DE GRUYTER Open Mathematics 2025; 23: 20250153

Research Article

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Classifying cubic symmetric graphs of order $88p$ and $88p^2$

https://doi.org/10.1515/math-2025-0153
received June 13, 2024; accepted April 10, 2025

Abstract: For a simple graph Γ, Γ is said to be s-regular, provided that the automorphism group of Γ regularly acts on the set consisting of s-arcs of Γ. Given a positive integer n, the question on finding all s-regular graphs of order n and degree 3 has received considerable attention. An s-regular graph with degree 3 is so-called a cubic symmetric graph. Let p be a prime. We show that if Γ is a cubic symmetric graph of order $88p$, then $p \in \{5, 11, 23\}$; if Γ is a cubic symmetric graph of order $88p^2$, then $p = 11$. Moreover, we classify all cubic symmetric graphs of order $88p$ and $88p^2$.

Keywords: cubic symmetric graph, simple group, simple group

MSC 2020: 05C25

1 Introduction

In the short note, it is assumed that every graph being discussed is finite and connected. For a given graph, say X, $V(X)$ always denotes the vertex set of X, and $E(X)$ denotes the edge set of X. In this graph X, the ordered $(s+1)$-tuple $(w_0, w_1, \ldots, w_s)$, where $\{w_0, w_1, \ldots, w_s\} \subseteq V(X)$ that satisfies what w_{i-1} is adjacent to w_i for any $1 \le i \le s$ and $w_{i-1} \ne w_{i+1}$ for every $1 \le i < s$ is called an *s-arc*. For given graph X, the automorphism group of X is denoted by $\mathrm{Aut}(X)$. If $\mathrm{Aut}(X)$ acting on the set consisting of all s-arcs of X is transitive, then X is said to be *s-arc-transitive*. Particularly, 0-arc-transitive is also said to be *vertex-transitive*. Also, a 1-arc-transitive graph is said to be *arc-transitive*, and sometimes, we also called it as a *symmetric graph*.

Assume that G is a permutation group. G that acts on the set Ω is called *semiregular*, provided that no nontrivial element of G can fix a point of Ω. Thus, from orbit-stabilizer theorem, it follows that if G is a semi-regular group, then the length of any orbit is $|G|$. In particular, if G is transitive and semi-regular, then G is said to be a *regular group*. Recall that X is graph. If a subgroup of $\mathrm{Aut}(X)$ acts regularly on the set consisting of all s-arcs of X, then this subgroup is called a *s-regular subgroup*. In particular, if we take the subgroup which is $\mathrm{Aut}(X)$, then the given graph X is called a *s-regular graph*. Given a group G and a subset S of G, if $S^{-1} \coloneqq \{s^{-1} : s \in S\} = S$, then S is called a *inverse-closed* subset. Now, suppose that e is the identity of G, and S is a subset of G and is inverse-closed with $e \notin S$. The *Cayley graph*, denoted by $\mathrm{Cay}(G, S)$, is a simple graph with vertex set G where two distinct a, b are adjacent, provided that $ba^{-1} \in S$.

Suppose that N is a subgroup of $\mathrm{Aut}(X)$ where X is a graph. The *quotient graph* with N, denoted by X_N, is a graph whose vertex set is the set of all orbits of N, and two distinct orbits A and B are adjacent in X_N if and only if in graph X, there is an edge whose one vertex is in A and another vertex in B. Recall that in a graph, $N(v)$ is the set consisting of all vertices adjacent to this vertex v. For two graphs $\tilde{X}$ and X, $\tilde{X}$ is said to be a *covering* of X (with a mapping, say ρ, from $\tilde{X}$ to X, provided that this mapping ρ is surjective from $V(\tilde{X})$ to $V(X)$ satisfying that $\rho\,|_{N(\tilde{v})}$ from $N(\tilde{v})$ to $N(v)$ is bijective for each two vertices $v \in V(X)$ and $\tilde{v} \in \rho^{-1}(v)$.

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Assume that $\rho : \tilde{X} \to X$ is a covering. If this group $\mathrm{Aut}(\tilde{X})$ has a semi-regular subgroup, say N, so that $X \cong \tilde{X}_N$, then ρ is said to be a *regular covering*.

The next result from [1, Theorem 9] will be used frequently in this article.

