Research Article

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On SI_2-convergence in T_0-spaces

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Abstract: Recently, Shen et al. showed that the SI_2-topology on a T_0-space can be described completely in terms of SI_2-convergence, and the SI_2-convergence is topological whenever the given space is SI_2-continuous. In this article, we give a characterization of T_0-spaces for the SI_2-convergence being topological by introducing the notion of strongly I_2-continuous spaces, which are strictly weaker than SI_2-continuous spaces but are more closely related to the SI_2-convergence. Moreover, as a common generalization of the irr-convergence and the $\mathcal{S}$-convergence, we introduce the concept of SI_2^*-convergence in T_0-spaces and the related concept of SI_2^*-continuous spaces. It is proved that if a T_0-space X is SI_2^*-continuous, then the SI_2^*-convergence in X is topological.

Keywords: irreducible set, SI_2-convergence, SI_2-continuous space, strongly I_2-continuous space, $SI_2{}^*$-convergence, $SI_2{}^*$-continuous space

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1 Introduction

Convergence and convergence class play an important role in both order theory and general topology [1,2]. For a topological space (X, τ) and a class $\mathcal{L}$ consisting of pairs $((x_i)_{i\in I}, x)$, where $(x_i)_{i\in I}$ is a net in X and x a point of X, the topology τ can naturally induce a convergence class as follows:

$$C(\tau) = \{((x_i)_{i\in I}, x) : (x_i)_{i\in I} \text{ is a net in } X, x \in X \text{ and for any } U \in \tau,$$
$$x \in U \text{ implies that } (x_i)_{i\in I} \text{ is eventually in } U\}.$$

And we can define a topology on X associated with $\mathcal{L}$:

$$O(\mathcal{L}) = \{U \subseteq X : ((x_i)_{i\in I}, x) \in \mathcal{L} \text{ and } x \in U \text{ imply } x_i \in U \text{ eventually}\}.$$

It is easy to verify that $\tau = O(C(\tau))$. However, if $\mathcal{L}$ is not a convergence class in the sense of Kelley [2], then the convergence class $C(O(\mathcal{L})) \neq \mathcal{L}$, that is, the class $\mathcal{L}$ is not topological.

Numerous researchers have studied various types of convergences [1,3–11]. With different convergence, they have not only proposed the corresponding continuity of posets (more generally, topological spaces) but also presented some links between order theory and topology. In [1], it was proved that the lim-inf convergence in a dcpo P is topological iff the poset P is a continuous domain. This result was generalized to partially ordered sets (posets) in [3]. In [4], using the cut operator instead of joins, Ruan and Xu introduced and discussed $\mathcal{S}$-convergence and $\mathcal{GS}$-convergence in posets. They proved that a poset P is s_2-continuous (resp., s_2-quasicontinuous) iff the $\mathcal{S}$-convergence (resp., the $\mathcal{GS}$-convergence) in P is topological.

In the invited talk at the Sixth International Symposium on Domain Theory in 2013, Jimmie Lawson emphasized the need to develop the core of domain theory directly in T_0-spaces to instead posets. In this direction, by using irreducible sets instead of directed sets, Zhao and Ho [12] introduced the SI-topology on

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T_0-spaces as a generalization of the Scott topology on posets. In [5], Andradi et al. defined SI-convergence in T_0-spaces and proved that for any T_0-space X having condition (I*), X is an I-continuous space iff SI-convergence in X is topological. Later, Lu and Zhao [6] gave a characterization of T_0-spaces for the SI-convergence being topological. In [7], Zhao et al. provided a different way to define irreducible convergence in T_0-spaces, which can be seen as a topological counterpart of lim-inf convergence in posets, and presented a sufficient and necessary condition for irreducible convergence to be topological in T_0-spaces. By using the cut operator of irreducible sets and the specialization order of a given T_0-space, Shen et al. [8] defined the SI_2-topology on T_0-spaces and proved that SI_2-convergence on a T_0-space X is topological whenever the space X is SI_2-continuous. This naturally raises a question whether there is a characterization of T_0-spaces for the SI_2-convergence to be topological.

In this article, we introduce a new way-below relation on T_0-spaces, called the I_2-way-below relation. By using the I_2-way-below relation, we introduce the notions of I_2-continuity and strongly I_2-continuity of T_0-spaces, both of them are strictly weaker than the SI_2-continuity but are more closely related to the SI_2-convergence. We prove that the SI_2-convergence in a T_0-space X is topological iff X is strongly I_2-continuous, giving an positive answer to the aforementioned question. Moreover, we define and study the SI_2^*-convergence in T_0-spaces, which can be seen as topological counterparts of the $\mathcal{S}$-convergence and the irr-convergence in posets. The related concept of SI_2^*-continuous spaces is also introduced. It is proved that if a T_0-space X is SI_2^*-continuous, then the SI_2^*-convergence in X is topological.

2 Preliminaries

In this section, we briefly recall some basic concepts and results about ordered structures and T_0-spaces that will be used in the article. For further details, we refer the reader to [1–2,13–14].

For a poset P and $A \subseteq P$, define $\uparrow A = \{x \in P : a \le x \text{ for some } a \in A\}$ and $\downarrow A = \{x \in P : x \le a \text{ for some } a \in A\}$. For $x \in X$, let $\uparrow x = \uparrow\{x\}$ and $\downarrow x = \downarrow\{x\}$. A subset A is called a *lower set* (resp., an *upper set*) if $A = \downarrow A$ (resp., $A = \uparrow A$). Define $A^{\uparrow} = \{u \in P : A \subseteq \downarrow u\}$ (the sets of all upper bounds of A in P) and $A^{\downarrow} = \{v \in P : A \subseteq \uparrow v\}$ (the sets of all lower bounds of A in P). The set $A^{\delta} = (A^{\uparrow})^{\downarrow}$ is called the *cut* of A in P. If the set of upper bounds of A has a unique smallest element (that is, the set of upper bounds contains exactly one of its lower bounds), we call this element the *least upper bound* and write it as $\vee A$ or sup A (for supremum). Similarly the greatest lower bound is written as $\bigwedge A$ or inf A (for infimum).

The set of all natural numbers is denoted by $\mathbb{N}$. When $\mathbb{N}$ is regarded as a poset (in fact, a chain), the order on $\mathbb{N}$ is the usual order of natural numbers. A nonempty subset D of a poset P is called *directed* if every finite subset of D has an upper bound in D. The set of all directed sets of P is denoted by $\mathcal{D}(P)$. The poset P is called a *directed complete poset*, or *dcpo* for short, if for any $D \in \mathcal{D}(P)$, $\vee D$ exists in P.

Let P be a poset and $a, b \in P$. We say that a is *way below* b, in symbols $a \ll b$, if for all $D \in \mathcal{D}(P)$ for which $\vee D$ exists in P, $b \le \vee D$ implies $a \in \downarrow D$. The poset P is called a *continuous poset* if for any $a \in P$, the set $\twoheaddownarrow a \coloneqq \{b \in P : b \ll a\}$ is directed and $a = \vee \twoheaddownarrow a$. A subset U of P is *Scott open* if (i) $U = \uparrow U$, and (ii) for any directed subset D for which $\vee D$ exists, $\vee D \in U$ implies $D \cap U \neq \emptyset$. The topology formed by all the Scott open sets of P is called the *Scott topology*, written as $\sigma(P)$. The upper sets of P form the (*upper*) *Alexandroff topology* $\alpha(P)$. The topology generated by the collection of sets $P \backslash \uparrow x$ (as subbasic open subsets) is called the *lower topology* and denoted it by $\omega(P)$; dually, the *upper topology* on a poset P, generated by the complements of the principal ideals of P, is denoted by $\upsilon(P)$.

A *net* $(x_i)_{i \in I}$ in a set X is a mapping from a directed set I to X. For each $x \in X$, one can define a *constant net with the value* x by $x_i = x$ for all $i \in I$. We denote this constant net by $(x)_{i \in I}$. If $Q(x)$ is a property of the elements $x \in X$, we say that $Q(x)$ *holds eventually* in the net $(x_i)_{i \in I}$ if there is a $i_0 \in J$ such that $Q(x_i)$ is true whenever $i_0 \le i$.

