Research Article

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Recurrence for probabilistic extension of Dowling polynomials

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Abstract: Spivey found a remarkable recurrence relation for Bell numbers, which was generalized to that for Bell polynomials by Gould-Quaintance. The aim of this article is to generalize their recurrence relation for Bell polynomials to that for the probabilistic Dowling polynomials associated with Y and also that for the probabilistic r-Dowling polynomials associated with Y. Here Y is a random variable whose moment generating function exists in a neighborhood of the origin.

Keywords: probabilistic Whitney numbers of the second kind, probabilistic Dowling polynomials, probabilistic r-Whitney numbers of the second, probabilistic r-Dowling polynomials

MSC 2020: 11B73, 11B83

1 Introduction

Assume that Y is a random variable whose moment generating function exists in a neighborhood of the origin (see (11)). We consider the probabilistic Whitney numbers of the second kind associated with Y, $W_m^Y(n,k)$ (see (14)), as a probabilistic extension of the Whitney numbers of the second kind $W_m(n,k)$ (see (5), (7)). Here we note that the Whitney numbers of the second kind amount to the Stirling numbers of the second kind. Then, as a polynomial extension of $W_m^Y(n,k)$, we introduce the probabilistic Dowling polynomials associated with Y, $D_m^Y(n,x)$ (see (16)), which is a probabilistic extension of the Dowling polynomials $D_m(n,x)$ (see (9)). The aim of this article is to generalize the Gould-Quaintance's recurrence relation for Bell polynomials (see (3), (4)) to that for $D_m^Y(n,x)$, which is given by

$$D_m^Y(n+k,x) = \sum_{l=0}^{n}\binom{n}{l}m^{n-l}D_m^Y(l,x)\sum_{i=0}^{k}\sum_{j=0}^{i}\binom{k}{i}\frac{x^j m^{i-j}}{j!}\sum_{\substack{l_1+\dots+l_j=i\\ l_1,l_2,\dots,l_j\ge 1}}\binom{i}{l_1,\dots,l_i}E[Y_1^{l_1}\cdots Y_j^{l_j}S_j^{n-l}]. \tag{1}$$

We note here that (1) boils down to the following recurrence relation when $Y=1$:

$$D_m(n+k,x) = \sum_{l=0}^{n}\sum_{j=0}^{k}\binom{n}{l}m^{n-l}j^{n-l}W_m(k,j)x^jD_m(l,x), \quad (n,k\ge 0).$$

We also consider their probabilistic r-Whitney numbers of the second kind associated with Y, $W_{m,r}^Y(n,k)$ (see (21)) and their polynomial extension, namely the probabilistic r-Dowling polynomials associated with Y,

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$D^{Y}_{m,r}(n,x)$ (see (23)). Then we derive a recurrence relation that generalizes Gould-Quaintance's for Bell polynomials (see (3), (4)). For the rest of this article, we recall the facts that are needed throughout the article.

It is known that the Bell polynomials are defined by

$$\phi_n(x) = \sum_{k=0}^{n} \left\{ n \atop k \right\} x^k \quad (\text{see } [1\text{–}18]), \tag{2}$$

with the Bell numbers given by $\phi_n = \phi_n(1)$, where $\left\{ n \atop k \right\}$ are the Stirling numbers of the second kind.

Spivey found an interesting recurrence relation for ϕ_n given in the following:

$$\phi_{l+n} = \sum_{k=0}^{l}\sum_{i=0}^{n} k^{n-i}\binom{n}{i}\left\{ l \atop k \right\}\phi_i, \quad (l,n \geq 0) \quad (\text{see } [19]). \tag{3}$$

In [20], Gould-Quaintance extended the recurrence relation for Bell numbers in (3) to that for Bell polynomials, which is given by

$$\phi_{l+n}(x) = \sum_{k=0}^{n}\sum_{i=0}^{n} k^{n-i}\binom{n}{i}\left\{ l \atop k \right\}\phi_i(x)x^i. \tag{4}$$

It is well known that the Whitney numbers of the second kind are defined by

$$(mx+1)^n = \sum_{k=0}^{n} m^k W_m(n,k)(x)_k, \quad (m \in \mathbb{N}) \quad (\text{see } [21,22,23]), \tag{5}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\ldots(x-n+1)$, $(n \geq 1)$.