Proposition 1.1. *Suppose that X is cubic and symmetric. For* $s \geq 1$, *suppose that G is an s-regular subgroup of* $\mathrm{Aut}(X)$. *If G has a normal subgroup, say N, which has more than two orbits acting on* $V(X)$, *then N is semi-regular and this quotient group* G/N *is s-regular subgroup of* $\mathrm{Aut}(X_N)$. *In particular, in this case, X is a regular covering of* X_N.

In [2,3], Tutte proved that any cubic symmetric graph is also an s-regular graph, where $1 \leq s \leq 5$. All cubic symmetric graphs with order $2p$ were classified by Cheng and Oxley [4]. Since then, for a given number n, the classifying cubic symmetric graphs of order n became an interesting research topic, see [5–9].

We use $\mathbb{Z}_n$ to denote a cyclic group of order n. Let $G_1 = \mathbb{Z}_2^2 \ltimes (\mathbb{Z}_{11}^3 \times \mathbb{Z}_2)$ and $G_2 = \mathbb{Z}_2 \ltimes ((\mathbb{Z}_{11} \ltimes \mathbb{Z}_{11}^2) \times \mathbb{Z}_2^2)$. Write

$$\begin{aligned} \mathcal{G}_1 &\coloneqq \mathrm{Cay}(G_1, \{(1,0,-1,0,0,1), (0,1,0,-1,0,1), (1,1,0,0,-1,1)\}), \\ \mathcal{G}_2 &\coloneqq \mathrm{Cay}(G_2, \{(1,0,0,0,0,0), (1,0,1,0,1,0), (1,1,0,0,0,1)\}). \end{aligned} \tag{1}$$

In 2014, Feng et al. [5] proved that both $\mathcal{G}_1$ and $\mathcal{G}_2$ are cubic 2-regular graphs of order 10648.

In 2011, with the help of a computer, Conder obtained all cubic symmetric graphs of order at most 10000 and uploaded the results on the website [10]. With the help of [10], this article classifies the cubic symmetric graphs with order $88p$ and $88p^2$, where p is a prime. The main results of our article are the following:

Theorem 1.2. *Suppose that p is a prime and X is a cubic symmetric graph with order* $88p$. *Then, X is isomorphic to one of these graphs in* Table 1 *from* [10].

Theorem 1.3. *Assume that p is a prime and X is a cubic symmetric graph with order* $88p^2$. *Then, X is isomorphic to either* $\mathcal{G}_1$ *or* $\mathcal{G}_2$ *in* (1).

We will prove Theorems 1.2 and 1.3 in Sections 2 and 3, respectively.

Table 1: All cubic symmetric graphs of order $88p$

Graph	Order	$\lvert\mathrm{Aut}(X)\rvert$	Girth	Diameter	s-Regular
C440.1	$88 \cdot 5$	2640	10	12	2
C440.2	$88 \cdot 5$	2640	10	11	2
C440.3	$88 \cdot 5$	5280	12	10	3
C968.1	$88 \cdot 11$	5808	6	29	2
C2024.1	$88 \cdot 23$	12144	14	13	2
C2024.2	$88 \cdot 23$	12144	11	14	2
C2024.3	$88 \cdot 23$	12144	14	13	2
C2024.4	$88 \cdot 23$	12144	12	14	2
C2024.5	$88 \cdot 23$	12144	8	15	2
C2024.6	$88 \cdot 23$	12144	12	14	2
C2024.7	$88 \cdot 23$	12144	11	14	2
C2024.8	$88 \cdot 23$	12144	8	14	2
C2024.9	$88 \cdot 23$	12144	11	12	2
C2024.10	$88 \cdot 23$	12144	11	15	2
C2024.11	$88 \cdot 23$	12144	12	13	2
C2024.12	$88 \cdot 23$	12144	12	12	2
C2024.13	$88 \cdot 23$	24288	16	16	3

2 Proof of Theorem 1.2

In the following, we always assume that graph X is cubic and symmetric. Recall that Aut(X) acting on this set consisting of all s-arcs of X is transitive, where $1 \le s \le 5$. Note that the size of the set consisting of all s-arcs of X is equal to $2^{s-1} \cdot 3 \cdot |V(\Gamma)|$, we have

$$|\mathrm{Aut}(X)| = 2^{s-1} \cdot 3 \cdot |V(X)|. \tag{2}$$

In the following, we first prove Theorem 1.2.

