Research Article

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Singular direction of meromorphic functions with finite logarithmic order

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Abstract: In this article, we construct filling disks for meromorphic functions of order zero and that way we prove the existence of Borel directions of these functions. In the latter part of this article, we demonstrate the existence of filling disks using the Borel direction of meromorphic functions.

Keywords: meromorphic function, finite logarithmic order, Borel direction, filling disks

MSC 2020: 30D30, 30D35

1 Introduction

We assume the reader is familiar with the basic notion of Nevanlinna's value distribution theory such as $T(r,f), m(r,f), N(r,f), n(r,f)$, and so on, which can be found, for instance, in [1–4]. The theory of finite positive-order meromorphic functions is well known to be more affluent than the theory of zero-order meromorphic functions. Techniques that work well for functions of finite positive-order often do not work for functions of zero order.

There are many results on the Borel direction of meromorphic functions [4]. At the end of the nineteenth century, Picard and Borel obtained Picard's theorem and Borel's theorem, respectively. In studying the behavior of an entire or meromorphic function near a ray, Milloux [5] introduced the concept of filling disks.

In 1928, Valiron [6] obtained the Borel direction for meromorphic functions of finite positive order based on Nevanlinna's theory.

Rauch [7] proved in 1933 that a sequence of filling disks can be obtained from the Borel direction of meromorphic functions. Shortly after this, Hiong [8] obtained the result about the Borel direction for meromorphic functions of infinite order.

In 1982, Yang [4] improved the proof of these theorems. The natural question is whether a sequence of filling disks exists for meromorphic functions of finite logarithmic order and whether a sequence of filling disks can be obtained from the Borel direction of the function. Results on the Borel direction of meromorphic function have also been studied by many scholars in recent decades [9–16].

Valiron [17] was the first to investigate the Borel direction for meromorphic functions of zero order. In 1995, Rossi [18] studied filling disks for meromorphic functions satisfying the growth condition (1.1). In order to facilitate the study of meromorphic functions of zero order, Chern [19,20] obtained some results on meromorphic functions of zero order using the concept of logarithmic order

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log\log r}.$$

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In 2004, he used the Ahlfors theory to obtain the existence of the Borel direction for meromorphic functions of zero order.

Theorem A. [21] *Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ with finite logarithmic order ρ, if $f(z)$ satisfies the growth condition*

$$\overline{\lim_{r\to\infty}}\frac{T(r,f)}{(\log r)^2}=\infty, \tag{1.1}$$

then there exists a direction $\arg z=\theta_0$ $(0\le\theta_0<2\pi)$, *such that for every small positive number ε and every $a\in\hat{\mathbb{C}}$, the equation*

$$\overline{\lim_{r\to\infty}}\frac{\log n(r,\theta_0,\varepsilon,f=a)}{\log\log r}=\rho-1$$

holds with at most two possible exceptional values of a, where $n(r,\theta_0,\varepsilon,f=a)$ denotes the number of zeros of $f(z)-a$ in the angular region $\{z:|\arg z-\theta_0|\le\varepsilon,|z|<r\}$.

The ray $\arg z=\theta_0$ in Theorem A is called a Borel direction of finite logarithmic order $\rho-1$ for $f(z)$. It is natural to wonder if we can determine the existence of Borel directions for meromorphic functions of zero order by constructing the filling disks.

In 2004, Wang [22] obtained the following results by way of constructing the filling disks.

Theorem B. [22] *Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ with finite logarithmic order ρ, then there exists a direction* $\arg z=\theta_0$ $(0\le\theta_0<2\pi)$, *such that for every small positive number ε and every $a\in\hat{\mathbb{C}}$, the equation*

$$\overline{\lim_{r\to\infty}}\frac{\log n(r,\theta_0,\varepsilon,f=a)}{\log\log r}\ge\rho-2$$

holds with at most two possible exceptional values of a.

However, we found that his result is inaccurate compared to Chern's. Next, we construct a sequence of filling disks using the method of Yang [4] to obtain the following two theorems.

Theorem 1.1. *Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ with finite logarithmic order, that for every sufficiently large R and $k>1$, satisfies the following inequality in the annulus $r<|z|<R$:*

$$T(R,f)\ge\max\left\{240,\frac{240\log\frac{R}{r}}{\log k},12T(r,f),\frac{12T(kr,f)}{\log k}\log\frac{R}{r}\right\}. \tag{1.2}$$

Then, there must exist a point z_j in the annulus $r<|z|<R$ such that $f(z)$ takes every complex number at least $n=C\frac{T(R,f)}{q^2\log q\log\frac{R}{r}\log\log\frac{R}{r}}$ times in $|z-z_j|<\frac{4\pi}{\log q-1}|z_j|$, where C is a constant and q is sufficiently large positive integer, except possibly for those numbers contained in two spherical disks each with radius e^{-n}.

