Research Article

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The number of rational points of some classes of algebraic varieties over finite fields

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Abstract: Let $\mathbb{F}_q$ be the finite field of characteristic p and $\mathbb{F}_q^* = \mathbb{F}_q\backslash\{0\}$. In this article, we use Smith normal form of exponent matrices to present exact formulas for the numbers of rational points on suitable affine algebraic varieties defined by the following systems of equations over $\mathbb{F}_q$:

$$\begin{cases} a_1 x_1^{e_{11}} \dots x_{m_1}^{e_{1m_1}} + \dots + a_{m_1} x_1^{e_{m_1,1}} \dots x_{m_1}^{e_{m_1,m_1}} = b_1, \\ a_{m_1+1} x_1^{e_{m_1+1,1}} \dots x_{m_2}^{e_{m_1+1,m_2}} + \dots + a_{m_2} x_1^{e_{m_2,1}} \dots x_{m_2}^{e_{m_2,m_2}} = b_2 \end{cases}$$

and

$$\begin{cases} c_1 x_1^{d_{11}} \dots x_{n_1}^{d_{1n_1}} + \dots + c_{n_1} x_1^{d_{n_1,1}} \dots x_{n_1}^{d_{n_1,n_1}} = l_1, \\ c_{n_1+1} x_1^{d_{n_1+1,1}} \dots x_{n_2}^{d_{n_1+1,n_2}} + \dots + c_{n_2} x_1^{d_{n_2,1}} \dots x_{n_2}^{d_{n_2,n_2}} = l_2, \\ c_{n_2+1} x_1^{d_{n_2+1,1}} \dots x_{n_3}^{d_{n_2+1,n_3}} + \dots + c_{n_3} x_1^{d_{n_3,1}} \dots x_{n_3}^{d_{n_3,n_3}} = l_3 \end{cases}$$

when the determinants of exponent matrices are coprime to $q-1$, where $e_{ij}, d_{i'j'} \in \mathbb{Z}^+$ (the set of positive integers), $a_i, c_{i'} \in \mathbb{F}_q^*$, $1 \le i, j \le m_2$, $1 \le i', j' \le n_3$, and $b_1, b_2, l_1, l_2, l_3 \in \mathbb{F}_q$. These formulas extend the theorem obtained by Wang and Sun (*An explicit formula of solution of some special equations over a finite field*, Chinese Ann. Math. Ser. A **26** (2005), 391–396, https://www.cqvip.com/doc/journal/977048790. (in Chinese)). Our results also give a partial answer to an open problem of Hu et al. raised in (*The number of rational points of a family of hypersurfaces over finite fields*, J. Number Theory **156** (2015), 135–153, doi: https://doi.org/10.1016/j.jnt.2015.04.006).

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1 Introduction

Let $\mathbb{Z}^+$ denote the set of positive integers. Let $\mathbb{F}_q$ be the finite field of characteristic p, and let $\mathbb{F}_q^* := \mathbb{F}_q\backslash\{0\}$ be its multiplicative group. We denote $|\mathcal{S}|$ by the number of elements of the finite set $\mathcal{S}$. For any $m \in \mathbb{Z}^+$, we define $\langle m\rangle := \{1, \ 2, \dots, m\}$. Let $f_i(x_1, \dots, x_n) (i \in \langle m\rangle)$ be a polynomial with n variables over $\mathbb{F}_q$ and let $V(f_1, \dots, f_m)$

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denote the algebraic variety defined by the simultaneous vanishing of $f_i(x_1, \dots, x_n)(i \in \langle m\rangle)$. Let $N(V)$ stand for the number of $\mathbb{F}_q$-rational points on the algebraic variety $V(f_1, \dots, f_m)$ in $\mathbb{F}_q^n$. That is,

$$N(V) = |\{(x_1, \dots, x_n) \in \mathbb{F}_q^n : f_i(x_1, \dots, x_n) = 0, i \in \langle m\rangle\}|.$$

Particularly, we denote $N(f)$ for $N(V)$ if $m = 1$. Determining the explicit value of $N(V)$ is an important subject in finite fields. In general, it is difficult to present an explicit formula for $N(V)$. It is well known that there is an exact formula for the number $N(f)$ when $\deg(f) \le 2$ (see, for example, pp. 275–289 of [1]). Finding the formula for $N(V)$ and relevant topics has attracted a lot of scholars in recent decades [2–24].