Proof of Theorem 1.2. Note that p is a prime. If $p < 127$, then X has order at most 10000, and so by Conder [10], X is isomorphic to one of these graphs in Table 1, as desired.

Suppose next that $p \ge 127$. In order to show Theorem 1.2, it suffices to prove that such a graph X does not exist. Suppose, for a contradiction, that X is a cubic symmetric graph with order $88p$. Note that it follows from (2) that we have

$$|\mathrm{Aut}(X)| = 2^{s+2} \cdot 3 \cdot 11 \cdot p, \quad 1 \le s \le 5.$$

Let $A = \mathrm{Aut}(X)$ and N be a minimal normal subgroup of A. We shall divide our proof into four steps.

Step 1. A has no normal p-subgroups.

Suppose that P is a normal p-subgroup of A. Then, $|P| = p$, which implies that P has more than two orbits on $V(X)$. By Proposition 1.1, we have that X_P is a cubic symmetric graph of order 88, this is in contradiction to [10].

Step 2. A has no normal 2-subgroups.

Suppose that H is a normal 2-subgroup of A. By Proposition 1.1, X_N is a cubic symmetric graph, and H on $V(X)$ is semiregular. As a consequence, $|H|$ is a divisor of $|X|$, and therefore, we have $|H| = 2$, 4, or 8. If $|H| = 2$, then X_H is of order $44p$, which is impossible by [8, Theorem 2.4]. If $|H| = 4$, then X_H is of order $22p$, which is impossible by [9, Theorem 3.3]. If $|H| = 8$, then $|V(X_H)|$ is odd, which is impossible.

Step 3. N is solvable.

Suppose that N is non-solvable. Then, N is a direct product of several copies of a non-abelian simple group T. Observe that any prime factor of $|T|$ belongs to $\{2, 3, 11, p\}$. If $p > 2^7 \cdot 3 \cdot 11$, then A has a normal p-subgroup, it obtains a contradiction by Step 1. Therefore, $|T| \le |A| < 10^{25}$. By [11, pp. 239–242], the simple group T does not exist, it obtains a contradiction.

Step 4. Final contradiction.

By Step 3, we see that N must be an elementary abelian r-group where r is prime. It follows from Proposition 1.1 that N acting on $V(X)$ must be semi-regular, which implies that $|N|$ is a divisor of $|X|$. Therefore, $N \cong \mathbb{Z}_{11}$ by Steps 1 and 2. Let J/N be a minimal normal subgroup of A/N. Since $|A/N| = 2^{s+2} \cdot 3 \cdot p$, it is similar to Step 3, one has that J/N is elementary abelian. Since J/N is semi-regular on $V(X_N)$, $8p$ is divided by $|J/N|$. It follows that $J/N \cong \mathbb{Z}_p$ or J/N is a 2-group. If $J/N \cong \mathbb{Z}_p$, then $|J| = 11p$ and so J has a normal Sylow p-subgroup P. Since P is characteristic in J and J is normalized by A, P is normal in A, contrary to Step 1. So, J/N is a 2-group. If $|J/N| = 8$, then X_J is a cubic graph on p vertices, which is impossible. So, $|J/N| = 2$ or 4. If $|J/N| = 2$, then X_J is a cubic symmetric graph of order $4p$, contradicting [6, Theorem 6.2]. Now, suppose $|J/N| = 4$. Then, X_J is a cubic symmetric graph of order $2p$. Let K/J be a minimal normal subgroup of A/J. It is similar to Step 3, K/J is elementary abelian. Then, $|K/J| = p$ and $|K| = 44p$. Also, by the proof of Case 1, we have a contradiction.

By the above discussion, we end the proof of Theorem 1.2. □

3 Proof of Theorem 1.3

Given a group G, let H be a subgroup of G. The center of G is denoted by $Z(G)$. We use G' to denote the derived subgroup of G. The symbol $[G : H]$ denotes the index of subgroup H in G, which is equal to the number of all right (or left) cosets of H of G. Two integers a and b are said to be *coprime* if the only positive factor that divides both of them is 1, and we denote it by $(a, b) = 1$.

