Research Article

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Generalized quandle polynomials and their applications to stuquandles, stuck links, and RNA folding

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Abstract: We introduce a generalization of the quandle polynomial. We prove that our polynomial is an invariant of stuquandles. Furthermore, we use the invariant of stuquandles to define a polynomial invariant of stuck links. As a byproduct, we obtain a polynomial invariant of RNA foldings. Finally, we provide explicit computations of our polynomial invariant for both stuck links and RNA foldings.

Keywords: quandles, quandle polynomial, stuquandles, stuquandle polynomial, stuck links, arc diagrams, RNA folding

MSC 2020: 57K12

1 Introduction

Stuck links can be considered a generalization of singular links. They were introduced in [1]. Stuck links are physical links where the strands are stuck together in a fixed position, with one strand above the other. As a result, their projections have two types of crossings: the classical crossings from classical knot theory and a different type of crossing called a stuck crossing. Stuck knots and links have applications to RNA folding through a transformation relating stuck link diagrams to arc diagrams of an RNA folding, see [1–3]. Additionally, the use of this generalization of classical knot theory is uniquely equipped to model both the entanglement and intra-chain interactions of a biomolecule as described in [1].

In [2], a generating set of the oriented stuck Reidemeister moves for oriented stuck links was introduced. The generating set of oriented stuck Reidemeister moves was used to define an algebraic structure called *stuquandle*. The motivation of the stuquandle algebraic structure was to axiomatize the oriented stuck Reidemeister moves, thus allowing the construction of the fundamental stuquandle associated with a given stuck link. Using the fundamental quandle, the coloring counting invariant of stuck links was defined. As a consequence, the coloring counting invariant for arc diagrams of RNA foldings was constructed through the use of stuck link diagrams. The coloring counting invariant of stuck links is defined as the cardinality of the set of homomorphisms from the fundamental stuquandle to a finite stuquandle. Although the stuquandle counting invariant is a useful invariant of stuck links, it is not strong enough, and thus, we define an enhancement of it

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Figure 1: (Left) Singular crossing in a singular link, and (right) singular crossing in a singular link diagram.

in this article. In the case of racks and quandles, the study of enhancements of the counting invariant is a very active area of research, see [4–7].

Specifically, in [8], a two-variable polynomial from finite quandles encodes a set with multiplicities arising from counting trivial actions of elements on other elements of the quandle. This polynomial was used to define a polynomial invariant of classical links and was shown to be an enhancement of the quandle coloring counting invariant. Additionally, the quandle polynomial invariant was extended to the case of singular knots in [9]. In this article, we generalize these polynomials to the case of stuck links. We then use these polynomials to define an enhancement of the coloring counting invariant of stuck links and of RNA foldings. Our approach in this article is different from the enhancement of the stuquandle counting invariant in [3], which was achieved by assigning Boltzmann weights at both classical and stuck crossings and thus leading to a single-variable, a two-variable, and a three-variable polynomial invariant of stuck links and applied to arc diagrams of RNA foldings.

This article is organized as follows. In Section 2, we review the basics of stuck knots and their diagrammatics. In Section 3, we recall the relationship between stuck links and arc diagrams. Specifically, we review the transformation to obtain a stuck link diagram from an arc diagram and vice versa. In Section 4, we discuss the algebraic structures motivated by the diagrammatic representation of stuck knots and the fundamental stuquandle, leading to the stuquandle counting invariant. Section 5 reviews the definition of the quandle polynomial, the subquandle polynomial, and the link invariants obtained from the subquandle polynomial. A generalization of the quandle polynomial is introduced in Section 6. We end this section by proving that this generalization is an invariant of stuquandles and then use the generalization to define a polynomial invariant of stuck links. Finally, in Section 7, we provide explicit computations of our invariants for both stuck links and RNA foldings. In the case of RNA foldings, we give an example of two arc diagrams that are not distinguished by the stuquandle counting invariant but are distinguished by the substuquandle polynomial invariant.

2 Review of stuck knots and links

In [1], a generalization of singular knots and links was introduced. In this article, we will follow the definitions and conventions established in that paper. Similar to the case of the theory of classical and singular links, one may consider diagrams when studying stuck links. A stuck link diagram may include classical and stuck crossings. A stuck crossing is a singular crossing with additional information about the stuck position. Figure 1 shows a singular crossing, while Figure 2 illustrates the two types of stuck crossings.

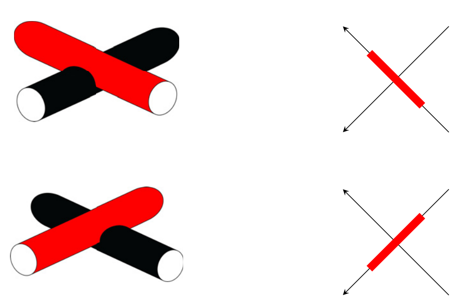


Figure 2: (Left) Stuck crossings in a stuck link, and (right) stuck crossings in a stuck link diagram.

To specify the stuck information in a diagram at a stuck crossing, we will use a thick bar on the over arc at a stuck crossing (Figure 2). The top crossings in Figure 2 are called positive stuck crossings, while the bottom crossings are negative stuck crossings.

In [2], the set of moves shown in Figure 3 was identified as a generating set of oriented stuck Reidemeister moves. We will follow the naming convention established in that paper. The oriented stuck Reidemeister moves are essential for studying stuck links through the use of stuck link diagrams. Specifically, two stuck link diagrams are equivalent if and only if one diagram can be transformed into the other by a finite sequence of planar isotopies and the moves in Figure 3. A *stuck link* is defined as an equivalence class of stuck link diagrams modulo the oriented stuck Reidemeister moves.

3 Stuck links and arc diagrams

In this section, we review arc diagrams and the relationship between stuck links and arc diagrams. Specifically, stuck links provide a way of studying the topology of RNA folding, as discussed in [1–3].

In [10], Kauffmann and Magarshak introduced arc diagrams as a combinatorial way of studying the topology of RNA folding. Arc diagrams, as noted in [10], were motivated by the fact that the RNA molecule is a long chain consisting of the bases A (adenine), C (cytosine), U (uracil), and G (guanine). In an RNA molecule, the pairs A–U and C–G can bond with each other. Therefore, an RNA molecule can be represented by a linear sequence of the letters A, C, U, and G, and folding the molecule involves pairing the bases in the sequence. Example 3.1 is taken from [10].

