Regarding Ulam-Hyers stability of the fixed-point inclusion, Petruşel et al. [8] established the following result.

Theorem 4.4. *Let (X, d) be a complete metric space, $\alpha \in (0,1)$, $\Phi : X \rightrightarrows X$ be a closed set-valued general graph $(\alpha, 0, 0, 0)$-contraction with proximinal values (i.e., for all $x, y \in X$, there exist $u \in \Phi(x)$ such that $d(y, u) = d(y, \Phi(x))$). Then, the fixed-point problem* (4.2) *is Ulam-Hyers stable.*

In [8], the authors proposed a question that, without the proximinal value assumption on Φ, whether one can obtain the Ulam-Hyers stability for fixed-point inclusion (4.2). With respect to this open question, the following theorem provides a positive answer.

Theorem 4.5. *For any given $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, +\infty)$ with $\lambda_3 < 1$, let s' be a positive root of the equation $\lambda_4 t^2 + (\lambda_1 + \lambda_2 + \lambda_4)t + \lambda_3 - 1 = 0$. Suppose that $1 \leq s < s'$, (X, d) be a complete b-metric space with constant s and $\Phi : X \rightrightarrows X$ be a closed set-valued mapping. If Φ is a general graph $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-contraction, then the fixed-point problem* (4.2) *is Ulam-Hyers stable.*

Proof. Let $\varepsilon > 0$ and z be an ε-solution of the fixed-point problem (4.2), then $d(z, \Phi(z)) \leq \varepsilon$, and hence, $z \in \mathrm{dom}(\Phi)$. By Theorem 4.2, there exists a sequence $\{x_n\}$ of successive approximations for Φ starting from z which converges to a point $x_* \in \mathrm{Fix}(\Phi)$ and

$$d(z, x_*) \leq \frac{s}{1 - s\alpha} d(z, \Phi(z)) \leq \frac{s\varepsilon}{1 - s\alpha}.$$

This shows the fixed-point problem (4.2) is Ulam-Hyers stable. □

Remark 4.6. The aforementioned theorem provides a sufficient condition for the Ulam-Hyers stability studied in [8], even in complete b-metric spaces. In particular, when $s = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$, the aforementioned theorem is still an improvement of [8, Theorem 4.8].

As a consequence of Theorems 4.2, we have the following quantitative results concerning the fixed-point sets. For more details, please refer to references [3,28].

Theorem 4.7. *For any given $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, +\infty)$ with $\lambda_3 < 1$, let s' be a positive root of the equation $\lambda_4 t^2 + (\lambda_1 + \lambda_2 + \lambda_4)t + \lambda_3 - 1 = 0$. Suppose that $1 \leq s < s'$ and (X, d) is a complete b-metric space with constant s.*

Suppose that the set-valued mapping $\Phi : X \rightrightarrows X$ *is a closed general graph* $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$*-contraction. Then the following conclusions hold*:

(*i*) $\mathrm{Fix}(\Phi)$ *is a closed set in* X;

(*ii*) *if* $\Psi : X \rightrightarrows X$ *is a set-valued mapping with* $\mathrm{Fix}(\Psi) \neq \emptyset$ *and* η *is a positive number such that*

$$e(\Psi(x), \Phi(x)) \leq \eta, \quad \forall x \in X,$$

then we have

$$e(\mathrm{Fix}(\Psi), \mathrm{Fix}(\Phi)) \leq \frac{\eta}{\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}};$$

(*iii*) *if the set-valued mapping* $\Psi : X \rightrightarrows X$ *is a closed general graph* $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$*-contraction and* η *is a positive number such that*

$$h(\Psi(x), \Phi(x)) \leq \eta, \quad \forall x \in X,$$

then

$$h(\mathrm{Fix}(\Psi), \mathrm{Fix}(\Phi)) \leq \frac{\eta}{\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}};$$

(*iv*) *if* $\Phi_k : X \rightrightarrows X$ $(k \in N)$ *is a sequence of closed general graph* $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$*-contractions such that* $\Phi_k(x) \xrightarrow{h} \Phi(x)$ *uniformly with respect to* $x \in X$ *as* $k \to \infty$, *then we have*

$$\mathrm{Fix}(\Phi_k) \xrightarrow{h} \mathrm{Fix}(\Phi) \text{ as } k \to \infty, \quad \text{i.e. } \lim_{k\to\infty} h(\mathrm{Fix}(\Phi_k), \mathrm{Fix}(\Phi)) = 0.$$

Proof.

(i) Pick any $x \in \overline{\mathrm{Fix}(\Phi)}$, then by definition of the closure, there exist $x_k \in \mathrm{Fix}(\Phi)$ such that $x_k \to x$. Note that $x_k \in \Phi(x_k)$, our conclusion follows from the closeness assumption on $\mathrm{gph}\Phi$.

(ii) Pick any $x \in \mathrm{Fix}(\Psi)$, then $x \in \Psi(x)$. By Theorem 4.2, we know that $\mathrm{Fix}(\Phi) \neq \emptyset$ and

$$d(x, \mathrm{Fix}(\Phi)) \leq \frac{d(x, \Phi(x))}{\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}} \leq \frac{e(\Psi(x), \Phi(x))}{\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}} \leq \frac{\eta}{\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}}.$$

Then our conclusion follows from the arbitrary choice of x in $\mathrm{Fix}(\Psi)$.