Sun [18] studied the number of rational points $(x_1, \dots, x_n) \in \mathbb{F}_q^n$ on the following affine hypersurface

$$a_1 x_1^{e_{11}} \dots x_n^{e_{1n}} + \dots + a_n x_1^{e_{n1}} \dots x_n^{e_{nn}} - b = 0$$

with $e_{ij} \in \mathbb{Z}^+$, $a_i \in \mathbb{F}_q^*$, $b \in \mathbb{F}_q$, $1 \le i, j \le n$ by proving that if $\gcd(\det(e_{ij}), q - 1) = 1$, then

$$N(f) = \begin{cases} B(n) + \dfrac{q-1}{q} A(n-1) & \text{if } b = 0, \\ \dfrac{1}{q} A(n) & \text{otherwise,} \end{cases}$$

where for any positive integers s, we have

$$B(s) := q^s - (q-1)^s \tag{1.1}$$

and

$$A(s) := (q-1)^s - (-1)^s. \tag{1.2}$$

Zhu et al. [24] considered a special variety defined by two or three equations, which is taken from the one investigated by Sun [18]. Sun's result [18] was extended by Wang and Sun [19] by presenting a formula for the number of rational points $(x_1, \dots, x_{n_2}) \in \mathbb{F}_q^{n_2}$ on the affine hypersurface

$$a_1 x_1^{d_{11}} \dots x_{n_1}^{d_{1n_1}} + \dots + a_{n_1} x_1^{d_{n_1,1}} \dots x_{n_1}^{d_{n_1,n_1}} + a_{n_1+1} x_1^{d_{n_1+1,1}} \dots x_{n_2}^{d_{n_1+1,n_2}} + \dots + a_{n_2} x_1^{d_{n_2,1}} \dots x_{n_2}^{d_{n_2,n_2}} = b$$

with $d_{ij} \in \mathbb{Z}^+$, $a_i \in \mathbb{F}_q^*$, $1 \le i, j \le n_2$. In 2015, Hu et al. [15] gave an uniform generalization to the results of [18] and [19]. In fact, they used the Smith normal form to present an explicit formula for $N(f)$ of rational points $(x_1, \dots, x_{n_t}) \in \mathbb{F}_q^{n_t}$ on the hypersurface defined by

$$f := f(x_1, \dots, x_{n_t}) = \sum_{j=0}^{t-1} \sum_{i=1}^{r_{j+1}-r_j} a_{r_j+i} x_1^{e_{r_j+i,1}} \dots x_{n_{j+1}}^{e_{r_j+i,n_{j+1}}} - b,$$

where the integers $t > 0$, $0 = r_0 < r_1 < r_2 < \dots < r_t$, $1 \le n_1 < n_2 < \dots < n_t$, $b \in \mathbb{F}_q$, $a_i \in \mathbb{F}_q^*$, and $e_{ij} \in \mathbb{Z}^+$, $i \in \langle r_t\rangle$, $j \in \langle n_t\rangle$. Zhu and Hong [23] followed the approach of [15] and gave an exact formula for the number of rational points on certain algebraic variety $V = V(f_1, f_2)$ over $\mathbb{F}_q$ as follows:

$$\begin{cases} f_1 := f_1(x_1, \dots, x_{n_t}) = \sum\limits_{i=1}^{r} a_i^{(1)} x_1^{e_{i1}^{(1)}} \dots x_n^{e_{in}^{(1)}} - b_1, \\ f_2 := f_2(x_1, \dots, x_{n_t}) = \sum\limits_{j'=0}^{t-1} \sum\limits_{i'=1}^{r_{j'+1}-r_{j'}} a_{r_{j'}+i'}^{(2)} x_1^{e_{r_{j'}+i',1}^{(2)}} \dots x_{n_{j'+1}}^{e_{r_{j'}+i',n_{j'+1}}^{(2)}} - b_2, \end{cases}$$

where $b_i \in \mathbb{F}_q$, $i = 1, 2$, $t \in \mathbb{Z}^+$, $0 = n_0 < n_1 < n_2 < \dots < n_t$, $n_{k-1} < n \le n_k$ for some $1 \le k \le t$, $0 = r_0 < r_1 < r_2 < \dots < r_t$, $a_i^{(1)}, a_{i'}^{(2)} \in \mathbb{F}_q^*$, $i \in \langle r\rangle$, $i' \in \langle r_t\rangle$, and the exponent of each variable is a positive integer.

Motivated by the works of [15,18,19,23], we consider the questions of counting $\mathbb{F}_q$-rational points of the variety $V(f_1, f_2)$ determined by

$$\begin{cases} f_1 := a_1 x_1^{e_{11}} \dots x_{m_1}^{e_{1m_1}} + \dots + a_{m_1} x_1^{e_{m_1,1}} \dots x_{m_1}^{e_{m_1,m_1}} - b_1, \\ f_2 := a_{m_1+1} x_1^{e_{m_1+1,1}} \dots x_{m_2}^{e_{m_1+1,m_2}} + \dots + a_{m_2} x_1^{e_{m_2,1}} \dots x_{m_2}^{e_{m_2,m_2}} - b_2 \end{cases} \tag{1.3}$$

and the variety $V(f_1, f_2, f_3)$ determined by

$$\begin{cases} f_1 := c_1 x_1^{d_{11}} \dots x_{n_1}^{d_{1n_1}} + \dots + c_{n_1} x_1^{d_{n_1,1}} \dots x_{n_1}^{d_{n_1,n_1}} - l_1, \\ f_2 := c_{n_1+1} x_1^{d_{n_1+1,1}} \dots x_{n_2}^{d_{n_1+1,n_2}} + \dots + c_{n_2} x_1^{d_{n_2,1}} \dots x_{n_2}^{d_{n_2,n_2}} - l_2, \\ f_3 := c_{n_2+1} x_1^{d_{n_2+1,1}} \dots x_{n_3}^{d_{n_2+1,n_3}} + \dots + c_{n_3} x_1^{d_{n_3,1}} \dots x_{n_3}^{d_{n_3,n_3}} - l_3, \end{cases} \tag{1.4}$$

where $e_{ij}, d_{i'j'} \in \mathbb{Z}^+, a_i, c_{i'} \in \mathbb{F}_q^*, 1 \le i, j \le m_2, 1 \le i', j' \le n_3$, and $b_1, b_2, l_1, l_2, l_3 \in \mathbb{F}_q$.

