Research Article

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Some new bounds on resolvent energy of a graph

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Abstract: Let G be a simple graph of order n with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. The resolvent energy of G is a spectrum-based graph invariant defined as $\mathrm{ER}(G) = \sum_{i=1}^{n}(n - \lambda_i)^{-1}$. In this work, we propose some new bounds for $\mathrm{ER}(G)$. As a direct consequence of these bounds, we present some (n, m)-type results for triangle-free graphs.

Keywords: graph spectrum, graph invariant, resolvent energy

MSC 2020: 05C50, 05C90, 05C09

1 Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple graph possessing n vertices and m edges, where $|V| = n$ and $|E| = m$. The $(0, 1)$-adjacency matrix of G is denoted by $A = A(G)$. Eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of A form the spectrum of G [1]. Some well-known properties on graph eigenvalues are [1]

$$\sum_{i=1}^{n}\lambda_i = 0, \quad \sum_{i=1}^{n}\lambda_i^2 = 2m, \quad \text{and} \quad \sum_{i=1}^{n}\lambda_i^3 = 6t, \tag{1}$$

where t is the number of triangles of G. In [2], the (ordinary) energy of the graph G is defined as

$$E(G) = \sum_{i=1}^{n}|\lambda_i|. \tag{2}$$

This spectrum-based graph invariant originated from theoretical chemistry [3,4]. There exists an exhaustive, mathematical, and mathematico-chemical literature on $E(G)$. For details on the theory and applications of $E(G)$ see the monograph [5] and references cited therein.

For an $n \times n$ matrix M, its resolvent matrix is defined as [6]

$$R_M(z) = (zI_n - M)^{-1},$$

where I_n is the $n \times n$ identity matrix and z is a complex variable, which differs from the eigenvalues of M. Then, the resolvent matrix of A, denoted by $R_A(z)$, is defined as [7]

$$R_A(z) = (zI_n - A)^{-1}.$$

Clearly, the numbers $\frac{1}{z - \lambda_i}$, $i = 1, 2, \ldots, n$, are the eigenvalues of $R_A(z)$ [7]. Since the eigenvalues of A cannot be greater than $n - 1$ [1], the matrix $R_A(n)$ is surely invertible [7]. Therefore, the matrix $R_A(n) = (nI_n - A)^{-1}$ has the

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eigenvalues $\frac{1}{n-\lambda_i}$, $i = 1, 2, \ldots, n$, and its determinant is $\det(R_A(n)) = \prod_{i=1}^{n}\frac{1}{n-\lambda_i}$ [7,8]. Motivated by the definition of graph energy, the resolvent energy of G is introduced as [7]

$$\mathrm{ER}(G) = \sum_{i=1}^{n}\frac{1}{n-\lambda_i}. \tag{3}$$

Gutman et al. [7] showed that $\mathrm{ER}(G)$ can be defined through the characteristic polynomial and the spectral moments of graph as well. The validity of some of the conjectures put forward in [7,9] on the resolvent energy of unicyclic, bicyclic, and tricyclic graphs was confirmed in [10]. Recently, in [11,12], relationships between ordinary and resolvent graph energy were demonstrated. Various mathematical properties and the bounds of $\mathrm{ER}(G)$ can be found in [7,8,12–14]. For more information on $\mathrm{ER}(G)$, refer [15–19].

In this study, we establish some new bounds for the resolvent energy of graphs. As a direct consequence of these bounds, we also give some (n, m)-type results for triangle-free graphs.

2 Preliminaries

For positive real numbers $p_1, p_2, \ldots, p_r$, it is well known that the kth elementary symmetric mean is the number

$$Q_k = \frac{\sum_{1\le i_1<i_2<\ldots<i_k\le r} p_{i_1}p_{i_2}\cdots p_{i_k}}{\binom{r}{k}}.$$

Obviously, Q_1 and $Q_r^{1/r}$ are, respectively, the arithmetic mean and the geometric mean of $p_1, p_2, \ldots, p_r$. This result is generalized in the following lemma [20]:

Lemma 2.1. (Maclaurin's symmetric mean inequality) [20] *Let $p_1, p_2, \ldots, p_r$ be positive real numbers. Then,*

$$Q_1 \ge Q_2^{1/2} \ge Q_3^{1/3} \ge \ldots \ge Q_r^{1/r}.$$

The equality holds if and only if $p_1 = p_2 = \ldots = p_r$.

Lemma 2.2. (Newton's inequality) [21] *Let $p_1, p_2, \ldots, p_r$ be positive real numbers and let Q_k, $k = 1, 2, \ldots, r$, be given as in Lemma* 2.1. *Then,*

$$Q_{k-1}Q_{k+1} \le Q_k^2,$$

where $k = 1, 2, \ldots, r-1$ and $Q_0 = 1$. Moreover, the equality holds if and only if $p_1 = p_2 = \ldots = p_r$.