Definition 2.1. [1] We say a net $(x_i)_{i \in I}$ *lim-inf converges* to x in a poset P if there exists a directed subset D of P such that

(i) $\vee D$ exists and $x \le \vee D$, and
(ii) for every $d \in D$, $d \le x_i$ holds eventually, i.e., there exists $i_0 \in I$ such that $d \le x_i$ for all $i \ge i_0$.

Definition 2.2. [4] Let P be a poset and $(x_j)_{j\in J}$ a net in P.
(1) A point $y \in P$ is called an *eventual lower bound* of a net $(x_j)_{j\in J}$ in P, if there exists a $k \in J$ such that $y \le x_j$ for all $j \ge k$, i.e., $(x_j)_{j\in J}$ is eventually in $\uparrow y$.
(2) Let $\mathcal{S}(P)$ denote the class of those pairs $((x_j)_{j\in J}, x)$ such that $x \in D^\delta$ for some directed set D of eventual lower bounds of the net $(x_j)_{j\in J}$. For each such pair, we again say that x is an *$\mathcal{S}$-limit* of $(x_j)_{j\in J}$ or $(x_j)_{j\in J}$ $\mathcal{S}$-converges to x, and write $(x_j)_{j\in J} \xrightarrow{\mathcal{S}} x$.

As in [9], an upper subset U of a poset P is called *s_2-open* if for any directed subset D of P, $D^\delta \cap U \neq \varnothing$ implies $D \cap U \neq \varnothing$. The collection of all s_2-open subsets of P forms a topology, called *s_2-topology*, and is denoted by $s_2(P)$. It is easy to see that $s_2(P) = O(\mathcal{S}(P)) = \{U \subseteq P : \text{whenever } x_i \xrightarrow{\mathcal{S}} x \text{ and } x \in U, \text{ then eventually } x_i \in U\}$. The *way-below relation* $\ll_2$ on P is defined by $x \ll_2 y$ iff for any directed subset D of P, $y \in D^\delta$ implies $x \in \downarrow D$.

Lemma 2.3. [5] *Let P be a poset, D a nonempty subset of P, and $(x_i)_{i\in I}$ a net in P. Then the following conditions are equivalent*:
(1) *D is a set of eventual lower bounds of $(x_i)_{i\in I}$.*
(2) *For every upper set U of P, $D \cap U \neq \varnothing$ implies $x_i \in U$ eventually.*

For a T_0-space X, let $\le_X$ denote the *specialization order* on X: $x \le_X y$ iff $x \in \overline{\{y\}}$. In the following, when a T_0-space X is considered as a poset, the order always refers to the specialization order if no other explanation. The pair $(X, \le_X)$ is denoted by ΩX or simply by X if no confusion arises, and sometimes we briefly write $\le$ instead $\le_X$. Let $O(X)$ (resp., $\Gamma(X)$) be the set of all open subsets (resp., closed subsets) of X. Clearly, each open set is an upper set and each closed set is a lower set with respect to the specialization order $\le_X$. For a subset of X, denote the closure of A in X by $\mathrm{cl}_X A$ or simply by $\mathrm{cl}A$ and the interior of A in X by $\mathrm{int}_X A$ in X or simply by $\mathrm{int}A$. We also simply use $\overline{A}$ to denote the the closure of A if no confusion arises.

A nonempty subset A of a T_0-space X is called an *irreducible* set if for any $F_1, F_2 \in \Gamma(X)$, $A \subseteq F_1 \cup F_2$ implies $A \subseteq F_1$ or $A \subseteq F_2$. We denote by $\mathsf{Irr}(X)$ (resp., $\mathsf{Irr}_c(X)$) the set of all irreducible (resp., irreducible closed) subsets of X. Clearly, every subset of X that is directed under $\le_X$ is irreducible and the nonempty irreducible sets of a poset equipped with the Alexandroff topology are exactly the directed sets of P. And we said that X is *irreducible complete space* if every irreducible subset of X has a sup.

Lemma 2.4. [15] *If $f : X \to Y$ is continuous and $A \in \mathsf{Irr}(X)$, then $f(A) \in \mathsf{Irr}(Y)$.*

For a set X and a class $\mathcal{L}$ consisting of pairs $((x_i)_{i\in I}, x)$, where $(x_i)_{i\in I}$ is a net in X and x is a point of X, the topology on X associated with $\mathcal{L}$ is denoted by $O(\mathcal{L})$, that is, $O(\mathcal{L}) = \{U \subseteq X : ((x_i)_{i\in I}, x) \in \mathcal{L} \text{ and } x \in U \text{ imply } x_i \in U \text{ eventually}\}$.

Definition 2.5. [5,7] Let X be a T_0-space.
(1) A net $(x_i)_{i\in I}$ of X is said to *irreducibly converge* to a point x of X, if there exists an irreducible set F of X with $\vee F$ existing such that $x \le \vee F$, and for each $e \in F$, $e \le x_i$ holds eventually. In this case, we write $(x_i)_{i\in I} \xrightarrow{\mathsf{Irr}} x$.
(2) A net $(x_i)_{i\in I}$ of X is said to *SI-converge* to a point x of X, if there exists an irreducible set F of X with $\vee F$ existing such that $x \le \vee F$, and for every $U \in O(X)$, $F \cap U \neq \varnothing$ implies $x_i \in U$ eventually. In this case, we write $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}} x$.

An open subset U of T_0-space X is called *SI-open* if for any $F \in \mathsf{Irr}(X)$, $\vee F \in U$ implies $F \cap U \neq \varnothing$ whenever $\vee F$ exists. The collection of all SI-open sets, denoted by $O_{\mathrm{SI}}(X)$, is a topology on X, called the *irreducibly-derived topology* (shortly *SI-topology*). The space $(X, O_{\mathrm{SI}}(X))$ will also be simply written as $\mathrm{SI}(X)$. In [7], Zhao et al. denoted by τ_{Irr} the topology induced by irr-convergence.

Proposition 2.6. [5] *For any T_0-space X, the SI-topology coincides with the topology induced by SI-convergence, namely,* $V \in \mathcal{O}_{\mathrm{SI}}(X)$ *iff for every net* $(x_i)_{i\in I}$ *in* X, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}} x$ *and* $x \in V$ *imply* $x_i \in V$ *eventually.*

Definition 2.7. [5,7,12] Let X be a T_0-space and $x, y \in X$. We say
(1) x is *SI-way-below* y, in symbols $x \ll_{\mathrm{SI}} y$, if for any irreducible set F of X, $y \le \vee F$ implies $x \in \downarrow F$ whenever $\vee F$ exists.
(2) x is *I-way-below* y, in symbols $x \ll_{\mathrm{I}} y$, if for every irreducible set F of X with $\vee F$ existing, $y \le \vee F$ implies $x \in \mathrm{cl}F$.
(3) x is *Irr-way-below* y, in symbols $x \ll_{\mathrm{Irr}} y$, if for every net $(x_i)_{i\in I}$ in X irreducibly converging to y, $x \le x_i$ holds eventually.

Definition 2.8. [8] Let X be a T_0-space. A subset U of X is called *SI_2-open* if the following two conditions are satisfied:
(1) U is an open set in X, and
(2) for any $F \in \mathsf{Irr}(X)$, $F^\delta \cap U \neq \varnothing$ implies $F \cap U \neq \varnothing$.
The set of all SI_2-open sets in X is denoted by $\mathcal{O}_{\mathrm{SI}_2}(X)$. It is straightforward to verify that $\mathcal{O}_{\mathrm{SI}_2}(X)$ is a topology on X, called the *SI_2-topology*. The space $(X, \mathcal{O}_{\mathrm{SI}_2}(X))$ will also be simply written as $\mathrm{SI}_2(X)$.

Definition 2.9. [8] Let X be a T_0-space and $x, y \in X$.
(1) We say that x is *SI_2-way-below* y, in symbols $x \ll_{\mathrm{SI}_2} y$, if for all irreducible set F of X, the relation $y \in F^\delta$ always implies $x \in \downarrow F$. We write $\Downarrow_{\mathrm{SI}_2} a = \{x \in X : x \ll_{\mathrm{SI}_2} a\}$ and $\Uparrow_{\mathrm{SI}_2} a = \{x \in X : a \ll_{\mathrm{SI}_2} x\}$.
(2) The space X is called *SI_2-continuous* if for any $x \in X$, $\Uparrow_{\mathrm{SI}_2} x \in \mathcal{O}(X)$, $\Downarrow_{\mathrm{SI}_2} x \in \mathsf{Irr}(X)$ and $x = \vee \Downarrow_{\mathrm{SI}_2} x$.