Theorem 1.2. *Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ with finite logarithmic order ρ. If $f(z)$ satisfies the growth condition* (1.1), *then there exists a sequence of disks*

$$\Gamma_j:|z-z_j|<\varepsilon_j|z_j|,\quad \lim_{j\to\infty}\varepsilon_j=0,\quad \lim_{j\to\infty}|z_j|=\infty\quad(j=1,2,\ldots),$$

such that $f(z)$ takes every complex number at least $(\log|z_j|)^{\rho-1-\delta_j}$ *times in Γ_j, where* $\lim_{j\to\infty}\delta_j=0$, *except possibly for those numbers contained in two spherical disks each with radius* $e^{-(\log|z_j|)^{\rho-1-\delta_j}}$.

Theorem A can be obtained by using the method of Yang in Theorem 3.8 of [4] from the aforementioned two theorems. This means that we obtain another proof for Theorem A. Conversely, given the Borel direction of the function, can we derive a sequence of filling disks? The answer is given in Theorem 1.5.

Theorem 1.3. *Let $f(z)$ be a meromorphic function in the angular domain $|\arg z - \theta_0| < \eta$, and there are three distinct values a_v $(v = 1, 2, 3)$ and a positive number σ, such that*

$$\sum_j (\log r_j(\theta_0, \eta, f = a_v))^{-\sigma} \tag{1.3}$$

converges, where $v = 1, 2, 3$, and $r_j(\theta_0, \eta, f = a_v)$ $(j = 1, 2, \ldots)$ denote the moduli of the zeros of $f(z) - a_v$ in the domain $\{z : |\arg z - \theta_0| < \eta, |z| > 1\}$, arranged by nondecreasing order and counted with their multiplicities. Then, $\sum_j (\log r_j(\theta_0, \eta - \varepsilon, f = a))^{-\sigma}$ is convergent, for any positive number $\varepsilon < \eta$ and all $a \in \hat{\mathbb{C}}$, possibly except at most for a set whose line measure is zero.

Theorem 1.4. *Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ with finite logarithmic order $\lambda + 1$, if $f(z)$ has no Borel direction of finite logarithmic order λ in the angular domain $\theta_1 < \arg z < \theta_2$, then for any small positive number α, there exist three distinct values a_v $(v = 1, 2, 3)$ and a positive number τ $(< \lambda)$, such that*

$$\sum_{v=1}^{3} n(r, \theta_1 + \alpha, \theta_2 - \alpha, f = a_v) < (\log r)^{\tau},$$

where $n(r, \theta_1 + \alpha, \theta_2 - \alpha, f = a_v)$ denotes the number of zeros of $f(z) - a$ in the domain $\{z : \theta_1 + \alpha \leq \arg z \leq \theta_2 - \alpha, 1 < |z| < r\}$.

Theorem 1.5. *Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ with finite logarithmic order $\lambda + 1 > 2$, if*

$$B : \arg z = \theta_0, \quad 0 \leq \theta_0 < 2\pi$$

is a Borel direction of finite logarithmic order λ for $f(z)$, then there exists a sequence of disks

$$\Gamma_j : |z - z_j| < \varepsilon_j |z_j|, \quad \lim_{j\to\infty} \varepsilon_j = 0, \quad \lim_{j\to\infty} |z_j| = \infty \quad (j = 1, \ 2, \ \ldots)$$

such that $f(z)$ takes every complex number at least $(\log|z_j|)^{\lambda - \delta_j}$ times in Γ_j, where $\lim_{j\to\infty}\delta_j = 0$, except possibly for those numbers contained in two spherical disks each with radius j^{-3}.

2 Some lemmas

The following lemmas will be needed in the sequel.