For $m = 1$, we have $W_1(n,k) = \left\{ n+1 \atop k+1 \right\}$. For $r \in \mathbb{N}$, the r-Whitney numbers of the second kind are defined by

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_{m,r}(n,k)(x)_k, \quad (n \geq 0) \quad (\text{see } [21,22,23]). \tag{6}$$

From (5) and (6), we note that

$$e^t\frac{1}{k!}\left(\frac{e^{mt}-1}{m}\right)^k = \sum_{n=k}^{\infty} W_m(n,k)\frac{t^n}{n!} \tag{7}$$

and

$$e^{rt}\frac{1}{k!}\left(\frac{e^{mt}-1}{m}\right)^k = \sum_{n=k}^{\infty} W_{m,r}(n,k)\frac{t^n}{n!} \quad (\text{see } [21,22,23]). \tag{8}$$

The Dowling polynomials are defined by

$$D_m(n,x) = \sum_{k=0}^{n} W_m(n,k)x^k, \quad (n \geq 0) \quad (\text{see } [22]), \tag{9}$$

and the r-Dowling polynomials are given by

$$D_{m,r}(n,x) = \sum_{k=0}^{n} W_{m,r}(n,k)x^k, \quad (n \geq 0) \quad (\text{see } [23]). \tag{10}$$

We assume that Y is a random variable satisfying the moment conditions

$$E[|Y|^n] < \infty, \quad n \in \mathbb{N} \cup \{0\}, \quad \lim_{n\to\infty}\frac{|t|^n E[|Y|^n]}{n!} = 0, \quad |t| < r, \tag{11}$$

for some r, where E stands for the mathematical expectation [24,25].

Let $(Y_j)_{j\geq 1}$ be a sequence of mutually independent copies of the random variable Y, and let

$$S_0 = 0, \quad S_k = Y_1 + Y_2 + \ldots + Y_k, \quad (k \geq 1). \tag{12}$$

Recently, the probabilisitic Stirling numbers of the second kind associated with Y are given by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_Y = \frac{1}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}E[S_k^n], \quad (0 \le m \le n) \quad (\text{see [24,25]}). \tag{13}$$

2 Recurrence for probabilistic extension of Dowling polynomials

Let $(Y_j)_{j\ge1}$ be a sequence of mutually independent copies of the random variable Y, and let

$$S_0 = 0, \quad S_k = Y_1 + Y_2 + \ldots + Y_k, \quad (k \ge 1).$$

In view of (7), we consider the *probabilistic Whitney numbers of the second kind associated with Y* given by

$$\frac{1}{k!}\left(\frac{E[e^{mYt}] - 1}{m}\right)^k e^t = \sum_{n=k}^{\infty} W_m^Y(n,k)\frac{t^n}{n!}, \quad (k \ge 0). \tag{14}$$

When $Y = 1$, we have $W_m^Y(n,k) = W_m(n,k)$.

By (14), we obtain

$$\begin{aligned}\sum_{n=k}^{\infty} W_m^Y(n,k)\frac{t^n}{n!} &= \frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[e^{(m(Y_1+Y_2+\ldots+Y_j)+1)t}] \\ &= \frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[e^{(mS_j+1)t}] \\ &= \sum_{n=0}^{\infty}\frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[(mS_j+1)^n]\frac{t^n}{n!}.\end{aligned} \tag{15}$$

Therefore, by (15), we obtain the following theorem.

Theorem 1. *For $n \ge k \ge 0$, we have*

$$W_m^Y(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[(mS_j+1)^n].$$

In view of (9), we define the *probabilistic Dowling polynomials associated with Y* by

$$D_m^Y(n,x) = \sum_{k=0}^{n} W_m^Y(n,k)x^k, \quad (n \ge 0). \tag{16}$$

From (16), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} D_m^Y(n,x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} W_m^Y(n,k)x^k\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} x^k e^t \frac{1}{k!}\left(\frac{E[e^{mYt}]-1}{m}\right)^k \\ &= e^t e^{x\left(\frac{E[e^{mYt}]-1}{m}\right)}.\end{aligned} \tag{17}$$

Theorem 2. *The generating function of Dowling polynomials is given by*

$$e^t e^{x\left(\frac{E[e^{mYt}]-1}{m}\right)} = \sum_{n=0}^{\infty} D_m^Y(n,x)\frac{t^n}{n!}.$$

Using Taylor series, we note that

$$f(x+t)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}t^{n}=\sum_{n=0}^{\infty}\frac{t^{n}D_{x}^{n}}{n!}f(x)=e^{tD_{x}}f(x), \tag{18}$$

where $D_x f(x)=\frac{\mathrm{d}}{\mathrm{d}x}f(x)$.