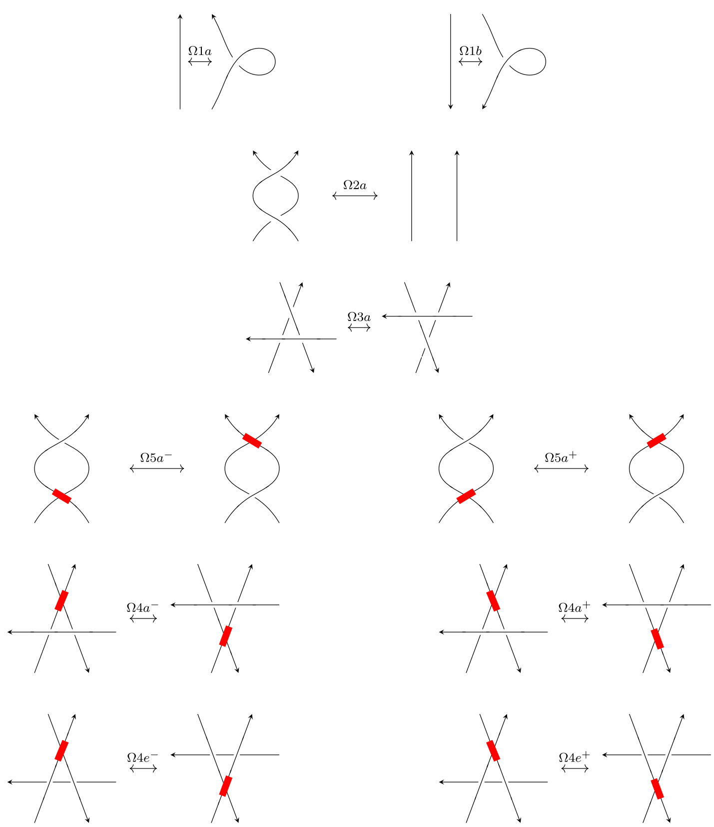


Figure 3: Generating set of oriented stuck Reidemeister moves.

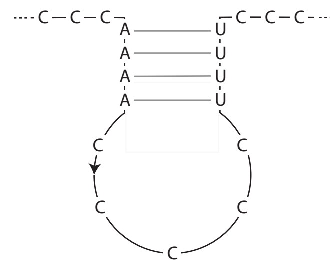


Figure 4: Folding of the sequence after the A-U pairing.

Example 3.1. In this example, we will see how to obtain an arc diagram from the description of RNA folding as a linear sequence with pairings A–U and C–G. In this case, consider the chain ··· CCCAAAACCCCCUUUUCCC ··· . From this linear sequence, the pairing of A–U and C–G will produce the folding in Figure 4. Furthermore, the folding in Figure 4 can be simplified and represented by the diagram in Figure 5.

The diagram in Figure 5 is called an *arc diagram* introduced in [10]. We note that in the original formulation of an arc diagram, the bonds were denoted with connecting arcs. To see examples of these arc diagrams, please refer to [10]. In this article, we will use the convention introduced in [1] and used in [2,7] by replacing the connecting arcs with just one solid gray stripe, as shown in Figure 5. We also note that in order to study the topology of RNA folding in three-dimensional space through the use of an arc diagram, a set of Reidemeister-type moves was introduced in [10]. Therefore, a specific RNA folding is an equivalence class of arc diagrams modulo the Reidemeister-type moves. For more information on the Reidemeister-type moves allowed on an arc diagram and the theory of arc diagrams, please refer to [10].

The following transformation was defined in [1] and formalizes the connection between stuck links and arc diagrams. To obtain a stuck link diagram from an arc diagram, we can apply the transformation, T (Figure 6). We note that the transformation, T, may also be used to obtain an arc diagram from a stuck link diagram; see [1] for examples.

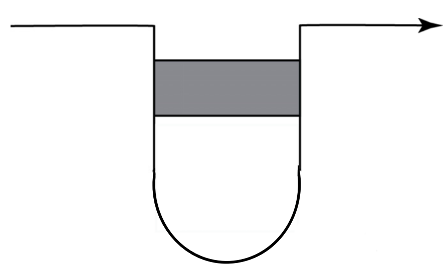


Figure 5: Arc diagram of the RNA folding in Figure 4.

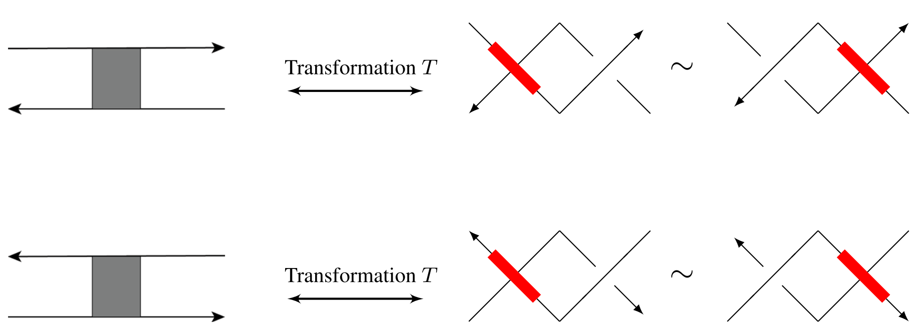


Figure 6: Transformation T.

In the following example, we will consider an arc diagram of an RNA folding and apply the transformation T to obtain a stuck link diagram. Since the endpoints of an arc diagram can be connected in several ways, we will use the convention from [1,11] of self-closure. Specifically, the self-closure of the arc diagram, will mean that each strand of the arc diagram is connected to itself.

Example 3.2. Consider the arc diagram of an RNA folding with two strands in Figure 7. In this case, the arc diagram contains two strands, so the self-closure means that we connect the endpoints of one strand to each other and the endpoints of the other strand to each other. Next, we replace the gray stripe by applying T to obtain the corresponding stuck link diagram; see the figure on the right in Figure 7.

The transformation T will play a key role in allowing us to define an invariant of RNA folding via an invariant of stuck links.

4 Algebraic structures from stuck knots

In this section, we discuss the algebraic structures motivated by the diagrammatics of stuck knots and links. For more details on quandles, singquandles, and stuquandles, the reader is referred to [2,12,13].

The following definition is motivated by the Reidemeister moves in classical knot theory.

Definition 4.1. A *quandle* is a set X with a binary operation $* : X \times X \to X$ satisfying the following three axioms:

(*right distributivity*) for all $x, y, z \in X$, we have $(x * y) * z = (x * z) * (y * z)$;
(*invertibility*) for all $x \in X$, the map $R_x : X \to X$ sending y to $y * x$ is a bijection;
(*idempotency*) for all $x \in X$, $x * x = x$.

If $S \subset X$ is itself a quandle, we call S a *subquandle* of X.

In the rest of the article, for all $x, y \in X$, we will denote $R_y^{-1}(x)$ by $x \bar{*} y$. The next definition is motivated by the generalized Reidemeister moves in singular knot theory.