Lemma 2.1. [4] *Assume that $f(z)$ is a meromorphic function in the complex plane $\mathbb{C}$, and D is a bounded region. Divide D into p subregions D_j $(j = 1, 2, \ldots, p)$. For each D_j, make a disk $K_j \supset D_j$, and then make a concentric disk K_j' of K_j, such that the radius is twice as large as K_j. If the set of complex number a such that $n(D, f = a) \geq N$ cannot be covered by a collection of spherical disks with radius total equal to $\frac{1}{2}$ on the Riemann sphere, then there must exist a circle K_j' with a constant C such that $f(z)$ takes every complex number at least $C\frac{N}{p}$ times in K_j', except possibly for those numbers contained in two spherical disks each with radius $e^{-C\frac{N}{p}}$.*

Lemma 2.2. [4] *Assume that $f(z)$ is a meromorphic function in the disk $|z| < R$, and let*

$$N = n(R, f = a_1) + n(R, f = a_2) + n(R, f = a_3),$$

where a_1, a_2, and a_3 are three distinct complex numbers, with their spherical distances larger than a positive number d. Then, there exists a point z_0, with $|z_0| < R$, such that for every $r \in (0, R)$, and any complex number a,

$$n(r, f = a) < \frac{CR^2}{(R-r)^2} \times \left\{(N+1)\log\frac{2R}{R-r} + \log^+\frac{1}{d} + \log\frac{1}{|f(z_0), a|}\right\},$$

where C is a positive constant, and $|f(z_0), a|$ denotes the spherical distance between $f(z_0)$ and a.

Lemma 2.3. [4] *Let a_μ ($\mu = 1, 2, \ldots, n$) be n complex numbers and h a positive number. Then, the points that satisfy the inequality*

$$\prod_{\mu=1}^{n} |z, a_\mu| \leq \left(\frac{h}{e}\right)^n$$

can be covered by a collection of spherical disks whose number does not exceed n and the sum of whose radii does not exceed 2h.

Lemma 2.4. *Assume that $f(z)$ is a meromorphic function in the complex plane $\mathbb{C}$, and let r_j ($j = 1, 2, \ldots$) be the moduli of the poles of $f(z)$ in the domain $\{z : |z| > 1\}$, with $r_j(a) \leq r_{j+1}(a)$. For any $r_0 > 1$, if $\sigma > 0$, then the series $\sum(\log r_j)^{-\sigma}$ and $\int_{r_0}^{\infty}\frac{n(t,f)}{(\log t)^{\sigma+1}t}\mathrm{d}t$ are either simultaneously convergent or simultaneously divergent.*

Proof. For any $R > r_0$ with $r_0 > 1$, let $z_{j_0}, \ldots, z_J$ be the poles of f within the annulus $\{z : r_0 < |z| < R\}$, and denote $r_{j_0} = |z_{j_0}|, \ldots, r_J = |z_J|$, arranged by nondecreasing order and counted with their multiplicities. It follows from the equation

$$\sum_{j=j_0}^{J}\frac{1}{(\log r_j)^\sigma} = \int_{r_0}^{R}\frac{1}{(\log t)^\sigma}\mathrm{d}n(t,f) = \frac{n(t,f)}{(\log t)^\sigma}\Bigg|_{r_0}^{R} + \int_{r_0}^{R}\frac{\sigma n(t,f)}{(\log t)^{\sigma+1}t}\mathrm{d}t, \tag{2.1}$$

if $\sum(\log r_j)^{-\sigma}$ converges, then $\int_{r_0}^{\infty}\frac{n(t,f)}{(\log t)^{\sigma+1}t}\mathrm{d}t$ also converges.

Conversely, if the series $\int_{r_0}^{\infty}\frac{n(t,f)}{(\log t)^{\sigma+1}t}\mathrm{d}t$ converges, combine

$$\lim_{R\to\infty}\frac{n(R,f)}{\sigma(\log R)^\sigma} = \lim_{R\to\infty} n(R,f)\int_{R}^{\infty}\frac{1}{(\log t)^{\sigma+1}t}\mathrm{d}t \leq \lim_{R\to\infty}\int_{R}^{\infty}\frac{n(t,f)}{(\log t)^{\sigma+1}t}\mathrm{d}t = 0.$$

with (2.1), $\sum(\log r_j)^{-\sigma}$ also converging.

So $\sum(\log r_j)^{-\sigma}$ and $\int_{r_0}^{\infty}\frac{n(t,f)}{(\log t)^{\sigma+1}t}\mathrm{d}t$ are either simultaneously convergent or simultaneously divergent. □

Similarly, the following lemma can be obtained in the angular domain.