By (17) and (18), we obtain

$$e^{zD_t}\left(e^{t}e^{x\left(\frac{E[e^{mYt}]-1}{m}\right)}\right)=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}D_{t}^{k}\sum_{n=0}^{\infty}D_{m}^{Y}(n,x)\frac{t^{n}}{n!}=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}D_{m}^{Y}(n+k,x)\frac{z^{k}}{k!}\frac{t^{n}}{n!}. \tag{19}$$

On the other hand, by (18), we obtain

$$\begin{aligned}
e^{zD_t}\left(e^{t}e^{x\left(\frac{E[e^{mYt}]-1}{m}\right)}\right)&=e^{t+z}e^{x\left(\frac{E[e^{mY(t+z)}]-1}{m}\right)}\\
&=e^{t}e^{x\left(\frac{E[e^{mYt}]-1}{m}\right)}e^{z}e^{x\left(\frac{E[e^{mYt}(e^{mYz}-1)]}{m}\right)}\\
&=\sum_{l=0}^{\infty}D_{m}^{Y}(l,x)\frac{t^{l}}{l!}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\sum_{j=0}^{i}\binom{k}{i}\frac{x^{j}}{j!}\frac{m^{i}}{m^{j}}\sum_{\substack{l_1+\ldots+l_j=i\\ l_1,l_2,\ldots,l_j\ge 1}}\binom{i}{l_1,\ldots,l_j}E[Y_1^{l_1}\ldots Y_j^{l_j}e^{mS_jt}]\frac{z^{k}}{k!}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}D_{m}^{Y}(l,x)\sum_{i=0}^{k}\sum_{j=0}^{i}\binom{k}{i}\frac{x^{j}m^{i-j}}{j!}\\
&\quad\times\sum_{\substack{l_1+\ldots+l_j=i\\ l_1,l_2,\ldots,l_j\ge 1}}\binom{i}{l_1,\ldots,l_i}E[Y_1^{l_1}\ldots Y_j^{l_j}S_j^{n-l}]m^{n-l}\frac{z^{k}}{k!}\frac{t^{n}}{n!}.
\end{aligned} \tag{20}$$

Therefore, by (19) and (20), we obtain the following theorem.

Theorem 3. *For $n,k\ge 0$, we have*

$$D_{m}^{Y}(n+k,x)=\sum_{l=0}^{n}\binom{n}{l}m^{n-l}D_{m}^{Y}(l,x)\sum_{i=0}^{k}\sum_{j=0}^{i}\binom{k}{i}\frac{x^{j}m^{i-j}}{j!}\sum_{\substack{l_1+\ldots+l_j=i\\ l_1,l_2,\ldots,l_j\ge 1}}\binom{i}{l_1,\ldots,l_i}E[Y_1^{l_1}\ldots Y_j^{l_j}S_j^{n-l}].$$

In view of (8), we consider the *probabilistic r-Whitney numbers of the second kind associated with Y* given by

$$\frac{1}{k!}\left(\frac{E[e^{mYt}]-1}{m}\right)^{k}e^{rt}=\sum_{n=k}^{\infty}W_{m,r}^{Y}(n,k)\frac{t^{n}}{n!},\quad (k\ge 0). \tag{21}$$

When $Y=1$, we have $W_{m,r}^{Y}(n,k)=W_{m,r}(n,k)$.

From (21), we note that

$$\begin{aligned}
\sum_{n=k}^{\infty}W_{m,r}^{Y}(n,k)\frac{t^{n}}{n!}&=\frac{1}{k!m^{k}}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[e^{m(Y_1+\ldots+Y_j)t}]e^{rt}\\
&=\frac{1}{k!m^{k}}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[e^{(mS_j+r)t}]\\
&=\sum_{n=0}^{\infty}\frac{1}{k!m^{k}}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[(mS_j+r)^{n}]\frac{t^{n}}{n!}.
\end{aligned} \tag{22}$$

Therefore, by (22), we obtain the following theorem.