(iii) By the assumptions, we have $\mathrm{Fix}(\Phi) \neq \emptyset, \mathrm{Fix}(\Psi) \neq \emptyset$ and, for all $x \in X$, $e(\Psi(x), \Phi(x)) \leq \eta$ and $e(\Phi(x), \Psi(x)) \leq \eta$. Then the conclusion follows immediately from (ii).

(iv) For any $\varepsilon > 0$, according to the assumptions, there exists an $N \in \mathbb{N}$ such that

$$h(\Phi_k(x), \Phi(x)) \leq \varepsilon, \quad \forall k \geq N, x \in X.$$

It follows from (iii) that

$$h(\mathrm{Fix}(\Phi_k), \mathrm{Fix}(\Phi)) \leq \frac{\varepsilon}{\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}}, \quad \forall k \geq N.$$

This implies $\lim_{k\to\infty} h(\mathrm{Fix}(\Phi_k), \mathrm{Fix}(\Phi)) = 0$. □

5 An application to the altering point problem

Let (X, d) and (Y, ρ) be two b-metric spaces, $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows X$ be two set-valued mappings. The altering point problem is to find a pair $(x_*, y_*) \in X \times Y$ such that

$$\begin{aligned} y_* &\in F(x_*) \\ x_* &\in G(y_*). \end{aligned} \tag{5.1}$$

If such (x_*, y_*) exists, then it is called an altering point of F and G (see Sahu [29] for the single-valued case). The alternating point theory has remarkable applications in best proximity pairs theory, systems of nonlinear/hierarchical variational inequalities, game theory, and so on. For more details, please refer to references [29,30]. We have the following existence result.

Theorem 5.1. *For any given $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0, +\infty)$ with $\lambda_3 < 1$, let s' be a positive root of the equation $\lambda_4 t^2 + (\lambda_1 + \lambda_2 + \lambda_4)t + \lambda_3 - 1 = 0$. Suppose that $1 \le s < s'$, (X, d), and (Y, ρ) are complete b-metric spaces with constant s. Let $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows X$ be two closed set-valued mappings. If, for all $(x_1, y_2) \in \operatorname{gph}(F)$ and $(y_1, x_2) \in \operatorname{gph}(G)$,*

$$e(F(x_1), F(x_2)) \le \lambda_1 d(x_1, x_2) + \lambda_2 \rho(y_1, F(x_1)) + \lambda_3 \rho(y_2, F(x_2)) + \lambda_4 \rho(y_1, F(x_2))$$

and

$$e(G(y_1), G(y_2)) \le \lambda_1 \rho(y_1, y_2) + \lambda_2 d(x_1, G(y_1)) + \lambda_3 d(x_2, G(y_2)) + \lambda_4 d(x_1, G(y_2)).$$

Then, for any $(x, y) \in \operatorname{dom}(G) \times \operatorname{dom}(F)$, there exists at least one solution (x_*, y_*) of (5.1) *such that*

$$d(x, x_*) + \rho(y, y_*) \le \frac{1}{\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}} (\rho(y, F(x)) + d(x, G(y))). \tag{5.2}$$

Proof. Let $Z \coloneqq X \times Y$ and the scalar metric $\tilde{d}$ on Z be given by

$$\tilde{d}((x, y), (u, v)) \coloneqq d(x, u) + \rho(y, v), \quad \forall (x, y), (u, v) \in Z.$$

Then Z is a complete b-metric space with constant s. Let $\Phi : Z \rightrightarrows Z$ be defined by

$$\Phi(x, y) \coloneqq G(y) \times F(x), \quad \forall (x, y) \in Z.$$

Then, for $z_1 \coloneqq (x_1, y_1)$, $z_2 \coloneqq (x_2, y_2) \in Z$ with $z_2 \in \Phi(z_1)$, we have $(x_1, y_2) \in \operatorname{gph}(F)$ and $(y_1, x_2) \in \operatorname{gph}(G)$. Hence,

$$\begin{aligned} e(\Phi(z_1), \Phi(z_2)) &\le e(F(x_1), F(x_2)) + e(G(y_1), G(y_2)) \\ &\le \lambda_1(d(x_1, x_2) + \rho(y_1, y_2)) + \lambda_2(\rho(y_1, F(x_1)) + d(x_1, G(y_1))) + \lambda_3(\rho(y_2, F(x_2)) + d(x_2, G(y_2))) \\ &\quad + \lambda_4(\rho(y_1, F(x_2)) + d(x_1, G(y_2))) \\ &\le \lambda_1 \tilde{d}(z_1, z_2) + \lambda_2 \tilde{d}(z_1, \Phi(z_1)) + \lambda_3 \tilde{d}(z_2, \Phi(z_2)) + \lambda_4 \tilde{d}(z_1, \Phi(x_2)). \end{aligned}$$

This shows that the set-valued mapping Φ is a general graph $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-contraction. It follows from Theorem 4.2 that, for any $(x, y) \in \operatorname{dom}(\Phi) = \operatorname{dom}(G) \times \operatorname{dom}(F)$, there exists $(x_*, y_*) \in \operatorname{Fix}(\Phi)$ such that

$$\tilde{d}((x, y), (x_*, y_*)) \le \frac{1}{\frac{1}{s} - \frac{\lambda_1 + \lambda_2 + s\lambda_4}{1 - \lambda_3 - s\lambda_4}} \tilde{d}((x, y), \Phi(x, y)).$$

This indicates that (x_*, y_*) is a solution of (5.1) and inequality (5.2) holds. □

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