Lemma 2.5. *Assume that $f(z)$ is a meromorphic function in the complex plane $\mathbb{C}$, $0 \leq \theta_0 < 2\pi$, $\eta > 0$, $a \in \mathbb{C}$. Let $r_j(\theta_0, \eta, f = a)$ be the moduli of the a-points of $f(z)$ in the domain $\{z : |\arg z - \theta_0| < \eta, |z| > 1\}$, with $r_j(a) \leq r_{j+1}(a)$. For any $r_0 > 1$, if $\sigma > 0$, then the series $\sum(\log r_j(\theta_0, \eta, f = a))^{-\sigma}$ and $\int_{r_0}^{\infty}\frac{n(t,\theta_0,\eta,f=a)}{(\log t)^{\sigma+1}t}\mathrm{d}t$ are either simultaneously convergent or simultaneously divergent.*

3 Proofs of Theorems 1.1 and 1.2

In this section, the proof method is based on the content of [4] (Section 3 in Chapter 3), but there are some differences in the details of constructing the filling disks. This construction method provides a different perspective for proving Theorem A.

Proof of Theorem 1.1. Assume that $f(z)$ is a meromorphic function in the complex plane $\mathbb{C}$, by the definitions of $n(r, f = a)$ and $N(r, f = a)$, we have

$$n(R, f = a) = \frac{n(R, f = a)}{\log\frac{R}{r}}\int_r^R \frac{dt}{t} \geq \frac{N(R, f = a) - N(r, f = a)}{\log\frac{R}{r}} \tag{3.1}$$

and

$$n(r, f = a) = \frac{n(r, f = a)}{\log k}\int_r^{kr} \frac{dt}{t} \leq \frac{1}{\log k}N(kr, f = a). \tag{3.2}$$

From the second fundamental theorem of Nevanlinna, for suitably large R and every complex number a, we have

$$N(R, f = a) > \frac{T(R, f)}{4}, \quad |f(0), a| > \frac{1}{3}, \tag{3.3}$$

except possibly for those numbers contained in a sequence of disks with total spherical radius $\frac{1}{2}$. Combining (1.2) with (3.3), we have

$$\begin{aligned}
n(R, f = a) - n(r, f = a) &\geq \frac{N(R, f = a)}{\log\frac{R}{r}} - \frac{N(r, f = a)}{\log\frac{R}{r}} - \frac{N(kr, f = a)}{\log k}\\
&\geq \frac{\frac{1}{4}T(R, f)}{\log\frac{R}{r}} - \frac{T(r, f = a)}{\log\frac{R}{r}} - \frac{T(kr, f = a)}{\log k}\\
&\geq \frac{\frac{1}{4}T(R, f)}{\log\frac{R}{r}} - \frac{T(r, f) + \log\frac{1}{|f(0), a|} + \log 2}{\log\frac{R}{r}} - \frac{T(kr, f) + \log\frac{1}{|f(0), a|} + \log 2}{\log k}\\
&\geq \frac{T(R, f)}{4\log\frac{R}{r}}\left\{1 - \frac{4T(r, f)}{T(R, f)} - \frac{4\log 6}{T(R, f)} - \frac{4T(kr, f)}{T(R, f)}\cdot\frac{\log\frac{R}{r}}{\log k} - \frac{4\log 6}{T(R, f)}\cdot\frac{\log\frac{R}{r}}{\log k}\right\}\\
&\geq \frac{T(R, f)}{15\log\frac{R}{r}}.
\end{aligned}$$

Taking a sufficiently large positive integer q, the annulus $r < |z| < R$ is divided as follows:

Make q rays from the origin, the angle between two consecutive rays is $\frac{2\pi}{q}$, and then make $1 + s[\log q]$ circles

$$\begin{aligned}
|z| = r, r + \frac{r(e^{(1+2\pi/q)} - 1)}{[\log q]}, r + \frac{2r(e^{(1+2\pi/q)} - 1)}{[\log q]}, \ldots, re^{(1+2\pi/q)},\\
re^{(1+2\pi/q)} + \frac{r(e^{(1+2\pi/q)^2} - e^{(1+2\pi/q)})}{[\log q]}, \ldots, re^{(1+2\pi/q)^2},\\
\ldots\\
re^{(1+2\pi/q)^{s-1}} + \frac{r(e^{(1+2\pi/q)^s} - e^{(1+2\pi/q)^{s-1}})}{[\log q]}, \ldots, \ldots, re^{(1+2\pi/q)^s},
\end{aligned}$$

where $s = \left\{\left[\frac{\log\log\frac{R}{r}}{\log(1 + \frac{2\pi}{q})}\right] + 1\right\}$.