Let

$$\mathscr{E} = \begin{pmatrix} e_{11} & \cdots & e_{1m_1} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ e_{m_1,1} & \cdots & e_{m_1,m_1} & 0 & \cdots & 0 \\ e_{m_1+1,1} & \cdots & e_{m_1+1,m_1} & e_{m_1+1,m_1+1} & \cdots & e_{m_1+1,m_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ e_{m_2,1} & \cdots & e_{m_2,m_1} & e_{m_2,m_1+1} & \cdots & e_{m_2,m_2} \end{pmatrix} \tag{1.5}$$

with $e_{ij} (1 \le i, j \le m_2)$ being given as in (1.3), and let

$$\mathscr{F} = \begin{pmatrix} d_{11} & \cdots & d_{1n_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ d_{n_1,1} & \cdots & d_{n_1,n_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ d_{n_1+1,1} & \cdots & d_{n_1+1,n_1} & d_{n_1+1,n_1+1} & \cdots & d_{n_1+1,n_2} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ d_{n_2,1} & \cdots & d_{n_2,n_1} & d_{n_2,n_1+1} & \cdots & d_{n_2,n_2} & 0 & \cdots & 0 \\ d_{n_2+1,1} & \cdots & d_{n_2+1,n_1} & d_{n_2+1,n_1+1} & \cdots & d_{n_2+1,n_2} & d_{n_2+1,n_2+1} & \cdots & d_{n_2+1,n_3} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ d_{n_3,1} & \cdots & d_{n_3,n_1} & d_{n_3,n_1+1} & \cdots & d_{n_3,n_2} & d_{n_3,n_2+1} & \cdots & d_{n_3,n_3} \end{pmatrix} \tag{1.6}$$

with $d_{i'j'} (1 \le i', j' \le n_3)$ being given as in (1.4).

The main results of this article can be stated as follows.

Theorem 1.1. *Let $V = V(f_1, f_2)$ be the variety determined by* (1.3). *If* $\gcd(q-1, \det(\mathscr{E})) = 1$, *then*

$$N(V) = \begin{cases} q^{m_2-m_1} B(m_1) + \dfrac{q-1}{q} A(m_1-1) B(m_2-m_1) \\ + \dfrac{(q-1)^2}{q^2} A(m_1-1) A(m_2-m_1-1), & \text{if } b_1 = b_2 = 0, \\ \dfrac{A(m_1)B(m_2-m_1)}{q} + \dfrac{q-1}{q^2} A(m_1) A(m_2-m_1-1), & \text{if } b_1 \ne 0, b_2 = 0, \\ \dfrac{q-1}{q^2} A(m_1-1) A(m_2-m_1), & \text{if } b_1 = 0, b_2 \ne 0, \\ \dfrac{A(m_1)A(m_2-m_1)}{q^2}, & \textit{if } b_1 \ne 0, b_2 \ne 0. \end{cases} \tag{1.7}$$

Theorem 1.2. *Let $V = V(f_1, f_2, f_3)$ be the variety determined by* (1.4). *If* $\gcd(q-1, \det(\mathscr{F})) = 1$, *then*

$$N(V) = \begin{cases} q^{n_3-n_1}B(n_1) + (q-1)q^{n_3-n_2-1}A(n_1-1)B(n_2-n_1) + \dfrac{(q-1)^2}{q^2}A(n_1-1)A(n_2-n_1-1) \\ \times B(n_3-n_2) + \dfrac{(q-1)^3}{q^3}A(n_1-1)A(n_2-n_1-1)A(n_3-n_2-1), \quad \text{if } l_1 = l_2 = l_3 = 0, \\ q^{n_3-n_2-1}A(n_1)B(n_2-n_1) + \dfrac{q-1}{q^2}A(n_1)A(n_2-n_1-1)B(n_3-n_2) \\ + \dfrac{(q-1)^2}{q^3}A(n_1)A(n_2-n_1-1)A(n_3-n_2-1), \quad \text{if } l_1 \neq 0, l_2 = l_3 = 0, \\ \dfrac{q-1}{q^2}A(n_1-1)A(n_2-n_1)B(n_3-n_2) \\ + \dfrac{(q-1)^2}{q^3}A(n_1-1)A(n_2-n_1)A(n_3-n_2-1), \quad \text{if } l_1 = 0, l_2 \neq 0, l_3 = 0, \\ \dfrac{1}{q^2}A(n_1)A(n_2-n_1)B(n_3-n_2) + \dfrac{q-1}{q^3}A(n_1)A(n_2-n_1)A(n_3-n_2-1), \quad \text{if } l_1 \neq 0, l_2 \neq 0, l_3 = 0, \\ \dfrac{(q-1)^2}{q^3}A(n_1-1)A(n_2-n_1-1)A(n_3-n_2), \quad \text{if } l_1 = l_2 = 0, l_3 \neq 0, \\ \dfrac{q-1}{q^3}A(n_1-1)A(n_2-n_1)A(n_3-n_2), \quad \text{if } l_1 = 0, l_2 \neq 0, l_3 \neq 0, \\ \dfrac{q-1}{q^3}A(n_1)A(n_2-n_1-1)A(n_3-n_2), \quad \text{if } l_1 \neq 0, l_2 = 0, l_3 \neq 0, \\ \dfrac{1}{q^3}A(n_1)A(n_2-n_1)A(n_3-n_2), \quad \text{if } l_1 \neq 0, l_2 \neq 0, l_3 \neq 0. \end{cases} \tag{1.8}$$