By Remark 5.1(1) and Proposition 5.6 of [8], we obtain the following result.

Proposition 2.10. *For a T_0-space X, the following conditions are equivalent*:
(1) *X is SI_2-continuous.*
(2) *For all* $x \in X$, $\Uparrow_{\mathrm{SI}_2} x \in \mathcal{O}(X)$, $\Downarrow_{\mathrm{SI}_2} x \in \mathsf{Irr}(X)$, *and* $x = (\Downarrow_{\mathrm{SI}_2} x)^\delta$.
(3) *For all* $x \in X$, $\Uparrow_{\mathrm{SI}_2} x$ *is* SI_2*-open*, $\Downarrow_{\mathrm{SI}_2} x \in \mathsf{Irr}(X)$ *and* $x = \vee \Downarrow_{\mathrm{SI}_2} x$.
(4) *For all* $x \in X$, $\Uparrow_{\mathrm{SI}_2} x$ *is* SI_2*-open*, $\Downarrow_{\mathrm{SI}_2} x \in \mathsf{Irr}(X)$ *and* $x = (\Downarrow_{\mathrm{SI}_2} x)^\delta$.

Throughout this article, when we say X is a space, it always means X is a T_0-space. For $x \in X$ and a net $(x_i)_{i\in I}$ in X, we use the symbols $(x_i)_{i\in I} \longrightarrow x$ to represent that the net $(x_i)_{i\in I}$ converges to x in the space X.

3 I_2-continuous spaces and strongly I_2-continuous spaces

In this section, we introduce the notions of I_2-continuous spaces and strongly I_2-continuous spaces, and discuss some basic properties of these spaces. Especially, we prove that a T_0-space X is strongly I_2-continuous iff SI_2-convergence on X is topological.

We first recall the definition of SI_2-convergence and give some its properties.

Definition 3.1. [8] We say a net $(x_i)_{i\in I}$ *SI_2-converges* to a point x in a T_0-space X if there exists an irreducible set F in X such that
(i) $x \in F^\delta$ and
(ii) for any $U \in \mathcal{O}(X)$, $F \cap U \neq \varnothing$ implies $x_i \in U$ eventually.
And in this case, we write $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$. Let $\mathcal{SI}_2(X) = \{((x_i)_{i\in I}, x) : (x_i)_{i\in I}$ is a net in $X, x \in X$ and $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x\}$.

Remark 3.2. For a T_0-space X, we have the following statements:
(1) The constant net $(x)_{i\in I}$ in X with value x SI_2-converges to x.
(2) If $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$ in X, then $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} y$ for any $y \le x$. Thus, the SI_2-convergence points of a net are generally not unique.
(3) Let P be a poset. Then the SI_2-convergence in $(P, \alpha(P))$ coincides with the $\mathcal{S}$-convergence in P.
(4) If X is irreducible complete, then for any net $(x_i)_{i\in I}$ in X, $(x_i)_{i\in I}$ SI-converges to $x \in X$ iff $(x_i)_{i\in I}$ SI_2-converges to x.

Lemma 3.3. [8] *For any T_0-space X, the two topologies $O(\mathcal{SI}_2(X))$ and $O_{\mathrm{SI}_2}(X)$ coincide, that is, $O_{\mathrm{SI}_2}(X) = \{U \subseteq P :$ whenever $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$ and $x \in U$, then eventually $x_i \in U\}$.*

Recall that a net $(y_j)_{j\in J}$ is a *subnet* of $(x_i)_{i\in I}$ if (i) there exists a function $g : J \to I$ such that $y_j = x_{g(j)}$ for all $j \in J$, and (ii) for each $i \in I$ there exists $j' \in J$ such that $g(j) \ge i$ whenever $j \ge j'$.

Proposition 3.4. *Let X be a T_0-space and $A \subseteq X$. Then the following conditions are equivalent*:
(1) *A is an SI_2-closed set.*
(2) *A is a closed subset of X, and for any irreducible set F in X, $F \subseteq A$ implies $F^\delta \subseteq A$.*
(3) *For any net $(x_i)_{i\in I}$ in A, if $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$, then $x \in A$.*

Proof. (1) ⇔ (2): See [8, Proposition 3.6].

(1) ⇒ (3): Let $(x_i)_{i\in I}$ be a net in A and $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$. If $x \notin A$, then $x \in X\backslash A$. Since A is an SI_2-closed set, $X\backslash A$ is SI_2-open, and hence, $X\backslash A \in O_{\mathrm{SI}_2}(X)$ by Lemma 3.3. Then the net $(x_i)_{i\in I}$ must be eventually in $X\backslash A$, being a contradiction with the fact that $(x_i)_{i\in I}$ is in A. Thus, $x \in A$.

(3) ⇒ (1): We show that $X\backslash A$ is SI_2-open. Let $x \in X\backslash A$ and $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$. Then the net $(x_i)_{i\in I}$ is eventually in $X\backslash A$. Otherwise, for each $i \in I$, there exists a $\varphi(i) \in I$ with $\varphi(i) \ge i$ such that $x_{\varphi(i)} \in A$. Let J be the subset of I consisting of all $j \in I$ such that $x_j \in A$. Then J is cofinal in I, and $(x_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$. As $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$, we have $(x_j)_{j\in J} \xrightarrow{\mathrm{SI}_2} x$, and hence, $x \in A$ by (3), which contradicts $x \in X\backslash A$. Then we conclude that the net $(x_i)_{i\in I}$ is eventually in $X\backslash A$. Hence, $X\backslash A \in O(\mathcal{SI}_2(X))$. By Lemma 3.3, A is SI_2-closed. □

Lemma 3.5. *Let X be a T_0-space and F be an irreducible set of X with $x \in F^\delta$. Then there exists a net $(x_i)_{i\in I}$ in X such that all of its terms are in F and $(x_i)_{i\in I}$ SI_2-converges to x.*

Proof. Let $I = \{(U, n, e) \in O(X) \times \mathbb{N} \times F : e \in U\}$ and define an order on I by the lexicographic order on the first two coordinates, that is, $(U, m, a) < (V, n, b)$ iff V is a proper subset of U or $U = V$ and $m < n$. For any $(U_1, n_1, e_1), (U_2, n_2, e_2) \in I$, we have $e_1 \in F \cap U_1$ and $e_2 \in F \cap U_2$. By the irreducibility of F, we have $F \cap U_1 \cap U_2 \ne \emptyset$. Select $e_3 \in F \cap U_1 \cap U_2$. Then $(U_1, n_1, e_1), (U_2, n_2, e_2) < (U_1 \cap U_2, n_1 + n_2 + 1, e_3)$. Hence, I is a directed set. We let $x_{(U,n,e)} = e$ for any $(U, n, e) \in I$. Now we show that the net $(e)_{(U,n,e)\in I}$$\mathrm{SI}_2$-converges to x. We firstly have that $F \in \mathrm{Irr}(X)$ and $x \in F^\delta$ by the assumption. For any $U \in O(X)$ with $F \cap U \ne \emptyset$, select a $d \in F \cap U$. Then $(U, 1, d) \in I$ and $e \in U$ for all $(V, n, e) \in I$ with $(V, n, e) \ge (U, 1, d)$, proving that $(e)_{(U,n,e)\in I}$ SI_2-converges to x. □

Proposition 3.6. *Let X, Y be T_0-spaces and f be a continuous mapping from X to Y. Then the following two conditions are equivalent*:
(1) *f is a continuous mapping from $\mathrm{SI}_2(X)$ to $\mathrm{SI}_2(Y)$.*
(2) *For any net $(x_i)_{i\in I}$ and $x \in X$, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$ in X implies $f(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} f(x)$ in Y.*

Proof. (1) ⇒ (2): First, f is order-preserving. In fact, if $x \le_X y$, i.e., $x \in \mathrm{cl}\{y\}$, then we have $f(x) \in f(\overline{\{y\}}) \subseteq \overline{f(\{y\})}$ by the continuity of $f : X \to Y$, whence $f(x) \le_Y f(y)$. Suppose that $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$ in X. Now we show that

$f(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} f(x)$ in Y. As $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$, there exists an irreducible set F in X such that conditions (i) and (ii) of Definition 3.1 are satisfied. Then $f(F) \in \mathsf{Irr}(Y)$ by Lemma 2.3. Since f is order-preserving, we obtain $f(x) \in f(F^\delta) = f((F^\uparrow)^\downarrow) \subseteq (f(F^\uparrow))^\downarrow \subseteq (f(F)^\uparrow)^\downarrow = (f(F))^\delta$. For $V \in \mathcal{O}(Y)$, if $f(F) \cap V \neq \emptyset$, then $F \cap f^{-1}(V) \neq \emptyset$ and $f^{-1}(V) \in \mathcal{O}(X)$ by the continuity of $f : X \to Y$, and consequently, $x_i \in f^{-1}(V)$ eventually. Hence, $f(x_i) \in V$ eventually. Thus, $f(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} f(x)$ in Y.