Thus, the annulus $r < |z| < R$ is divided into p curved quadrilaterals D_j, where

$$p \leq q\left\{\left[\frac{\log\log\frac{R}{r}}{\log\left(1 + \frac{2\pi}{q}\right)}\right] + 1\right\}[\log q] < \frac{q^2\log q\log\log\frac{R}{r}}{2}.$$

Let z_j $(j = 1, \dots, p)$ be the center of D_j, then $D_j \subset K_j = \{z : |z - z_j| < \frac{2\pi}{\log q - 1}|z_j|\}$. The division of the annulus is shown in Figure 1.

By Lemma 2.1, then there must exist a concentric circle $K_j' = \{z : |z - z_j| < \frac{4\pi}{\log q - 1}|z_j|\}$ with a constant C, such that $f(z)$ takes every complex number at least $n = C\frac{T(R,f)}{q^2 \log q \log \frac{R}{r} \log\log \frac{R}{r}}$ times in K_j', except possibly for those numbers contained in two spherical disks each with radius e^{-n}. □

Proof of Theorem 1.2. Since $f(z)$ is a meromorphic function with finite logarithmic order ρ and (1.1) holds, there exists a sequence $\{r_k\}$ that tends to infinity such that

$$\lim_{k\to\infty} \frac{T(r_k, f)}{(\log r_k)^2} = \infty,$$
$$\lim_{k\to\infty} \frac{\log T(r_k, f)}{\log\log r_k} = \rho.$$

Taking a suitably large r_{k_1}, such that

$$T(r_{k_1}, f) \geq \max\left\{240, \frac{240\log(r_{k_1})}{\log 2}, 12T(1, f), \frac{12T(2, f)}{\log 2}\log r_{k_1}\right\},$$

and

$$q_1 = \log\log r_{k_1}.$$

By Theorem 1.1, there exists a point $|z_1|$ in the annulus $1 < |z| < r_{k_1}$ such that $f(z)$ takes every complex number at least $n_1 = C\frac{T(r_{k_1}, f)}{\log r_{k_1}(\log\log r_{k_1})^3 \log\log\log r_{k_1}}$ times in $\Gamma_1 : |z - z_1| < \frac{4\pi}{\log\log\log r_{k_1} - 1}|z_1|$, except possibly for those numbers contained in two spherical disks each with radius e^{-n_1}, where C is a constant.

Take $r_{k_1'}$ such that $r_{k_1'} > 2r_{k_1}$. Then, take r_{k_2} such that

$$T(r_{k_2}, f) \geq \max\left\{240, \frac{240\log r_{k_2}}{\log 2}, 12T(r_{k_1'}, f), \frac{12T(2r_{k_1'}, f)}{\log 2}\log r_{k_2}\right\},$$

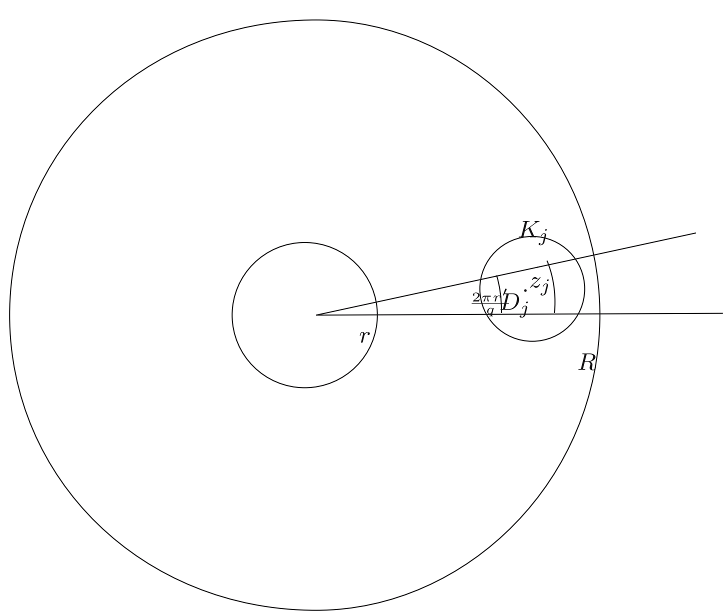

Figure 1: The division of the annulus.

and

$$q_2 = \log\log r_{k_2}.$$

By Theorem 1.1, there exists a point $|z_2|$ in the annulus $r_{k_1'} < |z| < r_{k_2}$ such that $f(z)$ takes every complex number at least $n_2 = C\frac{T(r,f)}{\log r_{k_2}(\log\log r_{k_2})^3 \log\log\log r_{k_2}}$ times in $\Gamma_2 : |z - z_2| < \frac{4\pi}{\log\log\log r_{k_2} - 1}|z_2|$, except possibly for those numbers contained in two spherical disks each with radius e^{-n_2}, where C is a constant.