Definition 4.2. Let $(X, *)$ be a quandle and R_1 and R_2 be maps from $X \times X$ to X. The quadruple $(X, *, R_1, R_2)$ is an *oriented singquandle* if for all $x, y, z \in X$:

$$R_1(x \bar{*} y, z) * y = R_1(x, z * y), \tag{1}$$

$$R_2(x \bar{*} y, z) = R_2(x, z * y) \bar{*} y, \tag{2}$$

$$(y \bar{*} R_1(x, z)) * x = (y * R_2(x, z)) \bar{*} z, \tag{3}$$

$$R_2(x, y) = R_1(y, x * y), \tag{4}$$

$$R_1(x, y) * R_2(x, y) = R_2(y, x * y). \tag{5}$$

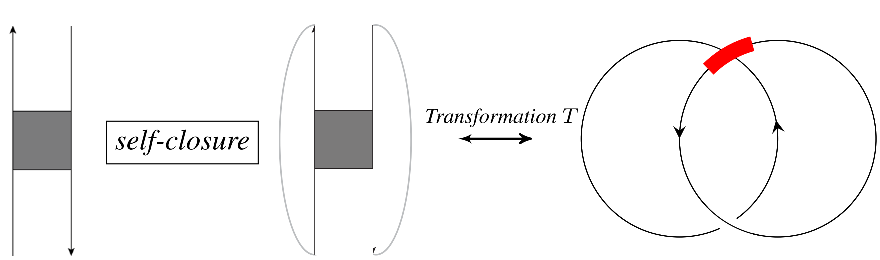


Figure 7: Arc diagram, self-closure, and corresponding stuck link diagram.

Definition 4.3. Let $(X, *, R_1, R_2)$ be a singquandle and R_3 and R_4 be maps from $X \times X$ to X. The six-tuple $(X, *, R_1, R_2, R_3, R_4)$ is called an *oriented stuquandle* if the following axioms are satisfied for all $x, y, z \in X$:

$$R_3(y, x) * R_4(y, x) = R_4(x * y, y), \tag{6}$$

$$R_4(y, x) = R_3(x * y, y), \tag{7}$$

$$R_3(y * x, z) = R_3(y, z \,\bar{*}\, x) * x, \tag{8}$$

$$R_4(y, z \,\bar{*}\, x) = R_4(y * x, z) \,\bar{*}\, x, \tag{9}$$

$$(x * R_4(y, z)) \,\bar{*}\, y = (x \,\bar{*}\, R_3(y, z)) * z. \tag{10}$$

Let $(X, *, R_1, R_2, R_3, R_4)$ be a stuquandle, and let L be a stuck link with diagram D. A coloring of D by X is an assignment of elements of X to the semiarcs at stuck crossings and to the arcs at classical crossings of D, obeying the coloring rules in Figure 8.

We note that the stuquandle axioms correspond to the oriented stuck Reidemeister moves, following the coloring rules in Figure 8. We will now review some key concepts about stuquandles, including basic examples.

Definition 4.4. Let $(X, *, R_1, R_2, R_3, R_4)$ be a stuquandle. A subset $S \subset X$ is called a *substuquandle* if $(S, *, R_1, R_2, R_3, R_4)$ is itself a stuquandle.

Definition 4.5. [2] Let $(X, *, R_1, R_2, R_3, R_4)$ and $(Y, \triangleright, S_1, S_2, S_3, S_4)$ be two stuquandles. A map $f : X \longrightarrow Y$ that satisfies the following conditions:

$$f(x * y) = f(x) \triangleright f(y), \tag{11}$$

$$f(R_1(x, y)) = S_1(f(x), f(y)), \tag{12}$$

$$f(R_2(x, y)) = S_2(f(x), f(y)), \tag{13}$$

$$f(R_3(x, y)) = S_3(f(x), f(y)), \tag{14}$$

$$f(R_4(x, y)) = S_4(f(x), f(y)), \tag{15}$$

is called a *stuquandle homomorphism*. If, furthermore, f is a bijection, then it is called a *stuquandle isomorphism*.

Lemma 4.6. *Let* $(X, *, R_1, R_2, R_3, R_4)$ *and* $(Y, \triangleright, S_1, S_2, S_3, S_4)$ *be two stuquandles. If* $f : X \to Y$ *is a stuquandle homomorphism, then the image of* f*, denoted by* $\mathrm{Im}(f)$*, is a substuquandle of* Y.

Proof. By definition, the image of f is $\mathrm{Im}(f) = \{f(x) | x \in X\} \subseteq Y$. Since f is a stuquandle homomorphism, it preserves the stuquandle operations and maps. Specifically, for any $x, y \in X$ equations (11)–(15) from Definition 4.5 are satisfied. Therefore, for any $f(x), f(y) \in \mathrm{Im}(f)$, the elements $f(x) \triangleright f(y)$, $S_1(f(x), f(y))$, $S_2(f(x), f(y))$, $S_3(f(x), f(y))$, $S_4(f(x), f(y))$ are also in $\mathrm{Im}(f)$. Hence, $(\mathrm{Im}(f), \triangleright, S_1, S_2, S_3, S_4)$ satisfies all the stuquandle axioms and is a substuquandle of Y. □

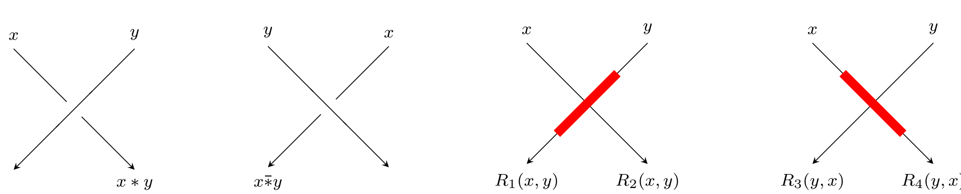


Figure 8: Coloring relations at classical and stuck crossings.

The following example was introduced in [2]. To see the construction details of the following stuquandle, the reader should refer to [2].

Example 4.7. Consider the set $X = \mathbb{Z}_n$. Let a be an invertible element of $\mathbb{Z}_n$, and let b and e be any elements of $\mathbb{Z}_n$. Define the operations as follows:

$$x * y = ax + (1 - a)y, \tag{16}$$

$$R_1(x, y) = bx + (1 - b)y, \tag{17}$$

$$R_2(x, y) = a(1 - b)x + (1 - a(1 - b))y, \tag{18}$$

$$R_3(x, y) = (1 - e)x + ey, \tag{19}$$

$$R_4(x, y) = (1 - a(1 - e))x + a(1 - e)y. \tag{20}$$

Then, the six-tuple $(\mathbb{Z}_n, *, R_1, R_2, R_3, R_4)$ is an oriented stuquandle.