Theorem 4. *For $n\ge k\ge 0$, we have*

$$W_{m,r}^{Y}(n,k)=\frac{1}{k!m^{k}}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E[(mS_j+r)^{n}].$$

When $Y = 1$, we have

$$W_{m,r}(n,k) = \frac{1}{m^k k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(mj+r)^n.$$

In view of (16), we define the *probabilistic r-Dowling polynomials associated with Y* as

$$D_{m,r}^{Y}(n,x) = \sum_{k=0}^{n} W_{m,r}^{Y}(n,k)x^k, \quad (n \geq 0). \tag{23}$$

When $Y = 1$, we have $D_{m,r}^{Y}(n,x) = D_{m,r}(n,x)$.

From (23), we have

$$\begin{aligned}\sum_{n=0}^{\infty} D_{m,r}^{Y}(n,x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} W_{m,r}^{Y}(n,k)x^k\frac{t^n}{n!}\\ &= \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} W_{m,r}^{Y}(n,k)\frac{t^n}{n!}\\ &= e^{x\left(\frac{E[e^{mYt}]-1}{m}\right)}e^{rt}.\end{aligned} \tag{24}$$

Therefore, by (24), we obtain the following theorem.

Theorem 5. *The generating function of probabilistic r-Dowling polynomials is given by*

$$e^{rt}e^{x\left(\frac{E[e^{mYt}]-1}{m}\right)} = \sum_{n=0}^{\infty} D_{m,r}^{Y}(n,x)\frac{t^n}{n!}. \tag{25}$$

By (25), we obtain

$$D_{m,r}^{Y}(n,x) = e^{-\frac{x}{m}}\sum_{k=0}^{\infty}\frac{x^k}{k!m^k}E[(mS_k + r)^n], \quad (n \geq 0).$$

Theorem 6. *For $n \geq 0$, we have*

$$D_{m,r}^{Y}(n,x) = e^{-\frac{x}{m}}\sum_{k=0}^{\infty}\frac{x^k}{k!m^k}E[(mS_k + r)^n].$$

Now, we observe that

$$e^{zD_t}\left(e^{rt}e^{x\left(\frac{E[e^{mYt}]-1}{m}\right)}\right) = \sum_{k=0}^{\infty}\frac{z^k}{k!}D_t^k\sum_{n=0}^{\infty}D_{m,r}^{Y}(n,x)\frac{t^n}{n!} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}D_{m,r}^{Y}(n+k,x)\frac{t^n}{n!}\frac{z^k}{k!}. \tag{26}$$

On the other hand, by (18), we obtain

$$\begin{aligned}e^{zD_t}\left(e^{rt}e^{x\left(\frac{E[e^{mYt}]-1}{m}\right)}\right) &= e^{r(z+t)}e^{x\left(\frac{E[e^{mY(z+t)}]-1}{m}\right)}\\ &= e^{rz}e^{x\left(\frac{E[e^{mYz}]-1}{m}\right)}e^{rt}e^{x\left(\frac{E[e^{mYz}(e^{mYt}-1)]-1}{m}\right)}\\ &= \sum_{l=0}^{\infty}D_{m,r}^{Y}(l,x)\frac{z^l}{l!}e^{rt}\sum_{j=0}^{\infty}\frac{x^j}{j!}\left(\frac{E[e^{mYz}(e^{mYt}-1)]}{m}\right)^j\\ &= \sum_{l=0}^{\infty}D_{m,r}^{Y}(l,x)\frac{z^l}{l!}e^{rt}\sum_{j=0}^{\infty}\frac{x^j}{j!m^j}E[e^{mS_jz}(e^{mY_1t}-1)\dots(e^{mY_jt}-1)]\\ &= \sum_{l=0}^{\infty}D_{m,r}^{Y}(l,x)\frac{z^l}{l!}e^{rt}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{m^{i-j}x^j}{j!}\sum_{\substack{l_1+\dots+l_j=i\\ l_1,l_2,\dots,l_j\geq 1}}\binom{i}{l_1,\dots,l_j}\times E[Y_1^{l_1}Y_2^{l_2}\dots Y_j^{l_j}e^{mS_jz}]\frac{t^i}{i!}\end{aligned} \tag{27}$$