By induction, we can obtain a sequence of disks

$$\Gamma_j : |z - z_j| < \varepsilon_j|z_j|, \quad \lim_{j\to\infty}\varepsilon_j = 0, \quad \lim_{j\to\infty}|z_j| = \infty \quad (j = 1, 2, \ldots)$$

such that $f(z)$ takes every complex number at least

$$n_j = C\frac{T(r,f)}{\log r_{k_j}(\log\log r_{k_j})^3 \log\log\log r_{k_j}} > (\log r_{k_j})^{\rho-1-\delta_j} = (\log|z_{k_j}|)^{\rho-1-\delta_j}$$

times in Γ_j, except possibly for those numbers contained in two spherical disks each with radius $e^{-(\log|z_j|)^{\rho-1-\delta_j}}$, where C is a constant and $\lim_{j\to\infty}\delta_j = 0$. □

4 Proof of Theorems 1.3 and 1.5

In [4] (Section 4 in Chapter 3), the relevant results about finite-order meromorphic functions are given. Here, we use the identical method but the quadrangles we split are different.

Proof of Theorem 1.3. Split the angular domain $|\arg z - \theta_0| < \eta - \varepsilon$ with rays from the origin so that the angle of each small angular domain does not exceed $\frac{\varepsilon}{4}$, the total number of these small angular domains is $J \geq \left[\frac{4(\eta-\varepsilon)}{\frac{\varepsilon}{4}} + 1\right]$. Then, take a suitably large positive number $r_0 > 1$, take the origin as the center, and make circles $|z| = r_0, r_0\exp\{1 + \frac{\varepsilon}{4}\}, r_0\exp\{(1 + \frac{\varepsilon}{4})^2\}, \ldots$. such that $(|z| > r_0) \cap (|\arg z - \theta_0| < \eta - \varepsilon)$ is divided into some small curved quadrilateral Ω_{jl} $(j = 1, 2, \ldots, J;\ l = 1, 2, \ldots, L)$. For each Ω_{jl}, there exists a small disk Γ_{jl} and a disk Γ'_{jl} of radius two times its radius, such that $\Omega_{jl} \subset \Gamma_{jl} \subset \Gamma'_{jl} \subset (|\arg z - \theta_0| < \eta)$. It is easy to see that for each $\Gamma'_{j_0 l_0}$, there is a fixed upper bound for the number of other Γ'_{jl} intersecting it. At the same time, combining (1.3) with Lemma 2.5, we obtain

$$\int_{r_0}^{\infty}\frac{n(t,\theta_0,\eta,f=a_\nu)}{(\log t)^{\sigma+1}t}\mathrm{d}t < \infty \quad (\nu = 1, 2, 3). \tag{4.1}$$

By Lemmas 2.2 and 2.3, for any $a \in \hat{\mathbb{C}}$, the equation

$$n(\Omega_{jl}, f = a) < C\left\{\sum_{\nu=1}^{3} n(\Gamma'_{jl}, f = a_\nu) + \log\frac{1}{\exp\left\{-\frac{\sigma}{2}\left\{\log|r_0| + (1 + \frac{\varepsilon}{4})^l\right\}\right\}}\right\} \tag{4.2}$$

holds in each Γ'_{jl}, except possibly for those numbers contained in the disk D_{jl} with spherical radius $\exp\left\{-\frac{\sigma}{2}\left\{\log|r_0| + (1 + \frac{\varepsilon}{4})^l\right\}\right\}$.

Let $D_l = \cup_{j=1}^{J} D_{jl}$, $D = \cap_{k=1}^{\infty}(\cup_{l=k}^{\infty} D_l)$, and we see that $\mathrm{mes}D = \lim_{k\to\infty}\mathrm{mes}(\cup_{l=k}^{\infty} D_l) = 0$.

For any complex $a \notin D$, there exists l_0, such that $a \notin \cup_{l=l_0}^{\infty} D_l$. So when $l \geq l_0$, (4.2) holds for $j = 1, 2, \ldots, J$.