Definition 4.8. [2] Let D be a stuck link diagram of a stuck link L and let $S = \{a_1, a_2, \ldots, a_m\}$ be the set of labels of the arcs in D at classical crossings and semiarcs in D at stuck crossings. In a similar way to the case of classical knot theory, we define the *fundamental stuquandle* of D as follows:

- The set of stuquandle words, $W(S)$, is recursively defined.
  - $S \subset W(S)$,
  - If $a_i, a_j \in W(S)$, then

$$a_i * a_j,\ a_i \,\overline{*}\, a_j,\ R_1(a_i, a_j),\ R_2(a_i, a_j),\ R_3(a_i, a_j),\ R_4(a_i, a_j) \in W(S).$$

- The set Y is the set of *free stuquandle words* which are equivalent classes of $W(S)$ determined by the conditions in Definition 4.3.
- Let $c_1, \ldots, c_n$ be the crossings of D. Each crossing c_i in D determines a relation r_i on the elements of Y.
- The *fundamental stuquandle* of D, $\mathcal{STQ}(D)$, is the set of equivalence class of words in $W(S)$ determined by the stuquandle conditions and the relations given by the crossings of D.

In the following example, we use the notation and naming convention of stuck knots and links from [2].

Example 4.9. In this example, we compute the fundamental stuquandle of the following oriented stuck link. Consider the stuck trefoil, denoted by 2_1^{k-}, with one negative stuck crossing and two negative classical crossings (Figure 9).

We will label the arcs of the diagram D by $a, b, c,$ and d. Then, the fundamental stuquandle of 2_1^{k-} is defined by

$$\mathcal{STQ}(2_1^{k-}) \cong \langle a, b, c, d \mid a = d \,\overline{*}\, b,\ b = R_3(a, c),\ c = b \,\overline{*}\, a,\ d = R_4(a, c)\rangle.$$

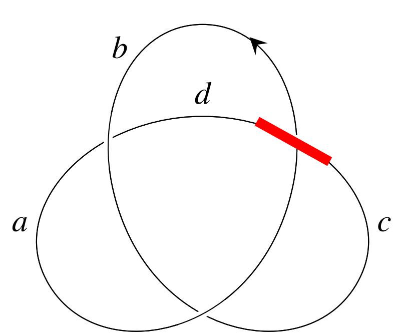


Figure 9: Diagram D of the stuck trefoil 2_1^{k-}.

The fundamental stuquandle can be used to define the following computable and effective invariant of stuck links. Given a finite stuquandle $(X, \triangleright, R_1', R_2', R_3', R_4')$, the set of stuquandle homomorphisms from $\mathcal{STQ}(L)$ to X, denoted by $\mathrm{Hom}(\mathcal{STQ}(L), X)$, may be used to define computable invariants. Specifically, by computing the cardinality $|\mathrm{Hom}(\mathcal{STQ}(L), X)|$, we obtain an integer value invariant called the *stuquandle counting invariant*, denoted by $Col_X(L)$ introduced in [2].

We can think of each $f \in \mathrm{Hom}(\mathcal{STQ}(L), X)$ as assigning an element of X to each arc in D at a classical crossing and to each semiarc in D at each stuck crossing, satisfying the coloring rules in Figure 8. Therefore, each $f \in \mathrm{Hom}(\mathcal{STQ}(L), X)$ can be represented by the m-tuple $(f(a_1), f(a_2), \dots, f(a_m))$, where $a_1, a_2, \dots, a_m$ are the arc labels of any diagram of D. Furthermore, the image of each element of $\mathrm{Hom}(\mathcal{STQ}(L), X)$ is a substuquandle of X by Lemma 4.6.

5 Review of the quandle polynomial

In this section, we recall the definition of the quandle polynomial, the subquandle polynomial, and the link invariants obtained from the subquandle polynomial. For a detailed construction of these polynomials, see [8,12].

Definition 5.1. Let $(Q, *)$ be a finite quandle. For any element $x \in Q$, let

$$C(x) = \{y \in Q \ : \ y * x = y\} \quad \text{and} \quad R(x) = \{y \in Q \ : \ x * y = x\}$$

and set $r(x) = |R(x)|$ and $c(x) = |C(x)|$. Then, the *quandle polynomial of* Q, $qp_Q(s,t)$, is

$$qp_Q(s,t) = \sum_{x \in Q} s^{r(x)} t^{c(x)}.$$

In [8], the quandle polynomial was shown to be an effective invariant of finite quandles. In addition to being an invariant of finite quandles, the quandle polynomial was generalized to give information about how a subquandle is embedded in a quandle.

Definition 5.2. Let $S \subset Q$ be a subquandle of Q. The *subquandle polynomial of* S, $qp_{S \subset Q}(s,t)$, is

$$qp_{S \subset Q}(s,t) = \sum_{x \in S} s^{r(x)} t^{c(x)},$$

where $r(x)$ and $c(x)$ are defined above.

Note that for any knot or link K, there is an associated fundamental quandle, $Q(K)$, and for any given finite quandle T the set of quandle homomorphisms, denoted by $\mathrm{Hom}(Q(K), T)$, has been used to define computable link invariants, for example, the cardinality of the set is known as the *quandle counting invariant*. In [8], the subquandle polynomial of the image of each homomorphism was used to enhance the counting invariant.

Definition 5.3. Let K be a link and T be a finite quandle. Then, for every $f \in \mathrm{Hom}(Q(K), T)$, the image of f is a subquandle of T. The *subquandle polynomial invariant*, $\Phi_{qp}(K, T)$, is the set with multiplicities

$$\Phi_{qp}(K,T) = \{qp_{\mathrm{Im}(f) \subset T}(s,t) \mid f \in \mathrm{Hom}(Q(K),T)\}.$$

Alternatively, the multiset can be represented in polynomial form by

$$\phi_{qp}(K,T) = \sum_{\bar{s}f \in \mathrm{Hom}(Q(K),T)} u^{qp_{\mathrm{Im}(f) \subset T}(s,t)}.$$

6 Generalized quandle polynomials

In this section, we introduce a generalization of the quandle polynomial in [8]. We show that this generalization is an invariant of stuquandles. We then use the generalization to define a polynomial invariant of stuck links.