The following inequality can be found in [12].

Lemma 2.3. [12] *Let G be a simple graph of order n with m edges. Then,*

$$\frac{n}{n^2 - 2m} \ge (\det(R_A(n)))^{1/n}.$$

Let $\overline{K}_n$ denote the complement graph of the complete graph K_n on n vertices.

Lemma 2.4. [12] *Let G be a simple graph of order n with m edges. Then,*

$$E(G) \le n^2\sqrt{\mathrm{ER}(G) - 1}.$$

The equality holds if and only if $G \cong \overline{K}_n$.

Lemma 2.5. [1] *A graph has one eigenvalue if and only if it is totally disconnected.*

3 Main results

In the following theorem, we present an upper bound on $\mathrm{ER}(G)$ in terms of n, m, t, and $\det(R_A(n))$.

Theorem 3.1. *Let G be a simple graph of order n with m edges and the number of triangles t. Then,*

$$\mathrm{ER}(G) \le n\det(R_A(n))\left[\frac{n^6 - 3n^5 + 2n^4 - 6n^2m + 12nm - 12t}{n(n-1)(n-2)}\right]^{(n-1)/3}. \tag{4}$$

The equality in (4) holds if and only if $G \cong \overline{K}_n$.

Proof. Let us choose $r = n$ and $p_i = n - \lambda_i$, $i = 1, 2, \ldots, n$ in Lemma 2.1. Then, we have

$$Q_3^{1/3} \ge Q_{n-1}^{1/(n-1)}, \tag{5}$$

where

$$\begin{aligned} Q_3 &= \frac{6\sum_{i<j<k}(n-\lambda_i)(n-\lambda_j)(n-\lambda_k)}{n(n-1)(n-2)} \\ &= \frac{2\sum_{i=1}^n(n-\lambda_i)^3 + \left(\sum_{i=1}^n(n-\lambda_i)\right)^3 - 3\sum_{i=1}^n(n-\lambda_i)\sum_{i=1}^n(n-\lambda_i)^2}{n(n-1)(n-2)}, \end{aligned} \tag{6}$$

and

$$Q_{n-1} = \frac{\sum_{i=1}^n\prod_{j=1, j\ne n-i+1}^n(n-\lambda_i)}{n} = \frac{\prod_{i=1}^n(n-\lambda_i)}{n}\sum_{i=1}^n\frac{1}{n-\lambda_i} = \frac{1}{n\det(R_A(n))}\mathrm{ER}(G). \tag{7}$$

On the other hand, by the identities given in (1), we have that

$$\begin{aligned} \sum_{i=1}^n(n-\lambda_i)^3 &= \sum_{i=1}^n(n^3 - 3n^2\lambda_i + 3n\lambda_i^2 - \lambda_i^3) \\ &= n^4 - 3n^2\sum_{i=1}^n\lambda_i + 3n\sum_{i=1}^n\lambda_i^2 - \sum_{i=1}^n\lambda_i^3 \\ &= n^4 + 6nm - 6t, \end{aligned}$$

$$\sum_{i=1}^n(n-\lambda_i)^2 = \sum_{i=1}^n(n^2 - 2n\lambda_i + \lambda_i^2) = n^3 - 2n\sum_{i=1}^n\lambda_i + \sum_{i=1}^n\lambda_i^2 = n^3 + 2m,$$

and

$$\sum_{i=1}^n(n-\lambda_i) = n^2 - \sum_{i=1}^n\lambda_i = n^2.$$

Considering the above results with (5)–(7), we arrive at

$$\frac{1}{n\det(R_A(n))}\mathrm{ER}(G) \le \left[\frac{n^6 - 3n^5 + 2n^4 - 6n^2m + 12nm - 12t}{n(n-1)(n-2)}\right]^{(n-1)/3}.$$

From the above, inequality (4) is obtained. By Lemma 2.1, the equality in (4) holds if and only if $n - \lambda_1 = n - \lambda_2 = \ldots = n - \lambda_n$, that is, if and only if $\lambda_1 = \lambda_2 = \ldots = \lambda_n$. In view of Lemma 2.5, we deduce that $G \cong \overline{K}_n$. □

Considering the relation between $\det(R_A(n))$, n, and m given in Lemma 2.3 with Theorem 3.1, we obtain the following upper bound on $\mathrm{ER}(G)$ involving the parameters n, m, and t.

Corollary 3.1. *Let G be a simple graph of order n with m edges and the number of triangles t. Then,*

$$\mathrm{ER}(G) \le n\left(\frac{n}{n^2-2m}\right)^n \left[\frac{n^6-3n^5+2n^4-6n^2m+12nm-12t}{n(n-1)(n-2)}\right]^{(n-1)/3}. \tag{8}$$

Remark 3.1. Although the upper bound (4) is stronger than the upper bound (8), we think that readers will prefer to use (8) for practical purposes.