(2) ⇒ (1): Let $V \in \mathcal{O}_{\mathrm{SI}_2}(Y)$. By the continuity of $f : X \to Y$, we have $f^{-1}(V) \in \mathcal{O}(X)$. For any $F \in \mathsf{Irr}(X)$, if $F^\delta \cap f^{-1}(V) \neq \emptyset$, then we can select a point $a \in F^\delta \cap f^{-1}(V)$. By Lemma 3.5, there exists a net $(a_i)_{i\in I_F}$ in F SI_2-converging to a. By the assumption, the net $(f(a_i))_{i\in I_F}$ SI_2-converges to $f(a)$ and $f(a) \in V$. Hence, by Lemma 3.3, $f(a_i) \in V$ eventually, or equivalently, $a_i \in f^{-1}(V)$ eventually. It follows that $F \cap f^{-1}(V) \neq \emptyset$. We conclude that $f^{-1}(V) \in \mathcal{O}_{\mathrm{SI}_2}(X)$, and therefore, (1) holds. □

In [8], Shen et al. proved that the SI_2-convergence in a T_0-space X is topological whenever the space X is SI_2-continuous. This naturally raises a question whether there is a characterization of T_0-spaces for the SI_2-convergence to be topological. In the remainder of this section, we shall give such a characterisation.

First, we introduce a new notion of way-below relation.

Definition 3.7. Let X be a T_0-space and $x, y \in X$. We say that x is *I_2-way-below* y, in symbols $x \ll_{\mathrm{I}_2} y$, if for any irreducible set F in X, $y \in F^\delta$ implies $x \in \mathrm{cl}F$.

For $a \in X$, we write $\Downarrow_{\mathrm{I}_2} a = \{x \in X : x \ll_{\mathrm{I}_2} a\}$ and $\Uparrow_{\mathrm{I}_2} a = \{x \in X : a \ll_{\mathrm{I}_2} x\}$.

Remark 3.8. For a T_0-space X, the following statements hold for all $u, x, y, z \in X$:
(i) $x \ll_{\mathrm{I}_2} y$ implies $x \leq y$;
(ii) $u \leq x \ll_{\mathrm{I}_2} y \leq z$ implies $u \ll_{\mathrm{I}_2} z$;
(iii) $x \ll_{\mathrm{I}_2} y$ iff for every irreducible closed set F, $y \in F^\delta$ implies $x \in F$;
(iv) $x \ll_{\mathrm{SI}_2} y$ implies $x \ll_{\mathrm{I}_2} y$. Hence, $\Downarrow_{\mathrm{SI}_2} x \subseteq \Downarrow_{\mathrm{I}_2} x \subseteq {\downarrow} x$.

One can easily see that when X is a poset P endowed with the Alexandroff topology, the I_2-way-below relation is exactly the way-below relation $\ll_2$ (cf. [15, Fact 2.6]). When X is irreducible complete, we have $x \ll_{\mathrm{I}_2} y$ iff $x \ll_{\mathrm{I}} y$.

The following example shows that $\ll_{\mathrm{I}_2}$ is different to $\ll_{\mathrm{SI}}$ and also different to $\ll_{\mathrm{I}}$ in general.

Example 3.9. Let $Q = \{a_1, a_2, \ldots, a_n, \ldots\} \cup \{b_1, b_2\} \cup \{c\}$ and define a partial order $\leq$ on Q as follows (see Figure 1):
(i) $a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots$;
(ii) $a_n < b_1, a_n < b_2$ for all $n \in \mathbb{N}$;
(iii) b_1 and b_2 are incomparable; and
(iv) $c < b_1$ and $c < b_2$.

Consider the Alexandroff topology space $(Q, \alpha(Q))$. Then $\mathsf{Irr}((Q, \alpha(Q))) = \mathcal{D}(Q)$ (cf. [15, Fact 2.6]). It is easy to verify that for any $D \in \mathcal{D}(Q)$, D has a largest element or $D \subseteq \{a_{n+1} : n \in \mathbb{N}\}$ is countable infinite. Hence for any $A \in \mathsf{Irr}((Q, \alpha(Q)))$ for which $\vee A$ exists, we have that $c \leq \vee A$ implies $c \in {\downarrow} A$. So $c \ll_{\mathrm{SI}} c$ and hence $c \ll_{\mathrm{I}} c$. Let $F = \{a_{n+1} : n \in \mathbb{N}\}$. Then $F \in \mathsf{Irr}((Q, \alpha(Q)))$ and $c \in F^\delta = F \cup \{c\}$ but $c \notin \mathrm{cl}A = A$. Thus, $c \not\ll_{\mathrm{I}_2} c$.

Example 3.15 shows that $\ll_{\mathrm{I}_2}$ is different to $\ll_{\mathrm{SI}_2}$ in general.

Proposition 3.10. *Let X be a T_0-space and $x, y \in X$. Then the following two conditions are equivalent*:
(1) $x \ll_{\mathrm{I}_2} y$.
(2) *For any net $(x_i)_{i\in I}$ of X, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} y$ implies $(x_i)_{i\in I} \longrightarrow x$.*

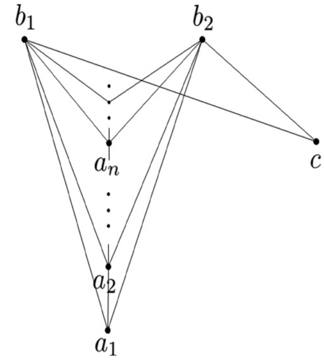

Figure 1: The poset Q in Example 3.9.

Proof. (1) ⇒ (2): Suppose that $x \ll_{\mathrm{I}_2} y$ and $(x_i)_{i\in I}$ is a net of X SI_2-converging to y. We show that $(x_i)_{i\in I}$ converges to x in the space X. As $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} y$, there exists an $F \in \mathsf{Irr}(X)$ such that $y \in F^\delta$, and $F \cap V \neq \emptyset$ implies $x_i \in V$ eventually for any $V \in \mathcal{O}(X)$. Then we have $x \in \mathrm{cl}F$ by $y \in F^\delta$ and $x \ll_{\mathrm{I}_2} y$. Hence, for any $U \in \mathcal{O}(X)$ with $x \in U$, it holds that $F \cap U \neq \emptyset$, and then $x_i \in U$ eventually. Therefore, $(x_i)_{i\in I} \to x$.

(2) ⇒ (1): Let $F \in \mathsf{Irr}(X)$ with $y \in F^\delta$. Then by Lemma 3.5, there exists a net $(x_i)_{i\in I_F}$ such that all of its terms are in F and it SI_2-converges to y. So $(x_i)_{i\in I_F} \to x$. Then for any $U \in \mathcal{O}(X)$ with $x \in U$, we have $x_i \in U$ eventually. Since $\{x_i : i \in I\} \subseteq F$, it holds that $x_i \in F \cap U$ eventually, and hence, $F \cap U \neq \emptyset$, proving that $x \in \mathrm{cl}F$. Thus, $x \ll_{\mathrm{I}_2} y$. □

Definition 3.11. A T_0-space X is called *I_2-continuous* if for every $x \in X$, $\twoheaddownarrow_{\mathrm{I}_2} x \in \mathsf{Irr}(X)$ and $x \in (\twoheaddownarrow_{\mathrm{I}_2} x)^\delta$.