This article is organized as follows. We present in Section 2 two preliminary lemmas that are needed in the proofs of our main results. Subsequently, we give the proof of Theorem 1.1 in Section 3. Section 4 is devoted to the proof of Theorem 1.2. In Section 5, we provide two examples to illustrate the validity of Theorems 1.1 and 1.2.

2 Auxiliary lemmas

In this section, we present two lemmas, which are needed in the proofs of our main results. We begin with a result due to Sun [18].

Lemma 2.1. [18] *Let $c_1, \ldots, c_k \in \mathbb{F}_q^*$ and $c \in \mathbb{F}_q$. Let $N(c)$ denote the number of rational points $(u_1, \ldots, u_k) \in (\mathbb{F}_q^*)^k$ on the equation $c_1u_1 + \ldots + c_ku_k = c$. Then*

$$N(c) = \begin{cases} \dfrac{q-1}{q}A(k-1) & \text{if } c = 0, \\ \dfrac{1}{q}A(k) & \text{otherwise} \end{cases}$$

with $A(k)$ being defined as in (1.2).

We also need the following result due to Zhu and Hong [23].

Lemma 2.2. [23, Lemma 2.6] *Let* $c_{ij} \in \mathbb{F}_q^*$ *for all* $i \in \langle m \rangle$ *and* $j \in \langle k_i \rangle$, *and let* $c_1, \ldots, c_m \in \mathbb{F}_q$. *Let* $N(c_1, \ldots, c_m)$ *denote the number of rational points*

$$(u_{11}, \ldots, u_{1k_1}, \ldots, u_{m1}, \ldots, u_{mk_m}) \in (\mathbb{F}_q^*)^{k_1 + \ldots + k_m}$$

on the variety defined by the following system of equations:

$$\begin{cases} c_{11}u_{11} + \ldots + c_{1k_1}u_{1k_1} = c_1 \\ \vdots \\ c_{m1}u_{m1} + \ldots + c_{mk_m}u_{mk_m} = c_m. \end{cases}$$

Then

$$N(c_1, \ldots, c_m) = \frac{(q-1)^{|\{1 \le i \le m : c_i = 0\}|}}{q^m} \prod_{\substack{i=1 \\ c_i = 0}}^{m} A(k_i - 1) \prod_{\substack{i=1 \\ c_i \neq 0}}^{m} A(k_i)$$

with $A(k_i)$ *being defined as in* (1.2).

3 Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. Firstly, we introduce some definitions and notations, which will be used in proving Theorem 1.1.

Let

$$E_1 = \begin{pmatrix} e_{11} & e_{12} & \cdots & e_{1m_1} \\ \vdots & \vdots & & \vdots \\ e_{m_1,1} & e_{m_1,2} & \cdots & e_{m_1,m_1} \end{pmatrix}$$

with $e_{ij}(1 \le i, j \le m_1)$ being given as in (1.3). It is well known [25] that there are *unimodular* matrices U_1, V_1, U_2, and V_2 such that

$$U_1 E_1 V_1 = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.1}$$

and

$$U_2 \mathscr{E} V_2 = \begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.2}$$

where $\mathscr{E}$ is given as in (1.5)

$$D_1 := \operatorname{diag}(g_1^{(E_1)}, \ldots, g_v^{(E_1)})$$

and

$$D_2 := \operatorname{diag}(g_1^{(\mathscr{E})}, \ldots, g_{v'}^{(\mathscr{E})}),$$

with v and v' being the ranks of the matrices E_1 and $\mathscr{E}$, respectively. All elements

$$g_1^{(E_1)}, \ldots, g_v^{(E_1)}, g_1^{(\mathscr{E})}, \ldots, g_{v'}^{(\mathscr{E})} \in \mathbb{Z}^+ \quad (\text{the set of positive integers})$$

satisfy that $g_i^{(E_1)} | g_{i+1}^{(E_1)} (i \in \langle v-1 \rangle)$ and $g_i^{(\mathscr{E})} | g_{i+1}^{(\mathscr{E})} (i \in \langle v'-1 \rangle)$. We say that the diagonal matrices $\begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}$ in (3.1) and $\begin{pmatrix} D_2 & 0 \\ 0 & 0 \end{pmatrix}$ in (3.2) are the *Smith normal form* of the matrices E_1 and $\mathscr{E}$ and are abbreviated as SNF(E_1) and SNF ($\mathscr{E}$), respectively.