Lemma 3.1. *Suppose that G is a group. Let H be a subgroup of G with* $([G : H], [H : H']) = 1$. *Then*

$$H \cap G' \cap Z(G) \subseteq H'.$$

Proof. Take $x \in H \cap G' \cap Z(G)$. According to the transfer from G to H/H' (cf. [12, Chapter 10]), one has $x^{[G:H]} \in H'$. Further, since $xH' \in H/H'$, it follows that $(xH')^{[H:H']} = H'$, which also implies $x^{[H:H']} \in H'$. Now in view of

$$([G : H], [H : H']) = 1,$$

we have $x \in H'$, as desired. □

In group G, the largest normal p-subgroup is denoted by $O_p(G)$.

Lemma 3.2. *Suppose that p is a prime at least* 13. *There is no cubic symmetric graph with order* $44p^2$.

Proof. Suppose, to the contrary, that there exists a cubic symmetric graph, say X, which has order $44p^2$. We write $A = \mathrm{Aut}(X)$. Assume that P is a Sylow p-subgroup of A. Then

$$|A| = 2^{s+1} \cdot 3 \cdot 11 \cdot p^2, \quad 1 \le s \le 5.$$

In view of [10], one has that P is not normal in A.

Since the number of orbits of $O_2(A)$ acting on $V(X)$ is greater than 2, $44p^2$ is divisible by $|O_2(A)|$. It follows that $|O_2(A)| = 1$, 2, or 4. If $|O_2(A)| = 2$, one has $|X_{O_2(A)}| = 22p^2$, which is a contradiction by [9, Theorem 3.4]. If $|O_2(A)| = 4$, then $X_{O_2(A)}$ is a cubic graph with odd order, which is a contradiction. Therefore, $O_2(A) = 1$. Similarly, by [6, Theorem 6.2], we obtain $O_{11}(A) = 1$.

Suppose, now, that T is a minimal normal subgroup of A. If T is nonsolvable, then, by [11, pp. 239–242], one has $T \cong \mathrm{PSL}(2, 23)$ or $\mathrm{PSL}(2, 23)$. Since T on $V(X)$ has more than two orbits, $|X|$ is divisible by $|T|$, which is impossible. As a result, T is an elementary abelian group, which implies that $T \cong \mathbb{Z}_p$.

Now note that X_T is cubic and symmetric, which has order $44p$. In view of [8, Theorem 2.4], one has that X_T is 2- or 3-regular and $p = 23$. Thus, A is at most 3-regular. As a consequence, $|A|$ is a divisor of $2^4 \cdot 3 \cdot 11 \cdot p^2$. Hence, A has $p + 1$ Sylow p-subgroups.

Suppose that N is the normalizer of P in A. Now, let A act on the set of all right cosets of N of A, by right multiplication. Then, A/N_A can be imbedded in the symmetric group on $p + 1$ letters, where N_A is the largest normal subgroup of A contained in N. It means that $|A/N_A|$ is a divisor of $(p + 1)!$. Since $p^2 \| |A|$, one has $p \| |N_A|$. If $p^2 \| |N_A|$, the fact that $|A/N| = 24$ implies that $|N_A| \le |N| \le 2 \cdot 11 \cdot p^2$. Therefore, N_A has a characteristic Sylow p-subgroup of order p^2. Since N_A is normalized by A, one has that P is normal in A, which is a contradiction. Thus, $|N_A|$ is not divisible by p^2. This forces that the number of orbits under the action N_A on $V(X)$ is greater than 2. Proposition 1.1 implies that $|N_A|$ is a divisor of $22p$.