Remark 3.12. By Remark 3.8 (i), we can easily see that a T_0-space X is I_2-continuous iff for any $x \in X$, $\twoheaddownarrow_{\mathrm{I}_2} x \in \mathsf{Irr}(X)$ and $x = \vee \twoheaddownarrow_{\mathrm{I}_2} x$.

Proposition 3.13. *For a T_0-space X, the following two conditions are equivalent*:
(1) *X is I_2-continuous.*
(2) *For any $x \in X$, there exists $F \in \mathsf{Irr}(X)$ such that $F \subseteq \twoheaddownarrow_{\mathrm{I}_2} x$ and $x = \vee F$.*

Proof. (1) ⇒ (2): Let $F = \twoheaddownarrow_{\mathrm{I}_2} x$. Then $F \in \mathsf{Irr}(X)$ and $x = \vee F$.

(2) ⇒ (1): For $x \in X$, by the assumption, there exists an irreducible subset $F \subseteq \twoheaddownarrow_{\mathrm{I}_2} x$ such that $x = \vee F$. Then $F^\delta = \downarrow \vee F = \downarrow x$, and hence, $x \in F^\delta$. It follows that $F \subseteq \twoheaddownarrow_{\mathrm{I}_2} x \subseteq \mathrm{cl}F$. So $\mathrm{cl} \twoheaddownarrow_{\mathrm{I}_2} x = \mathrm{cl}F \in \mathsf{Irr}_c(X)$. Then $\twoheaddownarrow_{\mathrm{I}_2} x \in \mathsf{Irr}(X)$ and $x = \vee F = \vee \mathrm{cl}F = \vee \mathrm{cl} \twoheaddownarrow_{\mathrm{I}_2} x = \vee \twoheaddownarrow_{\mathrm{I}_2} x$. Thus, X is I_2-continuous. □

By Remark 3.8(iv) and Proposition 3.13, we directly obtain the following corollary.

Corollary 3.14. *Every* SI_2*-continuous space is* I_2*-continuous.*

However, I_2-continuous spaces are not SI_2-continuous in general, as shown in the following example.

Example 3.15. Let X be a countable infinite set and X_{cof} the space equipped with the *co-finite topology* (the empty set and the complements of finite subsets of X are open). Then
(a) X_{cof} is a T_1-space, and hence, its specialization order is the discrete order on X.
(b) $\mathsf{Irr}(X_{\mathrm{cof}}) = \{\{x\} : x \in X\} \cup \{A : A$ is a countable infinite set of $X\}$ and $\mathsf{Irr}_c(X_{\mathrm{cof}}) = \{\{x\} : x \in X\} \cup \{X\}$.
(c) For any countable infinite set A of X, $\mathrm{cl}A = X$.
(d) For $x, y \in X$, $x \ll_{\mathrm{I}_2} y$ iff $x = y$ by (b) and (c). So X_{cof} is I_2-continuous.
(e) For any $x, y \in X$, $x \not\ll_{\mathrm{SI}_2} y$.

In fact, if $x \ll_{\mathrm{SI}_2} y$, then as $y \in \{y\} = \{y\}^\delta$, we have $x \in \downarrow\{y\} = \{y\}$ and hence, $x = y$. By (b), $X\backslash\{x\} \in \mathsf{Irr}(X_{\mathrm{cof}})$ and $x \in X = (X\backslash\{x\})^\delta$, but $x \notin \downarrow(X\backslash\{x\}) = X\backslash\{x\}$, which is a contradiction with $x \ll_{\mathrm{SI}_2} y$. Thus, $x \ll_{\mathrm{SI}_2} y$ for no $x, y \in X$.

(f) X_{cof} is not SI_2-continuous by (e).

Proposition 3.16. *Let X be a T_0-space, $y \in X$ and $(x_i)_{i\in I}$ be a net in X. Consider the following two conditions*:

(1) $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} y$.

(2) *For any $x \ll_{\mathrm{I}_2} y$, $(x_i)_{i\in I} \longrightarrow x$.*

Then (1) ⇒ (2), and two conditions are equivalent if X is I_2-continuous.

Proof. (1) ⇒ (2): By Proposition 3.10.

(2) ⇒ (1): Suppose that X is I_2-continuous. Then $\Downarrow_{\mathrm{I}_2} y \in \mathsf{Irr}(X)$ and $y \in (\Downarrow_{\mathrm{I}_2} y)^\delta$. For any $U \in \mathcal{O}(X)$, if $\Downarrow_{\mathrm{I}_2} y \cap U \neq \emptyset$, then there is $x \in U$ such that $x \ll_{\mathrm{I}_2} y$, and hence, $(x_i)_{i\in I} \longrightarrow x$ by (2). So $x_i \in U$ eventually. Thus, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} y$. □

Proposition 3.17. *Let X be a T_0-space. If SI_2-convergence in X is topological, then X is I_2-continuous.*

Proof. By Lemma 3.3, $\mathcal{O}(\mathcal{SI}_2(X)) = \mathcal{O}_{\mathrm{SI}_2}(X)$. Thus, if SI_2-convergence in X is topological, we must have

$$(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x \text{ iff } (x_i)_{i\in I} \text{ converges to } x \text{ with respect to the topology } \mathcal{O}_{\mathrm{SI}_2}(X).$$

Let $x \in X$. Define

$$I = \{(U, n, a) \in \mathcal{N}_{\mathrm{SI}_2}(x) \times \mathbb{N} \times X : a \in U\},$$

where $\mathcal{N}_{\mathrm{SI}_2}(x)$ consists of all open sets containing x in the space $(X, \mathcal{O}_{\mathrm{SI}_2}(X))$, and define an order on I by the lexicographic order on the first two coordinates, that is, $(U, m, a) < (V, n, b)$ iff V is a proper subset of U or $U = V$ and $m < n$. For any $(U_1, n_1, a_1), (U_2, n_2, a_2) \in I$, we have $U_1 \cap U_2 \in \mathcal{N}_{\mathrm{SI}_2}(x)$, and hence, $(U_1 \cap U_2, n_1 + n_2 + 1, x) \in I$. Clearly, $(U_1, n_1, a_1) < (U_1 \cap U_2, n_1 + n_2 + 1, x)$ and $(U_2, n_2, a_2) < (U_1 \cap U_2, n_1 + n_2 + 1, x)$. Thus, I is a directed set. Let $x_i = a$ for $i = (U, n, a) \in I$. It is easy to see that the net $(x_i)_{i\in I}$ converges to x in $(X, \mathcal{O}_{\mathrm{SI}_2}(X))$, and hence, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$. So there exists an irreducible set $F \in \mathsf{Irr}(X)$ such that $(x_i)_{i\in I}$ and F satisfy conditions (i) and (ii) of Definition 3.1. Now we show that $F \subseteq \Downarrow_{\mathrm{I}_2} x$.

Suppose that $s \in F$. We verify that $s \ll_{\mathrm{I}_2} x$. Let $E \in \mathsf{Irr}(X)$ and $x \in E^\delta$. Then by Lemma 3.5, there exists a net $(e_j)_{j\in J}$ such that all of its terms are in E and $(e_j)_{j\in J}$ SI_2-converges to x, and hence, it converges to x in the space $(X, \mathcal{O}_{\mathrm{SI}_2}(X))$ by Lemma 3.3.

For $V \in \mathcal{O}(X)$ with $s \in V$, we have $s \in F \cap V$, and hence, $F \cap V \neq \emptyset$. As $(x_i)_{i\in I}$ and F satisfy conditions (i) and (ii) of Definition 3.1, there is $i_0 = (U_0, m_0, z) \in I$ such that $x_i \in V$ for all $i \geq i_0$. For any $t \in U_0$, $(U_0, m_0 + 1, t) > (U_0, m_0, z)$, whence $t = x_{(U_0, m_0+1, t)} \in V$. So $x \in U_0 \subseteq V$. Since $U_0 \in \mathcal{N}_{\mathrm{SI}_2}(x)$ and $(e_j)_{j\in J}$ converges to x in $(X, \mathcal{O}_{\mathrm{SI}_2}(X))$, $e_j \in U_0$ eventually, and consequently, $e_j \in V$ eventually. Hence, $E \cap V \neq \emptyset$ (note that $(e_j)_{j\in J}$ is a net in E), proving that $s \in \mathrm{cl}_X E$.