Definition 6.1. Let $(X, *, R_1, R_2, R_3, R_4)$ be a finite stuquandle. For every $x \in X$, define

$$\begin{aligned} C^1(x) &= \{y \in X | y * x = y\} \quad \text{and} \quad R^1(x) = \{y \in X | x * y = x\}, \\ C^2(x) &= \{y \in X | R_1(y, x) = y\} \quad \text{and} \quad R^2(x) = \{y \in X | R_1(x, y) = x\}, \\ C^3(x) &= \{y \in X | R_2(y, x) = y\} \quad \text{and} \quad R^3(x) = \{y \in X | R_2(x, y) = x\}, \\ C^4(x) &= \{y \in X | R_3(y, x) = y\} \quad \text{and} \quad R^4(x) = \{y \in X | R_3(x, y) = x\}, \\ C^5(x) &= \{y \in X | R_4(y, x) = y\} \quad \text{and} \quad R^5(x) = \{y \in X | R_4(x, y) = x\}. \end{aligned}$$

Let $c^i(x) = |C^i(x)|$ and $r^i(x) = |R^i(x)|$ for i = 1, 2, 3, 4, 5. Then, the *stuquandle polynomial of* X is

$$stqp(X) = \sum_{x \in X} s_1^{r^1(x)} t_1^{c^1(x)} s_2^{r^2(x)} t_2^{c^2(x)} s_3^{r^3(x)} t_3^{c^3(x)} s_4^{r^4(x)} t_4^{c^4(x)} s_5^{r^5(x)} t_5^{c^5(x)}.$$

Proposition 6.2. *If* $(X, *, R_1, R_2, R_3, R_4)$ *and* $(Y, \triangleright, R_1', R_2', R_3', R_4')$ *are isomorphic finite stuquandles, then* $stqp(X) = stqp(Y)$.

Proof. Suppose $f : X \to Y$ is a stuquandle isomorphism and fix $x \in X$. For all $y \in C^1(x) = \{y \in X|\ y * x = y\}$, we have $f(y) \triangleright f(x) = f(y * x) = f(y)$, thus $f(y) \in C^1(f(x))$ and $|C^1(x)| \le |C^1(f(x))|$. Applying this argument to f^{-1}, we obtain $|C^1(f(x))| \le |C^1(x)|$, and therefore, $c^1(x) = c^1(f(x))$. By definition of a stuquandle isomorphism we have, $R_j'(f(y), f(x)) = f(R_j(y, x)) = f(y)$ for $j = 1, 2, 3, 4$. By applying a similar argument used to show $c^1(x) = c^1(f(x))$, we obtain $c^i(x) = c^i(f(x))$ for $i = 2, 3, 4, 5$. A similar argument also shows that $r^i(x) = r^i(f(x))$ for i = 1, 2, 3, 4, 5. These facts give the following:

$$\begin{aligned} stqp(X) &= \sum_{x \in X} s_1^{r^1(x)} t_1^{c^1(x)} s_2^{r^2(x)} t_2^{c^2(x)} s_3^{r^3(x)} t_3^{c^3(x)} s_4^{r^4(x)} t_4^{c^4(x)} s_5^{r^5(x)} t_5^{c^5(x)} \\ &= \sum_{f(x) \in Y} s_1^{r^1(x)} t_1^{c^1(x)} s_2^{r^2(x)} t_2^{c^2(x)} s_3^{r^3(x)} t_3^{c^3(x)} s_4^{r^4(x)} t_4^{c^4(x)} s_5^{r^5(x)} t_5^{c^5(x)} \\ &= \sum_{f(x) \in Y} s_1^{r^1(f(x))} t_1^{c^1(f(x))} s_2^{r^2(f(x))} t_2^{c^2(f(x))} s_3^{r^3(f(x))} t_3^{c^3(f(x))} s_4^{r^4(f(x))} t_4^{c^4(f(x))} s_5^{r^5(f(x))} t_5^{c^5(f(x))} \\ &= stqp(Y). \end{aligned}$$

□

Definition 6.3. Let X be a finite stuquandle and $S \subset X$ be a substuquandle. Then, the *substuquandle polynomial* is

$$Sstqp(S \subset X) = \sum_{x \in S} s_1^{r^1(x)} t_1^{c^1(x)} s_2^{r^2(x)} t_2^{c^2(x)} s_3^{r^3(x)} t_3^{c^3(x)} s_4^{r^4(x)} t_4^{c^4(x)} s_5^{r^5(x)} t_5^{c^5(x)}.$$

Note that for $i \in \{1, 2, 3, 4, 5\}$, $r^i(x)$ (respectively $c^i(x)$) is the number of elements of X that act trivially on x (respectively, is the number of elements of X on which x acts trivially) via $*$, R_1, R_2, R_3, and R_4. These values can be easily computed from the operation table of $*$, R_1, R_2, R_3, and R_4 by counting the occurrences of the row numbers. Please refer to Example 6.4 for further explanation.

Example 6.4. Let $X_1 = \mathbb{Z}_4$ be the stuquandle with operations $x * y = 3x + 2y$, $R_1(x, y) = 2x + 3y$, $R_2(x, y) = x$, $R_3(x, y) = 3x + 2y$, and $R_4(x, y) = y$. These operations have the following operation tables:

$*$	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

R_1	0	1	2	3
0	0	3	2	1
1	2	1	0	3
2	0	3	2	1
3	2	1	0	3

R_2	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

R_3	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	2	0	2	0
3	1	3	1	3

R_4	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

and the operations have the following $r^i(x)$ and $c^i(x)$ values for $i = 1, 2, 3, 4, 5$:

x	$r^1(x)$	$c^1(x)$
1	2	2
2	2	2
3	2	2
0	2	2

x	$r^2(x)$	$c^2(x)$
1	1	1
2	1	1
3	1	1
0	1	1

x	$r^3(x)$	$c^3(x)$
1	4	4
2	4	4
3	4	4
0	4	4

x	$r^4(x)$	$c^4(x)$
1	2	2
2	2	2
3	2	2
0	2	2

x	$r^5(x)$	$c^5(x)$
1	1	1
2	1	1
3	1	1
0	1	1

.

Thus, the stuquandle polynomial of X is

$$sqp(X_1) = 4s_1^2t_1^2s_2t_2s_3^4t_3^4s_4^2t_4^2s_5t_5.$$

Next, consider the stuquandle $X_2 = \mathbb{Z}_4$ with operations $x * y = x$, $R_1(x, y) = y$, $R_2(x, y) = x$, $R_3(x, y) = y$ and $R_4(x, y) = x$. The operations have the following operation tables:

$*$	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

R_1	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

R_2	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

R_3	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

R_4	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

and the operations have the following $r^i(x)$ and $c^i(x)$ values for $i = 1, 2, 3, 4, 5$:

x	$r^1(x)$	$c^1(x)$
1	4	4
2	4	4
3	4	4
0	4	4

x	$r^2(x)$	$c^2(x)$
1	1	1
2	1	1
3	1	1
0	1	1

x	$r^3(x)$	$c^3(x)$
1	4	4
2	4	4
3	4	4
0	4	4

x	$r^4(x)$	$c^4(x)$
1	1	1
2	1	1
3	1	1
0	1	1

x	$r^5(x)$	$c^5(x)$
1	4	4
2	4	4
3	4	4
0	4	4

.