Thus,

$$\sum_{l=l_0}^{L}\sum_{j=1}^{J} n(\Omega_{jl}, f=a) < C\left\{\sum_{l=l_0}^{L}\sum_{j=1}^{J}\sum_{\nu=1}^{3} n(\Gamma'_{jl}, f=a_\nu) + J\sum_{l=l_0}^{L}\frac{\sigma}{2}\left\{\log|r_0| + \left(1+\frac{\varepsilon}{4}\right)^l\right\}\right\},$$

which implies

$$n(r,\theta_0,\eta-\varepsilon, f=a) < C\left\{\sum_{\nu=1}^{3} n(2r,\theta_0,\eta-\varepsilon, f=a_\nu) + \log r\right\}.$$

By (4.1), for any complex $a \notin D$, we have

$$\int_{r_0}^{\infty}\frac{n(r,\theta_0,\eta-\varepsilon, f=a)}{(\log r)^{\sigma+1} r}\mathrm{d}r < \infty.$$

Furthermore, by Lemma 2.5, the theorem is proved. □

Proof of Theorem 1.4. According to the conditions of the theorem, for any $\varphi \in [\theta_1+\alpha, \theta_2-\alpha]$, $\arg z = \varphi$ is not a Borel direction of finite logarithmic order for $f(z)$. Nevertheless, there must exist three distinct values β_j $(j = 1, 2, 3)$ and positive numbers $\varepsilon(\varphi)$, $\tau(\varphi)(<\lambda)$, such that

$$n(r,\varphi,\varepsilon(\varphi), f=\beta_j) < (\log r)^{\tau(\varphi)} \quad (j=1,2,3).$$

It follows that, for $\tau_1(\varphi)(\tau(\varphi) < \tau_1(\varphi) < \lambda)$ and $r_0 > 1$, we have

$$\int_{r_0}^{\infty}\frac{n(t,\varphi,\varepsilon(\varphi), f=\beta_j)}{(\log t)^{\tau_1(\varphi)+1} t}\mathrm{d}t < \infty \quad (j=1,2,3).$$

By Lemma 2.5, the series $\sum(\log r(\varphi,\varepsilon(\varphi), f=\beta_j))^{-\tau_1(\varphi)}$ is convergent. As a result of applying Theorem 1.3, for any complex number a, the series $\sum(\log r(\varphi, \frac{\varepsilon(\varphi)}{2}, f=a))^{-\tau_1(\varphi)}$ is convergent possibly except for at most a set $D(\varphi)$ whose line measure is zero.

We note that $\{(\varphi - \frac{\varepsilon(\varphi)}{2}, \varphi + \frac{\varepsilon(\varphi)}{2}) : \theta_1+\alpha \le \varphi \le \theta_2-\alpha\}$ forms an open covering of the closed interval $[\theta_1+\alpha, \theta_2-\alpha]$. Therefore, there exists a finite set of intervals $(\varphi_l - \frac{\varepsilon_l}{2}, \varphi_l + \frac{\varepsilon_l}{2})$ $(l = 1, 2, \ldots, L)$ that are an open covering of $[\theta_1+\alpha, \theta_2-\alpha]$.

Corresponding to each φ_l, its exceptional zero measure set is $D_l = D(\varphi_l)$. Let $D = \cup_{l=1}^{L} D_l$, then it is still the zero measure set. If $\tau_1 = \max_{1\le l\le L}\{\tau_1(\varphi_l)\}$, then $0 < \tau_1 < \lambda$.

Hence, for every complex number a $(\notin D)$, we obtain

$$\sum\left(\log r_j\left(\varphi_l, \frac{\varepsilon_l}{2}, f=a\right)\right)^{-\tau_1} < \infty.$$

So

$$\sum\left(\log r_j\left(\frac{\theta_1+\theta_2}{2}, \frac{\theta_2-\theta_1}{2} - \alpha, f=a\right)\right)^{-\tau_1} < \infty.$$

By Lemma 2.5, we obtain

$$\int_{r_0}^{\infty}\frac{n(t,\theta_1+\alpha,\theta_2-\alpha, f=a)}{(\log t)^{\tau_1+1} t}\mathrm{d}t < \infty.$$

In particular, this holds for any three distinct complex numbers a_ν $(\nu = 1, 2, 3)$ that do not belong to D, and positive numbers $\tau(\tau_1 < \tau < \lambda)$. Hence, we obtain

$$\sum_{\nu=1}^{3} n(r,\theta_1+\alpha,\theta_2-\alpha, f=a_\nu) < (\log r)^{\tau}.$$

□

Proof of Theorem 1.5. Choose a small angular region with B as its bisector, which is defined by

$$\Omega := \left\{z : |\arg z - \theta_0| < \frac{\eta}{2}\right\}.$$

Using the arcs of circumferences $|z| = 2^j$ ($j = 1, 2, \ldots$), divide Ω into a sequence of small quadrangles

$$\Omega_j := \left\{z : (2^j \le |z| \le 2^{j+1}) \cup \left(|\arg z - \theta_0| < \frac{\eta}{2}\right)\right\}.$$