For triangle-free graphs, inequality (8) leads to the following (n, m)-type upper bound on resolvent energy.

Corollary 3.2. *Let G be a triangle-free graph of order n with m edges. Then,*

$$\mathrm{ER}(G) \le n\left(\frac{n}{n^2-2m}\right)^n \left[\frac{n^3(n-1)-6m}{n-1}\right]^{(n-1)/3}. \tag{9}$$

Considering the relation between ordinary and resolvent graph energy given in Lemma 2.4 with (9), we have the following upper bound for the energy of triangle-free graphs.

Corollary 3.3. *Let G be a triangle-free graph of order n with m edges. Then,*

$$E(G) \le \sqrt{n^5\left(\frac{n}{n^2-2m}\right)^n \left[\frac{n^3(n-1)-6m}{n-1}\right]^{(n-1)/3} - n^4}\,.$$

In the next theorem, we determine a lower bound on $\mathrm{ER}(G)$ involving the parameters n, m, and t.

Theorem 3.2. *Let G be a simple graph of order n with m edges and the number of triangles t. Then,*

$$\mathrm{ER}(G) \ge \frac{n(n-2)(n^4-n^3-2m)}{n^6-3n^5+2n^4-6n^2m+12nm-12t}. \tag{10}$$

The equality in (10) *holds if and only if* $G \cong \overline{K}_n$.

Proof. The following result was determined in [22] via Newton's inequality given in Lemma 2.2

$$\frac{Q_2}{Q_3} \le \ldots \le \frac{Q_{r-1}}{Q_r}.$$

From this result, it is clear that

$$Q_2 Q_r \le Q_{r-1} Q_3, \quad r \ge 3. \tag{11}$$

Putting $r = n$ and $p_i = n - \lambda_i$, $i = 1, 2, \ldots, n$ in (11), we have

$$\begin{aligned} Q_2 &= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1, i\neq j}^{n} (n-\lambda_i)(n-\lambda_j) \\ &= \frac{1}{n(n-1)}\left[\left(\sum_{i=1}^{n}(n-\lambda_i)\right)^2 - \sum_{i=1}^{n}(n-\lambda_i)^2\right] \\ &= \frac{n^4-n^3-2m}{n(n-1)}, \quad \text{by Eq. (1)}, \end{aligned}$$

and

$$Q_n = \prod_{i=1}^{n}(n-\lambda_i) = \frac{1}{\det(R_A(n))}.$$

From the proof of Theorem 3.1, we also have that

$$Q_3 = \frac{n^6 - 3n^5 + 2n^4 - 6n^2m + 12nm - 12t}{n(n-1)(n-2)},$$

and

$$Q_{n-1} = \frac{1}{n\det(R_A(n))}\mathrm{ER}(G).$$

Taking into account the above results with Eq. (11), we obtain

$$\frac{n^6 - 3n^5 + 2n^4 - 6n^2m + 12nm - 12t}{n^2(n-1)(n-2)\det(R_A(n))}.\mathrm{ER}(G) \geq \frac{n^4 - n^3 - 2m}{n(n-1)\det(R_A(n))},$$

from which inequality (10) follows. By Lemma 2.2, the equality in (10) holds if and only if $n - \lambda_1 = n - \lambda_2 = \ldots = n - \lambda_n$, which implies that $\lambda_1 = \lambda_2 = \ldots = \lambda_n$. Then, from Lemma 2.5, we conclude that $G \cong \overline{K}_n$. □

For triangle-free graphs, the inequality (10) yields the following (n, m)-type lower bound on resolvent energy.

Corollary 3.4. *Let G be a triangle-free graph of order n with m edges. Then,*

$$\mathrm{ER}(G) \geq 1 + \frac{4m}{n^3(n-1) - 6m}.$$

Example 1. Let us consider the triangle-free graph G with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and edge set $E = \{v_1v_7, v_1v_8\}$. Then, $\mathrm{ER}(G) \approx 1.008$. For this graph, at rounded three decimal places, the upper bound in Corollary 3.2 gives $\mathrm{ER}(G) \leq 1.662$ while the lower bound in Corollary 3.4 gives $\mathrm{ER}(G) \geq 1.002$.

4 Conclusion

Resolvent energy of a graph is a type of graph energy pertaining to its resolvent matrix. Recently, in [8,12], various lower and upper bounds for the resolvent energy, which depend on the parameters n, λ_1, λ_n, and $\det(R_A(n))$ have been presented. In this work, we have found some new estimates for the resolvent energy of graphs involving the number of vertices (n), the number of edges (m), and the number of triangles (t). For graphs possessing limited number of triangles, our bounds are more convenient than the bounds involving graph spectrum.

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