Thus, the stuquandle polynomial of X is

$$sqp(X_2) = 4s_1^4t_1^4s_2t_2s_3^4t_3^4s_4^1t_4^1s_5^4t_5^4.$$

We obtain that $sqp(X_1) \neq sqp(X_2)$. Therefore, the two stuquandle structures defined on $\mathbb{Z}_4$ are distinguished by the contrapositive of Proposition 6.2.

Example 6.5. Let $S = \{1, 3\}$ be a substuquandle of X from the previous example. Thus, the substuquandle polynomial of S is

$$Sstqp(S \subset X) = 2s_1^2t_1^2s_2t_2s_3^4t_3^4s_4^2t_4^2s_5t_5.$$

By Lemma 4.6, we know that the image of a stuquandle homomorphism is a substuquandle. Suppose that X is a stuquandle and L is a stuck link. For each $f \in \mathrm{Hom}(\mathcal{STQ}(L), X)$, the image $\mathrm{Im}(f)$ is a substuquandle of X. This allows us to define the following polynomial.

Definition 6.6. Let L be a stuck link, T be a finite stuquandle. Then, the multiset

$$\Phi_{Sstqp}(L, T) = \{Sstqp(\mathrm{Im}(f) \subset T) | f \in Hom(\mathcal{STQ}(L), T)\}$$

is the *substuquandle polynomial invariant of* L with respect to T. We can rewrite the multiset in a polynomial-style form by converting the multiset elements to exponents of a formal variable u and converting their multiplicities to coefficients:

$$\phi_{Sstqp}(L, T) = \sum_{\bar{s}f \in \mathrm{Hom}(\mathcal{STQ}(L), T)} u^{Sstqp(\mathrm{Im}(f) \subset T)}.$$

7 Examples

In this section, we present examples that demonstrate the effectiveness of the substuquandle polynomial invariant in distinguishing stuck links. Specifically, we include an example that illustrates how the substuquandle polynomial invariant enhances the stuquandle counting invariant. Additionally, we include an example of two stuck links that are not distinguished by the $\hat{X}$ polynomial, but can be distinguished by the substuquandle polynomial. Finally, we explicitly compute the substuquandle polynomial of two RNA foldings and differentiate them using the substuquandle polynomial.

Example 7.1. Consider coloring the stuck knots 2_1^{k-} and 0_1^{k+} in Figures 10 and 11, respectively, by using the stuquandle $X = \mathbb{Z}_4$ with operation defined by $x * y = 3x + 2y$ and maps $R_1(x, y) = x + 2y^2$, $R_2(x, y) = 2x^2 + y$, $R_3(x, y) = 3x$, and $R_4(x, y) = 2x + y$. It was found in [3] that the stuquandle counting invariant for both of these knots is equal to 4. Given the stuquandle defined above, the colorings for Figure 10 are $\mathrm{Hom}(\mathcal{STQ}(0_1^{k+}), X) = \{(0, 0), (0, 2), (2,0), (2, 2)\}$, and the colorings for Figure 11 are $\mathrm{Hom}(\mathcal{STQ}(2_1^{k-}), X) = \{(0, 0, 0, 0), (1, 3, 3, 1), (2, 2, 2, 2), (3, 1, 1, 3)\}$. Thus, the coloring invariant for both knots is 4. To calculate the

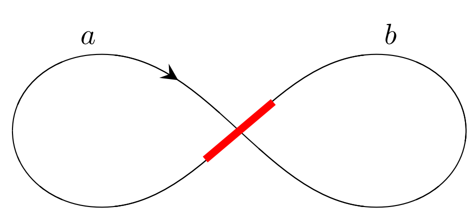


Figure 10: Diagram of the stuck knot 0_1^{k+}.

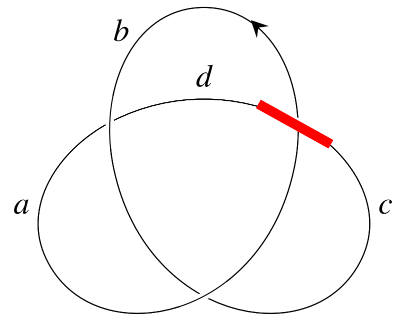


Figure 11: Diagram of the stuck trefoil 2_1^{k-}.

substuquandle polynomial invariant of these stuck knots, we show in the tables below the operations have the following $r^i(x)$ and $c^i(x)$ values for $i = 1, 2, 3, 4, 5$:

x	$r^1(x)$	$c^1(x)$
0	2	2
1	2	2
2	2	2
3	2	2

x	$r^2(x)$	$c^2(x)$
0	2	4
1	2	0
2	2	4
3	2	0

x	$r^3(x)$	$c^3(x)$
0	1	1
1	1	1
2	1	1
3	1	1

x	$r^4(x)$	$c^4(x)$
0	4	2
1	0	2
2	4	2
3	0	2

x	$r^5(x)$	$c^5(x)$
0	1	1
1	1	1
2	1	1
3	1	1

.

We collect the colorings of each stuck link and the substuquandle polynomial of the image of each coloring in Tables 1 and 2.

Using the coloring set of each stuck knot, we obtain the following substuquandle polynomial invariants:

$$\phi_{Sstqp}(0_1^{k+}, X) = 2u^{2s_1^2t_1^2s_2^2t_2^4s_3t_3s_4^4t_4^2s_5t_5} + 2u^{s_1^2t_1^2s_2^2t_2^4s_3t_3s_4^4t_4^2s_5t_5}$$

and

$$\phi_{Sstqp}(2_1^{k-}, X) = 2u^{s_1^2t_1^2s_2^2t_2^4s_3t_3s_4^4t_4^2s_5t_5} + 2u^{2s_1^2t_1^2s_2^2s_3t_3t_4^2s_5t_5}.$$

Table 1: Colorings and substuquandle polynomial of each coloring of 0_1^{k+}

$f(a)$	$f(b)$	$\mathrm{Im}(f) \subset X$	$Sstqp(\mathrm{Im}(f) \subset X)$
0	0	$\{0\}$	$s_1^2t_1^2s_2^2t_2^4s_3t_3s_4^4t_4^2s_5t_5$
0	2	$\{0, 2\}$	$2s_1^2t_1^2s_2^2t_2^4s_3t_3s_4^4t_4^2s_5t_5$
2	0	$\{0, 2\}$	$2s_1^2t_1^2s_2^2t_2^4s_3t_3s_4^4t_4^2s_5t_5$
2	2	$\{2\}$	$s_1^2t_1^2s_2^2t_2^4s_3t_3s_4^4t_4^2s_5t_5$