Subsequently, each Ω_j is further divided by $s-1$ arcs of $|z| = 2^j(1+\eta)^l$, $(l = 1, 2, \ldots, s-1)$, where s is a positive integer determined by $2^j(1+\eta)^{s-1} < 2^{j+1} \le 2^j(1+\eta)^s$. This results in s smaller quadrangles, denoted as Ω_{jl} ($l = 1, 2, \ldots, s$). Choose concentric disks Γ_{jl} and Γ'_{jl} with radius $C \cdot 2^{j+1}\eta$ and $2C \cdot 2^{j+1}\eta$, respectively, such that

$$\Omega_{jl} \subset \Gamma_{jl} \subset \Gamma'_{jl}.$$

Define n_{jl} as follows:

$$n_{jl} := \min\left\{\sum_{v=1}^{3} n(\Gamma'_{jl}, f = a_v)\right\},$$

where the minimum is taken over all the triples of complex numbers, provided that the mutual spherical distances among these three complex numbers are at least j^{-3}. By Lemma 2.2, $f(z)$ takes every complex number a at most $C\{n_{jl} + [\log j] + 1\}$ times in Ω_{jl}, except for those complex numbers contained in a disk with spherical radius j^{-3}.

For each fixed value of j, there exist s exceptional spherical disks. As j varies, the aggregate radius of these exceptional spherical disks is given by the series:

$$\sum_{j=1}^{\infty} \frac{s}{j^{-3}}.$$

By selecting a sufficiently large value for j_0, we can ensure that

$$\sum_{j=j_0}^{\infty} \frac{s}{j^{-3}} < \frac{1}{2}.$$

Since B is a Borel direction of logarithmic order λ of $f(z)$, by Lemma 2.5, the series

$$\sum\left(\log r_n\left(\theta_0, \frac{\eta}{2}, f = a\right)\right)^{-\tau}$$

diverges for any positive number $\tau < \lambda$ and every complex value a, except for at most two values. If a is not an exceptional value and does not belong to all the exceptional spherical disks, then we have

$$\sum\left(\log r_n\left(\theta_0, \frac{\eta}{2}, f = a\right)\right)^{-\tau} = \infty. \tag{4.3}$$

On the other hand, by setting $n_j = \max_{1\le l\le s} n_{jl}$, we have

$$\sum\left(\log r_n\left(\theta_0, \frac{\eta}{2}, f = a\right)\right)^{-\tau} < C\sum_j \frac{s(n_j + [\log j] + 1)}{(\log 2^j)^\tau} < C\left\{\sum_j \frac{n_j}{\eta(\log 2^j)^\tau} + \sum_j \frac{[\log j] + 1}{\eta(\log 2^j)^\tau}\right\}, \tag{4.4}$$

where $\tau < \lambda$.

Choose a decreasing sequence $\{\eta_l\}$ of positive numbers tending to zero and an increasing sequence of positive numbers $\{\tau_l\}$ tending to λ. Let $\tau_1 > 1$, then $\sum_j \frac{[\log j]+1}{\eta_1(\log 2^j)^{\tau_1}} < \infty$. Combining (4.3) with (4.4), we have $\sum_j \frac{n_j}{\eta_1(\log 2^j)^{\tau_1}} = \infty$. Thus, there is a positive integer j_1 such that

$$n_{j_1} > \frac{\eta_1(\log 2^{j_1+1})^{\tau_1}}{j_1(\log j_1)^2} > (\log 2^{j_1+1})^{2\tau_1-\lambda}.$$

Similarly, from $\sum_{j=j_1+1}^{\infty} \frac{[\log j]+1}{\eta_2(\log 2^j)^{\tau_2}} < \infty$, we have $\sum_{j=j_1+1}^{\infty} \frac{n_j}{\eta_2(\log 2^j)^{\tau_2}} = \infty$. Hence, $n_{j_2} > (\log 2^{j_2+1})^{2\tau_2-\lambda}$ for a certain positive integer j_2.

Thus, we obtain a sequence of disks $\Gamma_l : |z - z_l| < C\eta_l|z_l|$, where $\arg z_l = \theta_0$, such that $f(z)$ takes every complex value at least $(\log|z_l|)^{\lambda-2(\lambda-\tau_l)}$ times, except possibly for those values contained in two spherical disks each with radius j_l^{-3}. □

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