In summary, we have proved that for any $E \in \mathsf{Irr}(X)$ with $x \in E^\delta$, $s \in \mathrm{cl}_X E$. Hence, $s \ll_{\mathrm{I}_2} x$. Thus, $F \subseteq \Downarrow_{\mathrm{I}_2} x$. Therefore, $F \in \mathsf{Irr}(F)$, $F \subseteq \Downarrow_{\mathrm{I}_2} x$ and $x \in F^\delta$. So by Proposition 3.13, X is I_2-continuous. □

Proposition 3.18. *Let X be a T_0-space. If SI_2-convergence in X is topological, then for any $x, y \in X$ with $x \ll_{\mathrm{I}_2} y$ and $U \in \mathcal{O}(X)$ with $x \in U$, there exists an SI_2-open set W such that $y \in W \subseteq U$.*

Proof. Suppose that $x \ll_{\mathrm{I}_2} y, U \in \mathcal{O}(X)$ and $x \in U$. Then $y \in \Uparrow_{\mathrm{I}_2} x \subseteq U$. Consider the net $(y_j)_{j\in J}$ similarly defined in the proof of Proposition 3.17, where $J = \{(V, n, b) \in \mathcal{N}_{\mathrm{SI}_2}(y) \times \mathbb{N} \times X : b \in V\}$ with the lexicographic order on the first two coordinates and $y_{(V,n,b)} = b$ for any $(V, n, b) \in I$. Then $(y_j)_{j\in J} \xrightarrow{\mathrm{SI}_2} y$ (see the proof of Proposition 3.17). Hence, there exists an irreducible set M such that $y \in M^\delta$, and for any $O \in \mathcal{O}(X)$, $O \cap M \neq \emptyset$ implies

$y_j \in O$ eventually. By $x \ll_{\mathrm{I}_2} y$, we have $x \in \mathrm{cl}_X M$, and consequently, $U \cap M \neq \emptyset$. So $y_j \in U$ eventually, more precisely, there is $j_0 = (W, l, c) \in J$ such that $y_j \in U$ for all $j \geq j_0$. Then W is SI_2-open. For any $z \in W$, we have $(W, l+1, z) > (W, l, c)$, whence $z = y_{(W,l+1,z)} \in U$. So $y \in W \subseteq U$. □

Motivated by Propositions 3.17 and 3.18, we introduce the following concept.

Definition 3.19. A T_0-space X is called *strongly I_2-continuous* if the following two conditions hold:
(i) for any $x \in X$, $\Downarrow_{\mathrm{I}_2} x \in \mathsf{Irr}(X)$ and $x \in (\Downarrow_{\mathrm{I}_2} x)^{\delta}$ (i.e., X is I_2-continuous), and
(ii) for any $x, y \in X$ with $x \ll_{\mathrm{I}_2} y$ and $U \in \mathcal{O}(X)$ with $x \in U$, there exists an SI_2-open set W with $y \in W \subseteq U$.

Proposition 3.20. *Let X be an* I_2*-continuous space such that* $\Uparrow_{\mathrm{I}_2} x$ *is* SI_2*-open for all* $x \in X$*. Then X is a strongly* I_2*-continuous space.*

Proof. We only need to verify condition (ii) of Definition 3.19. Let $x, y \in X$ with $x \ll_{\mathrm{I}_2} y$ and $U \in \mathcal{N}(x)$. Then by the assumption $\Uparrow_{\mathrm{I}_2} x$ is SI_2-open. By Remark 3.8(i), we obtain $y \in \Uparrow_{\mathrm{I}_2} x \subseteq U$. Thus, X is strongly I_2-continuous. □

Proposition 3.21. *If X is an* SI_2*-continuous space, then X is strongly* I_2*-continuous.*

Proof. By Corollary 3.14, it is sufficient to verify condition (ii) of Definition 3.19. Let $x, y \in X$ with $x \ll_{\mathrm{I}_2} y$ and $U \in \mathcal{N}(x)$. Since X is SI_2-continuous, $\Downarrow_{\mathrm{SI}_2} y \in \mathsf{Irr}(X)$ and $y \in (\Downarrow_{\mathrm{SI}_2} y)^{\delta}$ (note that $y = \vee \Downarrow_{\mathrm{SI}_2} y$ is equivalent to $y \in (\Downarrow_{\mathrm{SI}_2} y)^{\delta}$). As $x \ll_{\mathrm{I}_2} y$, we have $x \in \mathrm{cl}_X \Downarrow_{\mathrm{SI}_2} y$, and hence, $\Downarrow_{\mathrm{SI}_2} y \cap U \neq \emptyset$ by $U \in \mathcal{N}(x)$. Select a point $z \in \Downarrow_{\mathrm{SI}_2} y \cap U$. Then $\Uparrow_{\mathrm{SI}_2} z \in \mathcal{O}_{\mathrm{SI}_2}(X)$ by Proposition 2.10 and $y \in \Uparrow_{\mathrm{SI}_2} z \subseteq U$. So X is strongly I_2-continuous. □

The converse of Proposition 3.21 may not be true, as shown in the following example.

Example 3.22. Let X_{cof} be the space in Example 3.15. Then by Example 3.15, we have the following conclusions:
(a) X_{cof} is an I_2-continuous T_1-space.
(b) X_{cof} is not SI_2-continuous.
(c) $\mathsf{Irr}(X_{\mathrm{cof}}) = \{\{x\} : x \in X\} \cup \{A : A$ is a countable infinite set of $X\}$.
(d) For any $s, t \in X$, $s \ll_{\mathrm{I}_2} t$ iff $s = t$.

Now we show that X_{cof} is strongly I_2-continuous. Suppose that $x \ll_{\mathrm{I}_2} y$ and $U \in \mathcal{N}(x)$. We first verify that U is SI_2-open. For $F \in \mathsf{Irr}(X_{\mathrm{cof}})$ with $F^{\delta} \cap U \neq \emptyset$, by (c) $F = \{z\}$ for some $z \in X$ or F is a countable infinite set of X. Then $F^{\delta} = \{z\}$ or $F^{\delta} = X$, and hence, $z \in F \cap U$ or $F \cap U \neq \emptyset$ by $|F| = \omega$ and U is an co-finite open set. So U is SI_2-open, and by (d), we have $y = x \in U \subseteq U$. Thus, X is strongly I_2-continuous.

Proposition 3.23. *If X is a strongly* I_2*-continuous space, then* SI_2*-convergence in X is topological.*

Proof. Let $(x_i)_{i\in I}$ be a net in X and $x \in X$. Obviously, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$ implies that $(x_i)_{i\in I}$ converges to x in $(X, \mathcal{O}(\mathcal{SI}_2(X)))$. Conversely, suppose that $(x_i)_{i\in I}$ converges to x in $(X, \mathcal{O}(\mathcal{SI}_2(X)))$. Then by Lemma 3.3, $(x_i)_{i\in I}$ converges to x with respect to the topology $\mathcal{O}_{\mathrm{SI}_2}(X)$. We will show that $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$. Let $F_x = \Downarrow_{\mathrm{I}_2} x$. Then by the strong I_2-continuity of X, we have that $F_x \in \mathsf{Irr}(X)$ and $x \in F_x^{\delta}$. For any $U \in \mathcal{O}(X)$, if $F_x \cap U \neq \emptyset$, then we can select a $u \in F_x \cap U$. Hence, $u \ll_{\mathrm{I}_2} x$ and $U \in \mathcal{N}(u)$. By the strong I_2-continuity of X again, there is an SI_2-open set W such that $x \in W \subseteq U$. Since $(x_i)_{i\in I}$ converges to x in $(X, \mathcal{O}_{\mathrm{SI}_2}(X))$, there is $i_0 \in I$ such that $x_i \in W \subseteq U$ for all $i \geq i_0$, proving that $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$. Thus, SI_2-convergence is topological. □

By Lemma 3.3, Propositions 3.17, 3.18, and 3.23, we obtain the main result of this article.