Table 2: Colorings and substuquandle polynomial of each coloring of 2_1^{k-}

$f(a)$	$f(b)$	$f(c)$	$f(d)$	$\mathrm{Im}(f) \subset X$	$Sstqp(\mathrm{Im}(f) \subset X)$
0	0	0	0	$\{0\}$	$s_1^2t_1^2s_2^2t_2^4s_3t_3s_4^4t_4^2s_5t_5$
1	3	3	1	$\{1, 3\}$	$2s_1^2t_1^2s_2^2s_3t_3t_4^2s_5t_5$
2	2	2	2	$\{2\}$	$s_1^2t_1^2s_2^2t_2^4s_3t_3s_4^4t_4^2s_5t_5$
3	1	1	3	$\{1, 3\}$	$2s_1^2t_1^2s_2^2s_3t_3t_4^2s_5t_5$

Example 7.2. Consider coloring the stuck knots K_1 and K_2 in Figures 12 and 13, respectively, by using the stuquandle $X = \mathbb{Z}_3$ with operations $x * y = x$, $R_1(x,y) = 2y^2$,$R_2(x,y) = 2x^2$, $R_3(x,y) = 2x + 2x^2$, and $R_4(x,y) = 2y + 2y^2$. The operation tables are given by

$*$	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

R_1	0	1	2
0	0	2	2
1	0	2	2
2	0	2	2

R_2	0	1	2
0	0	0	0
1	2	2	2
2	2	2	2

R_3	0	1	2
0	0	0	0
1	1	1	1
2	0	0	0

R_4	0	1	2
0	0	1	0
1	0	1	0
2	0	1	0

The $\hat{X}$ polynomial was defined in [1] for stuck links. Moreover, $\hat{X}(K_1) = 2xy - (x^2 + y^2)(A^2 + A - 2) = \hat{X}(K_2)$ as shown in Example 3 in [1]. Also, note that

$$\mathrm{Hom}(\mathcal{STQ}(K_1), X) = \{(0,0,0,0), (0,1,0,1), (1,0,1,0), (1,1,1,1)\}$$

and

$$\mathrm{Hom}(\mathcal{STQ}(K_2), X) = \{(0,0,0,0), (0,2,0,2),\ (2,0,2,0), (2,2,2,2)\}.$$

Thus, the coloring invariant for both is 4. To calculate the substuquandle polynomial invariant of these stuck knots, we show in the tables below the operations have the following $r^i(x)$ and $c^i(x)$ values for $i = 1, 2, 3, 4, 5$:

x	$r^1(x)$	$c^1(x)$
0	3	3
1	3	3
2	3	3

x	$r^2(x)$	$c^2(x)$
0	1	1
1	0	1
2	2	1

x	$r^3(x)$	$c^3(x)$
0	3	2
1	0	2
2	3	2

x	$r^4(x)$	$c^4(x)$
0	3	2
1	3	2
2	0	2

x	$r^5(x)$	$c^5(x)$
0	2	1
1	1	1
2	0	1

.

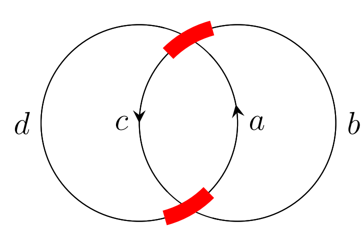


Figure 12: Diagram of the stuck link K_1.

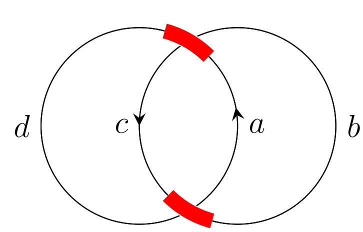


Figure 13: Diagram of the stuck link K_2.

Table 3: Colorings and substuquandle polynomial of each coloring of K_1

$f(a)$	$f(b)$	$f(c)$	$f(d)$	$\mathrm{Im}(f) \subset X$	$Sstqp(\mathrm{Im}(f) \subset X)$
0	0	0	0	$\{0\}$	$s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5$
1	0	1	0	$\{0,1,2\}$	$s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5 + s_1^3t_1^3t_2t_3^2s_4^3t_4^2s_5t_5 + s_1^3t_1^3s_2t_2s_3^3t_3^2t_4^2t_5$
0	1	0	1	$\{0,1,2\}$	$s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5 + s_1^3t_1^3t_2t_3^2s_4^3t_4^2s_5t_5 + s_1^3t_1^3s_2t_2s_3^3t_3^2t_4^2t_5$
1	1	1	1	$\{0,1,2\}$	$s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5 + s_1^3t_1^3t_2t_3^2s_4^3t_4^2s_5t_5 + s_1^3t_1^3s_2t_2s_3^3t_3^2t_4^2t_5$

We collect the colorings of each stuck link and the substuquandle polynomial of the image of each coloring in Tables 3 and 4.

Using the coloring set of each stuck link, we obtain the following substuquandle polynomial invariants:

$$\phi_{Sstqp}(K_1, X) = u^{s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5} + 3u^{s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5+s_1^3t_1^3t_2t_3^2s_4^3t_4^2s_5t_5+s_1^3t_1^3s_2t_2s_3^3t_3^2t_4^2t_5}$$

and

$$\phi_{Sstqp}(K_2, X) = u^{s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5} + 3u^{s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5+s_1^3t_1^3s_2t_2s_3^3t_3^2t_4^2t_5},$$

which shows that our invariant is stronger than the $\hat{X}$ polynomial of [1] in this case.

Now we will compute our new substuquandle polynomial invariant to distinguish the topology of RNA structure. First, we will consider an arc diagram of RNA folding, then use the transformation T and apply the self-closure to obtain a stuck link diagram. We will then compute the substuquandle polynomial invariant using the stuck link diagram corresponding to the arc diagram of an RNA folding. Note that since the substuquandle polynomial invariant is unchanged by the Reidemeister moves, the invariant only depends on the stuck link and not the diagram.

Example 7.3. Let $(\mathbb{Z}_4, *, R_1, R_2, R_3, R_4)$ be the stuquandle with operations $x * y = x$, $R_1(x,y) = 3x + y$, $R_2(x,y) = x + 3y$, $R_3(x,y) = x + 2y$, and $R_4(x,y) = 2x + y$. The operation tables are given by

$*$	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3

R_1	0	1	2	3
0	0	1	2	3
1	3	0	1	2
2	2	3	0	1
3	1	2	3	0

R_2	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

R_3	0	1	2	3
0	0	2	0	2
1	1	3	1	3
2	2	0	2	0
3	3	1	3	1

R_4	0	1	2	3
0	0	1	2	3
1	2	3	0	1
2	0	1	2	3
3	2	3	0	1

We will consider the two arc diagrams of RNA foldings and their corresponding stuck links (Figures 14 and 15).