Set $\alpha \in \mathbb{F}_q^*$ to be a primitive element of $\mathbb{F}_q$, for any $\beta \in \mathbb{F}_q^*$, there is a unique integer $\gamma \in [1, q-1]$ such that $\beta = \alpha^\gamma$, where γ is called the index of β with regard to the primitive element α and denoted by $\operatorname{ind}_\alpha \beta := \gamma$.

Let $\mathscr{M}_1$ denote the number of rational points $(u_1, \ldots, u_{m_1}) \in (\mathbb{F}_q^*)^{m_1}$ on the affine hypersurface

$$\sum_{i=1}^{m_1} a_i u_i = b_1 \tag{3.3}$$

under the following additional condition:

$$\begin{cases} \gcd(q-1, g_j^{(E_1)}) | h_j^{(E_1)} & \text{for } j \in \langle v \rangle \\ (q-1) | h_j^{(E_1)} & \text{for } j \in \langle m_1 \rangle \backslash \langle v \rangle, \end{cases} \tag{3.4}$$

where

$$(h_1^{(E_1)}, \ldots, h_{m_1}^{(E_1)})^{\mathrm{T}} := U_1(\mathrm{ind}_\alpha(u_1), \ldots, \mathrm{ind}_\alpha(u_{m_1}))^{\mathrm{T}}.$$

Let $\mathscr{M}_2$ denote the number of rational points $(u_1, \ldots, u_{m_2}) \in (\mathbb{F}_q^*)^{m_2}$ on the variety

$$\begin{cases} \sum_{i=1}^{m_1} a_i u_i = b_1, \\ \sum_{i=m_1+1}^{m_2} a_i u_i = b_2 \end{cases} \tag{3.5}$$

under the following additional condition:

$$\begin{cases} \gcd(q-1, g_j^{(\mathscr{E})}) | h_j^{(\mathscr{E})} & \text{for } j \in \langle v' \rangle \\ (q-1) | h_j^{(\mathscr{E})} & \text{for } j \in \langle m_2 \rangle \backslash \langle v' \rangle, \end{cases} \tag{3.6}$$

where

$$(h_1^{(\mathscr{E})}, \ldots, h_{m_2}^{(\mathscr{E})})^{\mathrm{T}} := U_2(\mathrm{ind}_\alpha(u_1), \ldots, \mathrm{ind}_\alpha(u_{m_2}))^{\mathrm{T}}.$$

Lemma 3.1. *Let $V = V(f_1, f_2)$ be the affine algebraic variety* (1.3) *and $m_1 < m_2$. Then*

$$N(V) = \begin{cases} q^{m_2-m_1} B(m_1) + \mathscr{M}_1 B(m_2 - m_1)\mathscr{A} + \mathscr{M}_2 \mathscr{C} & \text{if } b_1 = b_2 = 0, \\ \mathscr{M}_1 B(m_2 - m_1)\mathscr{A} + \mathscr{M}_2 \mathscr{C} & \text{if } b_1 \neq 0, b_2 = 0, \\ \mathscr{M}_2 \mathscr{C} & \text{if } b_2 \neq 0, \end{cases} \tag{3.7}$$

where $B(m_1)$ is given as in (1.1),

$$\mathscr{A} = (q-1)^{m_1 - v} \prod_{j=1}^{v} \gcd(q-1, g_j^{(E_1)}) \quad \textit{and} \quad \mathscr{C} = (q-1)^{m_2 - v'} \prod_{j=1}^{v'} \gcd(q-1, g_j^{(\mathscr{E})})$$

with v and v' being the ranks of the matrices E_1 and $\mathscr{E}$, respectively.

Proof. This follows immediately from [23, Theorem 1.2]. □

Proof of Theorem 1.1. The condition $\gcd(q-1, \det(\mathscr{E})) = 1$ means that $\det(\mathscr{E}) \neq 0$. Moreover, it is clear that $\det(E_1) | \det(\mathscr{E})$; thus, $\det(E_1) \neq 0$. Hence, the ranks of the matrices E_1 and $\mathscr{E}$ are m_1 and m_2, respectively. By taking determinants of both sides of (3.1) and (3.2), one can deduce that

$$\det(U_1)\det(E_1)\det(V_1) = g_1^{(E_1)} \ldots g_{m_1}^{(E_1)}$$

and

$$\det(U_2)\det(\mathscr{E})\det(V_2) = g_1^{(\mathscr{E})} \ldots g_{m_2}^{(\mathscr{E})}.$$