Theorem 3.24. *For a T_0-space, the following conditions are equivalent*:

(1) SI_2-*convergence X is topological.*

(2) *For any net $(x_i)_{i\in I}$ in X and $x \in X$, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$ iff $(x_i)_{i\in I}$ converges to x with respect to the* SI_2-*topology $O_{\mathrm{SI}_2}(X)$.*

(3) *X is strongly* I_2-*continuous.*

From Proposition 3.21 and Theorem 3.24 we directly deduce the following [8, Proposition 5.13].

Corollary 3.25. [8] *If X is an* SI_2-*continuous space, then the* SI_2-*convergence in X is topological.*

4 SI_2^*-continuous spaces

In this section, as a common generalization of the irr-convergence and the $\mathcal{S}$-convergence, we introduce the concept of SI_2^*-convergence in T_0-spaces and the related concept of SI_2^*-continuous spaces. Some basic properties of them are discussed. It is proved that if X is SI_2^*-continuous, then the SI_2^*-convergence in X is topological.

Definition 4.1. We say a net $(x_i)_{i\in I}$ SI_2^*-*converge* to a point x in a T_0-space X if there exists an irreducible set F in X such that

(i) $x \in F^\delta$, and

(ii) for each $e \in F$, $e \le x_i$ holds eventually.

In this case, we write $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$. Let $\mathcal{SI}_2^*(X) = \{((x_i)_{i\in I}, x) : (x_i)_{i\in I}$ is a net in $X, x \in X$ and $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x\}$.

Remark 4.2. For a T_0-space X a net $(x_i)_{i\in I}$ in X, we have the following statements:

(1) The constant net $(x)_{j\in J}$ in X with value x SI_2^*-converges to x.

(2) If $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$ in X, then $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} y$ for any $y \le x$. So the SI_2^*-convergence points of a net are generally not unique.

(3) $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$ implies $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$. In fact, if $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$, then there exists an irreducible set F of eventual lower bounds of $(x_i)_{i\in I}$ such that $x \in F^\delta$. For any $U \in O(X)$, if $F \cap U \neq \varnothing$, then we can select an $e \in F \cap U$. Hence, $e \le x_i$ holds eventually, and consequently, $x_i \in \uparrow U = U$ eventually. Thus, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$.

(4) Let P be a poset and $(s_j)_{j\in J}$ be a net in P. Then $(s_j)_{j\in J}$ SI_2^*-converges to s in $(P, \alpha(P))$ iff $(s_j)_{j\in J}\mathcal{S}$-converges to s iff $(s_j)_{j\in J}$ SI_2-converges to s by Lemma 2.3.

Definition 4.3. Let X be T_0-space. Then

$$O(\mathcal{SI}_2^*(X)) = \{U \subseteq X : \text{ whenever } (x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x \text{ and } x \in U, \text{ then eventually } x_i \in U\}$$

is a topology, called the SI_2^*-*topology* on X. A subset U of X is said to be SI_2^*-*open* if $U \in O(\mathcal{SI}_2^*)$. Complements of SI_2^*-open sets are called SI_2^*-*closed sets*.

Lemma 4.4. *Let X be T_0-space and $A \subseteq X$. Then the following two conditions are equivalent*:

(1) *A is* SI_2^*-*closed.*

(2) *For any net $(x_i)_{i\in I}$ in A, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$ implies $x \in A$.*

Proof. (1) ⇒ (2): Let $(x_i)_{i\in I}$ be a net in A and $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$. If $x \notin A$, then $x \in X\backslash A \in O(\mathcal{SI}_2^*(X))$. Hence, the net $(x_i)_{i\in I}$ must be eventually in $X\backslash A$, being a contradiction with the fact that $(x_i)_{i\in I}$ is in A. Thus, $x \in A$.

(2) ⇒ (1): We show that $X\backslash A$ is SI_2^*-open. Let $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$ and $x \in X\backslash A$. Then $x_i \in X\backslash A$ eventually. Otherwise, for each $i \in I$, there exists a $\varphi(i) \in I$ with $\varphi(i) \geq i$ such that $x_{\varphi(i)} \in A$. Let J be the subset of I consisting of all $j \in I$ such that $x_j \in A$. Then J is cofinal in I and $(x_j)_{j\in J}$ is a subnet of $(x_i)_{i\in I}$. As $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$, we have $(x_j)_{j\in J} \xrightarrow{\mathrm{SI}_2^*} x$, and hence, $x \in A$ by the assumption, which contradicts $x \in X\backslash A$. So $x_i \in X\backslash A$ eventually. Hence, $X\backslash A \in \mathcal{O}(\mathcal{SI}_2^*(X))$, that is, A is SI_2^*-closed. □

Remark 4.5. For a T_0-space X, we have the following statements:
(1) If $U \subseteq X$ is an SI_2-open set, then U is SI_2^*-open, that is, $\mathcal{O}_{\mathrm{SI}_2}(X) \subseteq \mathcal{O}(\mathcal{SI}_2^*(X))$.
(2) If $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$, then $(x_i)_{i\in I}$ converges to x with respect to the topology $\mathcal{O}_{\mathrm{SI}_2}(X)$.

Proof. (1) Let $U \in \mathcal{O}_{\mathrm{SI}_2}(X)$. Then $U \in \mathcal{O}(\mathcal{SI}_2(X))$ by Lemma 3.3. It follows from Remark 4.2(3) that $U \in \mathcal{O}(\mathcal{SI}_2^*(X))$.

(2) Suppose $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$. Then $(x_i)_{i\in I}$ converges to x in $(X, \mathcal{O}(\mathcal{SI}_2^*(X)))$. By (1), we have that $(x_i)_{i\in I}$ converges to x with respect to the topology $\mathcal{O}_{\mathrm{SI}_2}(X)$. □

The following example shows that for a T_0-space X, $\mathcal{O}_{\mathrm{SI}_2}(X)$ generally does not agree with $\mathcal{O}(\mathcal{SI}_2^*(X))$.

Example 4.6. Let X_{cof} be the space in Example 3.15. Then we have the following conclusions:
(a) X_{cof} is a T_1-space and hence the specialization order of X_{coc} is the discrete order.
(b) $\mathrm{Irr}(X_{\mathrm{cof}}) = \{\{x\} : x \in X\} \cup \{A : A$ is a countable infinite set of $X\}$.
(c) For any $x \in X$, $\{x\}$ is not open in X_{cof}, and hence, $\{x\} \notin \mathcal{O}_{\mathrm{SI}_2}(X)$.
(d) For any $x \in X$, $\{x\} \in \mathcal{O}(\mathcal{SI}_2^*(X))$.

Suppose $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$. Then there exist an $F \in \mathrm{Irr}(X_{\mathrm{cof}})$ such that conditions (i) and (ii) of Definition 4.1 hold. For any two points $e_1, e_2 \in F$, since F satisfies condition (ii) of Definition 4.1, there is $(i_1, i_2) \in I \times I$ such that $e_1 \leq x_i$ and $e_2 \leq x_j$ for any $i \geq i_1$ and $j \geq i_2$. As I is directed, there is $i_3 \in I$ such that $i_3 \in \uparrow i_1 \cap \uparrow i_2$. Then for any $i \geq i_3$, $e_1 = x_i = e_2$ (note that the specialization order of X_{coc} is the discrete order). Hence, F is a single point set. So $x \in F^\delta = F$ and $x_i \in \{x\}$ eventually. Thus, $\{x\} \in \mathcal{O}(\mathcal{SI}_2^*(X))$.

Now we give an example to show that for a T_0-space X, $\mathcal{O}(\mathcal{SI}_2^*(X))$ generally does not agree with $\mathcal{O}(X)$.

Example 4.7. Let $L = \mathbb{N} \cup \{\top\}$, where $\mathbb{N}$ is the set of all natural numbers $\mathbb{N} = \{1, 2, 3, \ldots, n, \ldots\}$, as a poset with the partial order defined by for any $n \in \mathbb{N}$, $n < n+1$ and $n < \top$. We consider the Alexandroff topological spaces $(L, \alpha(L))$. Obviously, $\{\top\} = \uparrow \top \in \alpha(L)$. For any $n \in \mathbb{N}$, let $x_n = n$. Set $F = \{n : n \in \mathbb{N}\}$. Then $F \in \mathrm{Irr}(L, \alpha(L))$, $\top \in L = F^\delta$, and for each $n \in F$, $n \leq x_m$ holds eventually. Thus, $(x_n)_{n\in\mathbb{N}} \xrightarrow{\mathrm{SI}_2^*} \top$. But $x_n \notin \{\top\}$ for any $n \in \mathbb{N}$. So $\{\top\} \notin \mathcal{O}(\mathcal{SI}_2^*(X))$.