With respect to the stuquandle defined above, the colorings for Figure 14 are $\mathrm{hom}(\mathcal{STQ}(K_1), X) = \{(0,0,0), (1,3,3), (2,2,2), (3,1,1)\}$ and the colorings for Figure 15 are $\mathrm{hom}(\mathcal{STQ}(K_2), X) = \{(0,0,0), (0,2,0), (2,0,2), (2,2,2)\}$. In this case, the stuquandle counting invariant cannot distinguish the two arc diagrams. Now, we will consider the substuquandle polynomial invariant of the two arc diagrams. In the tables below, we collect our operations r^i and c^i values:

x	$r^1(x)$	$c^1(x)$
0	4	4
1	4	4
2	4	4
3	4	4

x	$r^2(x)$	$c^2(x)$
0	1	2
1	1	0
2	1	2
3	1	0

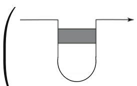
x	$r^3(x)$	$c^3(x)$
0	1	4
1	1	0
2	1	0
3	1	0

Table 4: Colorings and substuquandle polynomial of each coloring of 2_1^{k-}

$f(a)$	$f(b)$	$f(c)$	$f(d)$	$\mathrm{Im}(f) \subset X$	$Sstqp(\mathrm{Im}(f) \subset X)$
0	0	0	0	$\{0\}$	$s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5$
0	2	0	2	$\{0, 2\}$	$s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5 + s_1^3t_1^3s_2t_2s_3^3t_3^2t_4^2t_5$
2	0	2	0	$\{0, 2\}$	$s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5 + s_1^3t_1^3s_2t_2s_3^3t_3^2t_4^2t_5$
2	2	2	2	$\{0, 2\}$	$s_1^3t_1^3s_2t_2s_3^3t_3^2s_4^3t_4^2s_5^2t_5 + s_1^3t_1^3s_2t_2s_3^3t_3^2t_4^2t_5$

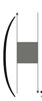
$$\begin{array}{c|cc} x & r^4(x) & c^4(x) \\ \hline 0 & 2 & 4 \\ 1 & 2 & 0 \\ 2 & 2 & 4 \\ 3 & 2 & 0 \end{array} \qquad \begin{array}{c|cc} x & r^5(x) & c^5(x) \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{array}.$$

We collect the colorings of each arc diagram and the substuquandle polynomial of the image of each coloring in Tables 5 and 6. Using the coloring set of each arc diagram, we obtain the following substuquandle polynomial invariants:

$\phi_{Sstqp}\Big($  $, X\Big)$

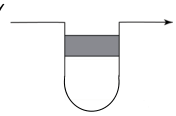
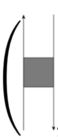
$$= u^{s_1^4t_1^4s_2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5} + u^{s_1^4t_1^4s2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5+s_1^4t_1^4s_2t_2^2s_3s_4^2t_4^4s_5t_5} + 2u^{2s_1^4t_1^4s_2s_3s_4^2s_5t_5+s_1^4t_1^4s2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5+s_1^4t_1^4s_2t_2^2s_3s_4^2t_4^4s_5t_5}$$

and

$\phi_{Sstqp}\Big($  $, X\Big)$

$$= u^{s_1^4t_1^4s_2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5} + 3u^{s_1^4t_1^4s2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5+s_1^4t_1^4s_2t_2^2s_3s_4^2t_4^4s_5t_5}.$$

Thus, the RNA foldings are distinguished since

$\phi_{Sstqp}\Big($  $, X\Big) \neq \phi_{Sstqp}\Big($  $, X\Big).$

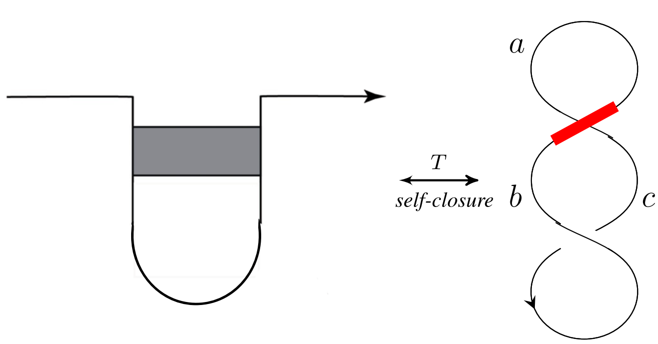


Figure 14: Arc diagram and corresponding stuck link K_3.

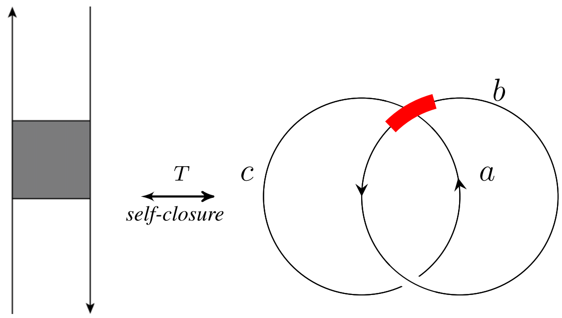


Figure 15: Arc diagram and corresponding stuck link diagram denoted by K_2.

Table 5: Colorings and substuquandle polynomial of each coloring of K_1

$f(a)$	$f(b)$	$f(c)$	$\mathrm{Im}(f) \subset X$	$Sstqp(\mathrm{Im}(f) \subset X)$
0	0	0	$\{0\}$	$s_1^4t_1^4s_2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5$
1	3	3	$\{0, 1, 2, 3\}$	$2s_1^4t_1^4s_2s_3s_4^2s_5t_5 + s_1^4t_1^4s2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5 + s_1^4t_1^4s_2t_2^2s_3s_4^2t_4^4s_5t_5$
2	2	2	$\{0, 2\}$	$s_1^4t_1^4s2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5 + s_1^4t_1^4s_2t_2^2s_3s_4^2t_4^4s_5t_5$
3	1	1	$\{0, 1, 2, 3\}$	$2s_1^4t_1^4s_2s_3s_4^2s_5t_5 + s_1^4t_1^4s2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5 + s_1^4t_1^4s_2t_2^2s_3s_4^2t_4^4s_5t_5$

Table 6: Colorings and substuquandle polynomial of each coloring of K_2

$f(a)$	$f(b)$	$f(c)$	$\mathrm{Im}(f) \subset X$	$Sstqp(\mathrm{Im}(f) \subset X)$
0	0	0	$\{0\}$	$s_1^4t_1^4s_2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5$
0	2	0	$\{0, 2\}$	$s_1^4t_1^4s2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5 + s_1^4t_1^4s_2t_2^2s_3s_4^2t_4^4s_5t_5$
2	0	2	$\{0, 2\}$	$s_1^4t_1^4s2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5 + s_1^4t_1^4s_2t_2^2s_3s_4^2t_4^4s_5t_5$
2	2	2	$\{0, 2\}$	$s_1^4t_1^4s2t_2^2s_3t_3^4s_4^2t_4^4s_5t_5 + s_1^4t_1^4s_2t_2^2s_3s_4^2t_4^4s_5t_5$

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