$$= \sum_{l=0}^{\infty} D_{m,r}^{Y}(l,x)\frac{z^l}{l!}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\binom{k}{i}r^{k-i}\sum_{j=0}^{i}\frac{m^{i-j}x^j}{j!}\sum_{\substack{l_1+\ldots+l_j=i\\ l_1,l_2,\ldots,l_j\geq 1}}\binom{i}{l_1,\ldots,l_j}\times E[Y_1^{l_1}Y_2^{l_2}\ldots Y_j^{l_j}e^{mS_j z}]\frac{t^k}{k!}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}D_{m,r}^{Y}(l,x)\sum_{i=0}^{k}\binom{k}{i}r^{k-i}\sum_{j=0}^{i}\frac{m^{i-j}}{j!}x^j\sum_{\substack{l_1+\ldots+l_j=i\\ l_1,l_2,\ldots,l_j\geq 1}}\binom{i}{l_1,\ldots,l_j}\times E[Y_1^{l_1}Y_2^{l_2}\ldots Y_j^{l_j}(mS_j)^{n-l}]\frac{z^n}{n!}\frac{t^k}{k!}.$$

Thus, by (26) and (27), we obtain

$$D_{m,r}^{Y}(n+k,x) = \sum_{l=0}^{n}\binom{n}{l}m^{n-l}D_{m,r}^{Y}(l,x)\sum_{i=0}^{k}\binom{k}{i}r^{k-i}\sum_{j=0}^{i}\frac{m^{i-j}}{j!}x^j \times \sum_{\substack{l_1+\ldots+l_j=i\\ l_1,l_2,\ldots,l_j\geq 1}}\binom{i}{l_1,\ldots,l_j}E\left[\prod_{m=1}^{j}Y_m^{l_m}S_j^{n-l}\right]. \tag{28}$$

When $Y = 1$, we have

$$D_{m,r}(n+k,x) = \sum_{l=0}^{n}\sum_{j=0}^{k}W_{m,r}(k,j)x^j\binom{n}{l}D_{m,r}(l,x)m^{n-l}j^{n-l}, \quad (n,k\geq 0).$$

For this, one has to observe that

$$W_{m,r}(k,j) = \frac{1}{m^j}\sum_{i=j}^{k}\binom{k}{i}r^{k-i}m^i\frac{1}{j!}\sum_{\substack{l_1+\ldots+l_j=i\\ l_1,\ldots,l_j\geq 1}}\binom{i}{l_1,\ldots,l_j} = \frac{1}{m^j}\sum_{i=j}^{k}\binom{k}{i}r^{k-i}m^i\left\{\begin{matrix} i\\ j\end{matrix}\right\},$$

where the last identity follows from (8).

Theorem 7. *For $n, k \geq 0$, we have*

$$D_{m,r}^{Y}(n+k,x) = \sum_{l=0}^{n}\binom{n}{l}m^{n-l}D_{m,r}^{Y}(l,x)\sum_{i=0}^{k}\binom{k}{i}r^{k-i}\sum_{j=0}^{i}\frac{m^{i-j}}{j!}x^j\sum_{\substack{l_1+\ldots+l_j=i\\ l_1,l_2,\ldots,l_j\geq 1}}\binom{i}{l_1,\ldots,l_j}E\left[\prod_{m=1}^{j}Y_m^{l_m}S_j^{n-l}\right].$$

3 Conclusion

Let Y be a random variable such that the moment generating function of Y exists in a neighborhood of the origin. We derived recurrence relations for the probabilistic Dowling polynomials associated with Y, $D_m^Y(n,x)$ and the probabilistic r-Dowling polynomials associated with Y, $D_{m,r}^Y(n,x)$, which generalized the recurrence relation for Bell polynomials due to Gould-Quaintance. In detail, an expression for $W_m^Y(n,k)$ was derived in Theorem 1. We obtained the generating function and a recurrence relation of $D_m^Y(n,x)$, respectively, in Theorem 2 and 3. An expression for $W_{m,r}^Y(n,k)$ was given in Theorem 4. We found the generating function and an expression for $D_{m,r}^Y(n,x)$, respectively, in Theorems 5 and 6. Finally, we derived a recurrence relation for $D_{m,r}^Y(n,x)$ in Theorem 7.

As one of our future projects, we would like to continue to study probabilistic extensions of many special polynomials and numbers and to find their applications to physics, science, and engineering as well as to mathematics.