Lemma 4.8. *Let X be an* SI_2*-continuous space,* $x \in X$ *and* $(x_i)_{i\in I}$ *be a net in X. Then the following three condition are equivalent*:
(1) $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$.
(2) $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$.
(3) $(x_i)_{i\in I}$ *converges to x with respect to the topology* $\mathcal{O}_{\mathrm{SI}_2}(X)$.

Proof. (1) ⇒ (2): By Remark 4.5.
(2) ⇔ (3): By Proposition 3.21 and Theorem 3.24.
(3) ⇒ (1): Let $F = \downarrow\!\!\downarrow_{\mathrm{SI}_2} x$. Then $F \in \mathrm{Irr}(X)$ and $x \in F^\delta$ by the SI_2-continuity of X. For any $e \in F$, by Proposition 2.10, we have $x \in \Uparrow_{\mathrm{SI}_2} e \in \mathcal{O}_{\mathrm{SI}_2}(X)$. As $(x_i)_{i\in I}$ converges to x in $(X, \mathcal{O}_{\mathrm{SI}_2}(X))$, $x_i \in \Uparrow_{\mathrm{SI}_2} e$ eventually. Since $\Uparrow_{\mathrm{SI}_2} e \subseteq \uparrow e$, we obtain that $x_i \geq e$ eventually. Thus, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$. □

By Lemmas 3.3 and 4.8, we obtain the following.

Corollary 4.9. *For any* SI_2*-continuous space* X, $O(\mathcal{SI}_2^*(X)) = O(\mathcal{SI}_2(X)) = O_{\mathrm{SI}_2}(X)$.

Lemma 4.10. *Let* X *be a* T_0*-space and* $x, y \in X$. *If* $x \ll_{\mathrm{SI}_2} y$, *then for any net* $(x_i)_{i\in I}$ *in* X, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} y$ *implies* $x_i \geq x$ *eventually.*

Proof. Suppose that $x \ll_{\mathrm{SI}_2} y$ and $(x_i)_{i\in I}$ is a net of X SI_2^*-converging to y. We show that $x \leq x_i$ holds eventually. Since $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} y$, there is an $F \in \mathsf{Irr}(X)$ such that $y \in F^\delta$, and for each $e \in F$, $e \leq x_i$ eventually. By $x \ll_{\mathrm{SI}_2} y$ and $y \in F^\delta$, we have $x \in {\downarrow} F$, and hence, there is $e_x \in F$ such that $x \leq e_x$. Consequently, $x \leq x_i$ eventually. □

Proposition 4.11. *Let* X *be a* T_0*-space,* $y \in X$ *and* $(x_i)_{i\in I}$ *be a net in X. Consider the following two conditions*:

(1) $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} y$.
(2) *For any* $x \ll_{\mathrm{SI}_2} y$, $x_i \geq x$ *eventually.*

Then (1) ⇒ (2), and two conditions are equivalent if X *is* SI_2*-continuous.*

Proof. (1) ⇒ (2): By Lemma 4.10.

(2) ⇒ (1): Suppose that X is SI_2-continuous. Let $F = \Downarrow_{\mathrm{SI}_2} y$. Then by the SI_2-continuity of X, $F \in \mathsf{Irr}(X)$ and $y \in F^\delta$. For any $e \in F$, $x_i \geq e$ eventually by the assumption. Thus, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} y$. □

Definition 4.12. A T_0-space X is called SI_2^**-continuous* if for every $x \in X$, the following two conditions hold:

(1) $\Uparrow_{\mathrm{SI}_2} x$ is an SI_2^*-open set in X.
(2) $\Downarrow_{\mathrm{SI}_2} x$ is irreducible and $x = \vee \Downarrow_{\mathrm{SI}_2} x$ (equivalently, $x \in (\Downarrow_{\mathrm{SI}_2} x)^\delta$).

Theorem 4.13. *If* X *is an* SI_2^**-continuous space, then* SI_2^**-convergence in* X *is topological.*

Proof. Suppose that $(x_i)_{i\in I}$ converging to x in $(X, O(\mathcal{SI}_2^*(X)))$. We need to show $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$. Let $F = \Downarrow_{\mathrm{SI}_2} x$. Then $F \in \mathsf{Irr}(X)$ and $x \in F^\delta$ by the SI_2^*-continuity of X. For any $e \in F$, by the SI_2^*-continuity of X again, we have $x \in \Uparrow_{\mathrm{SI}_2} e \in O(\mathcal{SI}_2^*(X))$. As $(x_i)_{i\in I}$ converges to x in $(X, O(\mathcal{SI}_2^*(X)))$, $x_i \in \Uparrow_{\mathrm{SI}_2} e$ eventually. Since $\Uparrow_{\mathrm{SI}_2} e \subseteq {\uparrow} e$, we obtain that $x_i \geq e$ eventually. Thus, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$. □

But we do not know whether the converse of Theorem 4.13 is true. So naturally we asks the following question.

Question 4.14. Characterize those T_0-spaces X for which the SI_2^*-convergence in X is topological.

Theorem 4.15. *For a* T_0*-space X, the following conditions are equivalent*:

(1) X *is* SI_2*-continuous.*
(2) X *is* SI_2^**-continuous and* $O(\mathcal{SI}_2^*(X)) = O_{\mathrm{SI}_2}(X)$.
(3) X *is* SI_2^**-continuous, and for any net* $(x_i)_{i\in I}$ in X *and* $x \in X$, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$ *iff* $(x_i)_{i\in I}$ *converges to x with respect to the topology* $O_{\mathrm{SI}_2}(X)$.
(4) X *is* SI_2^**-continuous, and for any net* $(x_i)_{i\in I}$ *in* X *and* $x \in X$, $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$ *iff* $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$.

Proof. (1) ⇒ (2): Suppose that X is SI_2-continuous. Then by Proposition 2.10 and Remark 4.5 (1), X is SI_2^*-continuous. By Corollary 4.9, $O(\mathcal{SI}_2^*(X)) = O(\mathcal{SI}_2(X)) = O_{\mathrm{SI}_2}(X)$.

(2) ⇒ (1): Trivial.

(2) ⇒ (3): If $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$, then $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$ by Remark 4.5(2), and hence, $(x_i)_{i\in I}$ converges to x with respect to the topology $\mathcal{O}_{\mathrm{SI}_2}(X)$ by Lemma 3.3. Conversely, if $(x_i)_{i\in I}$ converges to x with respect to the topology $\mathcal{O}_{\mathrm{SI}_2}(X)$, then it converges to x with respect to the topology $\mathcal{O}_{\mathrm{SI}_2^*}(X)$ by $O(\mathcal{SI}_2^*(X)) = \mathcal{O}_{\mathrm{SI}_2}(X)$. Then by Theorem 4.13, we obtain $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$.

(3) ⇒ (4): Suppose $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$. Then $(x_i)_{i\in I}$ converges to x with respect to the topology $O(\mathcal{SI}_2^*(X))$. It follows from Remark 4.5(1) that $(x_i)_{i\in I}$ converges to x with respect to the topology $\mathcal{O}_{\mathrm{SI}_2}(X)$. Conversely, if $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2} x$, then $(x_i)_{i\in I}$ converges to x with respect to the topology $\mathcal{O}_{\mathrm{SI}_2}(X)$ by Lemma 3.3. By (3), we obtain $(x_i)_{i\in I} \xrightarrow{\mathrm{SI}_2^*} x$.

(4) ⇒ (2): By (4) and Lemma 3.3, we have that $O(\mathcal{SI}_2^*(X)) = O(\mathcal{SI}_2(X)) = \mathcal{O}_{\mathrm{SI}_2}(X)$. □

By Theorems 4.13 and 4.15, we obtain the following corollary.

Corollary 4.16. *If X is an SI_2-continuous space, then SI_2^*-convergence in X is topological.*

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