Since $\det(U_i) = \pm 1$ and $\det(V_i) = \pm 1$ for all $i \in \{1, 2\}$, we have $\det(E_1) = \pm g_1^{(E_1)} \dots g_{m_1}^{(E_1)}$ and $\det(\mathscr{E}) = \pm g_1^{(\mathscr{E})} \dots g_{m_2}^{(\mathscr{E})}$. The condition $\gcd(q-1, \det(\mathscr{E})) = 1$ together with $\det(E_1) | \det(\mathscr{E})$ implies that

$$\gcd(q-1, g_j^{(E_1)}) = 1 \quad \text{for all } j \in \langle m_1 \rangle,$$

and

$$\gcd(q-1, g_j^{(\mathscr{E})}) = 1 \quad \text{for all } j \in \langle m_2 \rangle.$$

So (3.4), and (3.6) hold. It follows from (3.7) that

$$N(V) = \begin{cases} q^{m_2-m_1} B(m_1) + \mathscr{M}_1 B(m_2 - m_1) + \mathscr{M}_2 & \text{if } b_1 = b_2 = 0, \\ \mathscr{M}_1 B(m_2 - m_1) + \mathscr{M}_2 & \text{if } b_1 \neq 0, b_2 = 0, \\ \mathscr{M}_2 & \text{if } b_2 \neq 0. \end{cases} \tag{3.8}$$

From Lemmas 2.1 and 2.2, one derives that

$$\mathscr{M}_1 = \sum_{(u_1, \dots, u_{m_1}) \in (\mathbb{F}_q^*)^{m_1} \text{ such that (3.3) holds}} 1 = \begin{cases} \frac{q-1}{q} A(m_1 - 1) & \text{if } b_1 = 0, \\ \frac{1}{q} A(m_1) & \text{if } b_1 \neq 0, \end{cases} \tag{3.9}$$

and

$$\mathscr{M}_2 = \sum_{(u_1, \dots, u_{m_2}) \in (\mathbb{F}_q^*)^{m_2} \text{ such that (3.5) holds}} 1$$
$$= \begin{cases} \frac{(q-1)^2}{q^2} A(m_1 - 1) A(m_2 - m_1 - 1) & \text{if } b_1 = b_2 = 0, \\ \frac{q-1}{q^2} A(m_1) A(m_2 - m_1 - 1) & \text{if } b_1 \neq 0, b_2 = 0, \\ \frac{q-1}{q^2} A(m_1 - 1) A(m_2 - m_1) & \text{if } b_1 = 0, b_2 \neq 0, \\ \frac{1}{q^2} A(m_1) A(m_2 - m_1) & \text{if } b_1 \neq 0, b_2 \neq 0. \end{cases} \tag{3.10}$$

Putting (3.9) and (3.10) into (3.8) yields (1.7) as expected. This concludes the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

Let $h_{ij}, b_i (1 \leq i \leq z, 1 \leq j \leq a)$ and b be integers. For the vectors $Y = (y_1, \dots, y_a)^{\mathrm{T}}$ and $\mathscr{B} = (b_1, \dots, b_z)^{\mathrm{T}}$, and the $z \times a$ matrix $\mathscr{H} = (h_{ij})$, we can form system of congruences as follows:

$$\mathscr{H} Y \equiv \mathscr{B} \pmod{b}. \tag{4.1}$$

From [25], we can find unimodular matrices $\mathcal{U}$ of order z and $\mathcal{V}$ of order a such that

$$\mathcal{U} \mathscr{H} \mathcal{V} = \mathrm{SNF}(\mathscr{H}) = \begin{pmatrix} \mathscr{D} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathscr{D} := \mathrm{diag}(d_1, \dots, d_r)$ with all diagonal elements d_i being positive integers and satisfying that $d_i | d_{i+1}$ $(1 \leq i < r)$. The following result is known.

Lemma 4.1. [15, Lemma 2.3] *Let $B' = (b'_1, \dots, b'_z)^{\mathrm{T}} = \mathcal{U}\mathscr{B}$. Then the system* (4.1) *of linear congruences is solvable if and only if* $\gcd(b, d_i) | b'_i$ *for all $i \in \langle r \rangle$ and $b | b'_i$ for all integers i with $r + 1 \leq i \leq z$. Besides, the number of solutions of* (4.1) *is equal to* $b^{a-r} \prod_{i=1}^{r} \gcd(b, d_i)$.