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References

[1] M. Abbas and S. Bouroubi, *On new identities for Bell's polynomials*, Discrete Math. **293** (2005), no. 1–3, 5–10, DOI: https://doi.org/10.1016/j.disc.2004.08.023.
[2] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1992.
[3] H. Belbachir and I. E. Bousbaa *Translated Whitney and r-Whitney numbers: a combinatorial approach*, J. Integer Seq. **16** (2013), no. 8, 13.8.6.
[4] K. Boubellouta, A. Boussayoud, S. Araci, and M. Kerada, *Some theorems on generating functions and their applications*, Adv. Stud. Contemp. Math. (Kyungshang) **30** (2020), no. 3, 307–324.
[5] A. Z. Broder, *The r-Stirling numbers*, Discrete Math. **49** (1984), no. 3, 241–259, DOI: https://doi.org/10.1016/0012-365X(84)90161-4.
[6] L. Carlitz, *Some remarks on the Bell numbers*, Fibonacci Quart. **18** (1980), no. 1, 66–73, DOI: https://doi.org/10.1080/00150517.1980.12430191.
[7] L. Carlitz, *Weighted Stirling numbers of the first and second kind II*, Fibonacci Quart. **18** (1980), no. 3, 242–257, DOI: https://doi.org/10.1080/00150517.1980.12430154.
[8] L. Carlitz, *Weighted Stirling numbers of the first and second kind I*, Fibonacci Quart. **18** (1980), no. 2, 147–162, DOI: https://doi.org/10.1080/00150517.1980.12430168.
[9] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co., Dordrecht, 1974.
[10] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd edition, Addison-Wesley Publishing Company, Reading, MA, 1994.
[11] N. Kilar and Y. Simsek, *Combinatorial sums involving Fubini type numbers and other special numbers and polynomials: approach trigonometric functions and p-adic integrals*, Adv. Stud. Contemp. Math. (Kyungshang) **31** (2021), no. 1, 75–87.
[12] J. Riordan, *An Introduction to Combinatorial Analysis*, Dover Publications, Inc., Mineola, NY, 2002.
[13] J. Riordan, *Combinatorial Identities*, John Wiley & Sons, Inc., New York-London-Sydney, 1968.
[14] S. Roman, *The Umbral Calculus*, Pure and Applied Mathematics, vol. 111, Academic Press, Inc. [Harcourt Brace Jovanovich, publishers], New York, 1984, DOI: https://doi.org/10.1007/978-1-4757-2178-2_17.
[15] S. M. Ross, *Introduction to Probability Models*, 13th edition, Academic Press, London, 2024.
[16] S. Soni, P. Vellaisamy, and A. K. Pathak, *A probabilistic generalization of the Bell polynomials*, J. Anal. **32** (2024), 711–732, DOI: https://doi.org/10.1007/s41478-023-00642-y.
[17] B. Q. Ta, *Probabilistic approach to Appell polynomials*, Expo. Math. **33** (2015), no. 3, 269–294, DOI: https://doi.org/10.1016/j.exmath.2014.07.003.
[18] Y. Zheng and N. N. Li, *Bivariate extension of Bell polynomials*, J. Integer Seq. **22** (2019), no. 8, 19.8.8.
[19] M. Z. Spivey, *A generalized recurrence for Bell numbers*, J. Integer Seq. **11** (2008), no. 2, 08.2.5.
[20] H. W. Gould and J. Quaintance, *Implications of Spiveyas Bell number formula*, J. Integer Seq. **11** (2008), no. 3, 08.3.7.
[21] D. S. Kim and T. Kim, *Moment representations of fully degenerate Bernoulli and degenerate Euler polynomials*, Russ. J. Math. Phys. **31** (20224), no. 4, 682–690, DOI: https://doi.org/10.1134/S1061920824040071.
[22] T. Kim and D. S. Kim, *Degenerate Whitney numbers of first and second kind of Dowling lattices*, Russ. J. Math. Phys. **29** (2022), no. 3, 358–377, DOI: https://doi.org/10.1134/S1061920822030050.
[23] T. Kim and D. S. Kim, *Combinatorial identities involving degenerate harmonic and hyperharmonic numbers*, Adv. Appl. Math. **148** (2023), 102535, DOI: https://doi.org/10.1016/j.aam.2023.102535.
[24] J. A. Adell, *Probabilistic Stirling numbers of the second kind and applications*, J. Theoret. Probab. **35** (2022), no. 1, 636–652, DOI: https://doi.org/10.1007/s10959-020-01050-9.
[25] T. Kim and D. S. Kim, *Probabilistic degenerate Bell polynomials associated with random variables*, Russ. J. Math. Phys. **30** (2023), no. 4, 528–542, DOI: https://doi.org/10.1134/S106192082304009X.