Let K be a Sylow p-subgroup in N_A. It follows that K is normal in A. Also, in A, we say that C is the centralizer of K. By, N/C theorem (see, for example, [12, Theorem 1.6.13]), we know that $N_A(K)/C = A/C$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$. Since $K \cong \mathbb{Z}_p$, we have $\mathrm{Aut}(K) \cong \mathbb{Z}_{p-1}$, and so $|A/C|$ is a divisor of $p - 1$. So, $p^2 \| |C|$. It is straightforward that $C' \cap K = K$ or 1. If $C' \cap K = K$, then $K \le C'$. Since $K \le Z(C)$, $p \| |C' \cap Z(C)|$. Let P_1 be a Sylow p-subgroup of C. Then, $p \| |P_1 \cap C' \cap Z(C)|$. However, by Lemma 3.1, $P_1 \cap C' \cap Z(C) = 1$, which is a contradiction. Thus, $C' \cap K = 1$ and so C' is not divisible by p^2. It follows that C' is semiregular. As a result, $|C'| \| 44p$. Now, suppose that H/C' is a Sylow p-subgroup of C/C'. By $p^2 \| |C|$, one has $p^2 \| |H|$. It follows that $|H| \| 44p^2$. Thus, H has a normal Sylow p-subgroup. In view of the commutativity of C/C', one has that P is normal in A, which is impossible. □

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. If $p \leq 7$, in view of [10], there is no cubic symmetric graph with order $88p^2$. If $p = 11$, in view of [5, Theorem 6.1], X is isomorphic to $\mathcal{G}_1$ or $\mathcal{G}_2$.

Now, suppose that $p \geq 13$. Write $A = \mathrm{Aut}(X)$. Let P be a Sylow p-subgroup of A. Then, $|A| = 2^{s+2} \cdot 3 \cdot 11 \cdot p^2$, where $1 \leq s \leq 5$. Note that there is no cubic symmetric graph with order 88. It follows that P is non-normal in A.

Suppose $O_2(A) \neq 1$. Then, $|O_2(A)| = 2$ or 4. Thus, $X_{O_2(A)}$ is a cubic symmetric graph of order $44p^2$ or $22p^2$, contradicting Lemma 3.2 or [9, Theorem 3.4], respectively. Hence, $O_2(A) = 1$.

Suppose $O_{11}(A) \cong \mathbb{Z}_{11}$. Then, $X_{O_{11}(A)}$ is cubic and symmetric, which has order $8p^2$. In view of [7, Theorem 5.2], we have that $X_{O_{11}}(A)$ is either cyclic or elementary abelian cover of the hypercube. It means that $A/O_{11}(A)$ has a normal Sylow p-subgroup, say $M/O_{11}(A)$. Thus, $|M| = 11p^2$. Since M is normal in A, P is normal in A, which is a contradiction. As a result, $O_{11}(A) = 1$.

Now, let N be a minimal normal subgroup of A. If N is nonsolvable, in view of [11, p. 239], $N \cong \mathrm{PSL}(2, 23)$ or $\mathrm{PSL}(2, 32)$. Now, by Proposition 1.1, we have that $|X|$ must be divisible by $|N|$, which is impossible. As a result, N must be an elementary abelian group. As mentioned in the previous paragraphs, one has $N \cong \mathbb{Z}_p$. Thus, X_N is a cubic symmetric graph with order $88p$. In view of Theorem 1.2, one has $p = 23$ and X_N is 2- or 3-regular. Thus, A is at most 3-regular. Therefore, $|A|$ is a divisor of $2^5 \cdot 3 \cdot 11 \cdot p^2$. This forces that A has $p + 1$ Sylow p-subgroups. Now, it is similar to the last two paragraphs of the proof of Lemma 3.2, we can also obtain a contradiction.

Based on the discussion, we complete the proof of Theorem 1.3. □

4 Conclusions

For a positive integer n, the question on classifying s-regular graphs of order n and degree 3 has received considerable attention. A s-regular graph with degree 3 is so-called a cubic symmetric graph. It was proved that every cubic symmetric graph is also a s-regular graph, where $1 \leq s \leq 5$. For some prime p and a graph Γ, if Γ is a cubic symmetric graph of order $88p$, this article showed that $p \in \{5, 11, 23\}$. Moreover, if Γ is a cubic symmetric graph of order $88p^2$, this article showed that $p = 11$. In fact, this article classified all cubic symmetric graphs of order $88p$ and $88p^2$ for each prime p.

Acknowledgment: The author is deeply grateful to the reviewers for their valuable comments that improved the manuscript.

Funding information: Author states no funding involved.

Author contributions: The author confirms the sole responsibility for the conception of the study, presented results, and manuscript preparation.

Conflict of interest: The author declares that there are no conflicts of interest regarding the publication of this article.

Data availability statement: No datasets were generated or analyzed during the current study.

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