To state Lemma 4.2, we first introduce several relevant concepts and notations as follows. Let

$$F_1 = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n_1} \\ \vdots & \vdots & & \vdots \\ d_{n_1,1} & d_{n_1,2} & \cdots & d_{n_1,n_1} \end{pmatrix}$$

and

$$F_2 = \begin{pmatrix} d_{11} & \cdots & d_{1n_1} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ d_{n_1,1} & \cdots & d_{n_1,n_1} & 0 & \cdots & 0 \\ d_{n_1+1,1} & \cdots & d_{n_1+1,n_1} & d_{n_1+1,n_1+1} & \cdots & d_{n_1+1,n_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ d_{n_2,1} & \cdots & d_{n_2,n_1} & d_{n_2,n_1+1} & \cdots & d_{n_2,n_2} \end{pmatrix}$$

with $d_{ij}(1 \le i, j \le n_3)$ being given as in (1.4). By [25], we know that there are unimodular matrices M_1, W_1, M_2, W_2, M_3, and W_3 such that

$$M_1F_1W_1 = \begin{pmatrix} G_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.2}$$

$$M_2F_2W_2 = \begin{pmatrix} G_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.3}$$

and

$$M_3\mathscr{F}W_3 = \begin{pmatrix} G_3 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.4}$$

where $\mathscr{F}$ is given as in (1.6),

$$G_1 := \mathrm{diag}(g_1^{(F_1)}, \ldots, g_u^{(F_1)}),$$
$$G_2 := \mathrm{diag}(g_1^{(F_2)}, \ldots, g_{u'}^{(F_2)}),$$

and

$$G_3 := \mathrm{diag}(g_1^{(\mathscr{F})}, \ldots, g_{u''}^{(\mathscr{F})})$$

with u, u' and u'' being the ranks of the matrices F_1, F_2, and $\mathscr{F}$, respectively. All elements

$$g_1^{(F_1)}, \ldots, g_u^{(F_1)}, g_1^{(F_2)}, \ldots, g_{u'}^{(F_2)}, g_1^{(\mathscr{F})}, \ldots, g_{u''}^{(\mathscr{F})}$$

are positive integers and

$$g_i^{(F_1)} | g_{i+1}^{(F_1)} (i \in \langle u-1 \rangle), \quad g_i^{(F_2)} | g_{i+1}^{(F_2)} (i \in \langle u'-1 \rangle), \quad g_i^{(\mathscr{F})} | g_{i+1}^{(\mathscr{F})} (i \in \langle u''-1 \rangle).$$

Let $\mathscr{N}_1$ denote the number of rational points $(v_1, \ldots, v_{n_1}) \in (\mathbb{F}_q^*)^{n_1}$ on the affine hypersurface

$$\sum_{i=1}^{n_1} c_i v_i = l_1 \tag{4.5}$$

under the following extra condition:

$$\begin{cases} \gcd(q-1, g_j^{(F_1)}) | h_j^{(F_1)} & \text{for } j \in \langle u \rangle \\ (q-1) | h_j^{(F_1)} & \text{for } j \in \langle n_1 \rangle \backslash \langle u \rangle, \end{cases} \tag{4.6}$$

where

$$(h_1^{(F_1)}, \ldots, h_{n_1}^{(F_1)})^{\mathrm{T}} := M_1(\mathrm{ind}_\alpha(v_1), \ldots, \mathrm{ind}_\alpha(v_{n_1}))^{\mathrm{T}}.$$

Let $\mathcal{N}_2$ denote the number of rational points $(v_1, \dots, v_{n_2}) \in (\mathbb{F}_q^*)^{n_2}$ on the variety

$$\begin{cases} \sum_{i=1}^{n_1} c_i v_i = l_1, \\ \sum_{i=n_1+1}^{n_2} c_i v_i = l_2 \end{cases} \tag{4.7}$$

under the following extra condition:

$$\begin{cases} \gcd(q-1, g_j^{(F_2)}) | h_j^{(F_2)} & \text{for } j \in \langle u' \rangle \\ (q-1) | h_j^{(F_2)} & \text{for } j \in \langle n_2 \rangle \backslash \langle u' \rangle, \end{cases} \tag{4.8}$$

where

$$(h_1^{(F_2)}, \dots, h_{n_2}^{(F_2)})^{\mathrm{T}} := M_2(\mathrm{ind}_\alpha(v_1), \dots, \mathrm{ind}_\alpha(v_{n_2}))^{\mathrm{T}}.$$

Let $\mathcal{N}_3$ denote the number of rational points $(v_1, \dots, v_{n_3}) \in (\mathbb{F}_q^*)^{n_3}$ on the variety

$$\begin{cases} \sum_{i=1}^{n_1} c_i v_i = l_1, \\ \sum_{i=n_1+1}^{n_2} c_i v_i = l_2, \\ \sum_{i=n_2+1}^{n_3} c_i v_i = l_3 \end{cases} \tag{4.9}$$

under the following extra condition:

$$\begin{cases} \gcd(q-1, g_j^{(\mathscr{F})}) | h_j^{(\mathscr{F})} & \text{for } j \in \langle u'' \rangle, \\ (q-1) | h_j^{(\mathscr{F})} & \text{for } j \in \langle n_3 \rangle \backslash \langle u'' \rangle, \end{cases} \tag{4.10}$$

where

$$(h_1^{(\mathscr{F})}, \dots, h_{n_3}^{(\mathscr{F})})^{\mathrm{T}} := M_3(\mathrm{ind}_\alpha(v_1), \dots, \mathrm{ind}_\alpha(v_{n_3}))^{\mathrm{T}}.$$

We have the following result.

Lemma 4.2. *Let $n_1 < n_2 < n_3$. Then*

$$\sum_{\substack{(v_1, \dots, v_{n_1}) \in (\mathbb{F}_q^*)^{n_1} \\ (4.5) \text{ holds}}} |\{(x_1, \dots, x_{n_1}) \in (\mathbb{F}_q^*)^{n_1} : x_1^{d_{i1}} \dots x_{n_1}^{d_{in_1}} = v_i, i \in \langle n_1 \rangle\}| = \mathcal{N}_1 (q-1)^{n_1 - u} \prod_{i=1}^{u} \gcd(q-1, g_i^{(F_1)}), \tag{4.11}$$

$$\begin{aligned} &\sum_{\substack{(v_1, \dots, v_{n_2}) \in (\mathbb{F}_q^*)^{n_2} \\ (4.7) \text{ holds}}} \left| \left\{ (x_1, \dots, x_{n_2}) \in (\mathbb{F}_q^*)^{n_2} : \begin{array}{l} x_1^{d_{i1}} \dots x_{n_1}^{d_{in_1}} = v_i, i \in \langle n_1 \rangle \\ x_1^{d_{i1}} \dots x_{n_2}^{d_{in_2}} = v_i, i \in \langle n_2 \rangle \backslash \langle n_1 \rangle \end{array} \right\} \right| \\ &= \mathcal{N}_2 (q-1)^{n_2 - u'} \prod_{i=1}^{u'} \gcd(q-1, g_i^{(F_2)}) \end{aligned} \tag{4.12}$$

and

$$\sum_{\substack{(v_1,\ldots,v_{n_3})\in(\mathbb{F}_q^*)^{n_3}\\ (4.9)\ holds}}\left|\left\{(x_1,\ldots,x_{n_3})\in(\mathbb{F}_q^*)^{n_3}: \begin{array}{r} x_1^{d_{i1}}\ldots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_2}^{d_{in_2}}=v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_3}^{d_{in_3}}=v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{array}\right\}\right| \tag{4.13}$$
$$= \mathcal{N}_3(q-1)^{n_3-u''}\prod_{i=1}^{u''}\gcd(q-1, g_i^{(\mathscr{F})}).$$

Proof. For any given $(v_1,\ldots,v_{n_1})\in(\mathbb{F}_q^*)^{n_1}$ satisfying (4.5), one has the system of congruences:

$$\sum_{j=1}^{n_1} d_{ij}\mathrm{ind}_\alpha(x_i)\equiv \mathrm{ind}_\alpha(v_i) \pmod{q-1},\ i\in\langle n_1\rangle, \tag{4.14}$$

then

$$\begin{aligned}&|\{(x_1,\ldots,x_{n_1})\in(\mathbb{F}_q^*)^{n_1}: x_1^{d_{i1}}\ldots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle\}|\\ &=|\{(x_1,\ldots,x_{n_1})\in(\mathbb{F}_q^*)^{n_1}: \alpha^{\sum_{j=1}^{n_1} d_{ij}\mathrm{ind}_\alpha(x_i)}=\alpha^{\mathrm{ind}_\alpha(v_i)},\ i\in\langle n_1\rangle\}|\\ &=|\{(x_1,\ldots,x_{n_1})\in(\mathbb{F}_q^*)^{n_1}: \text{(4.14) holds}\}|.\end{aligned}$$

From Lemma 4.1, we know that (4.14) is solvable if and only if (4.6) holds. Further, Lemma 4.1 tells us that if (4.14) has a solution, then the number of n_1-tuples $(\mathrm{ind}_\alpha(x_1),\ldots,\mathrm{ind}_\alpha(x_{n_1}))\in\langle q-1\rangle^{n_1}$ satisfying (4.14) equals

$$(q-1)^{n_1-u}\prod_{i=1}^{u}\gcd(q-1, g_i^{(F_1)}).$$

In other words, if (4.6) holds, then

$$|\{(x_1,\ldots,x_{n_1})\in(\mathbb{F}_q^*)^{n_1}: x_1^{d_{i1}}\ldots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle\}| = (q-1)^{n_1-u}\prod_{i=1}^{u}\gcd(q-1, g_i^{(F_1)}).$$

So the left-hand side of (4.11) is equal to

$$(q-1)^{n_1-u}\prod_{i=1}^{u}\gcd(q-1, g_i^{(F_1)})\times\sum_{\substack{(v_1,\ldots,v_{n_1})\in(\mathbb{F}_q^*)^{n_1}\\ (4.5)\text{ and }(4.6)\text{ hold}}}1 \tag{4.15}$$

Notice that

$$\mathcal{N}_1=\sum_{(v_1,\ldots,v_{n_1})\in(\mathbb{F}_q^*)^{n_1}\quad \text{such that (4.5) and (4.6) hold}}1. \tag{4.16}$$

Thus, (4.15) and (4.16) yield (4.11).

Similarly, we can show that (4.12) and (4.13) hold. Lemma 4.2 is proved. □

Associated to l_1, l_2, and l_3, we define the set $T(l_1,l_2,l_3)$ of $\mathbb{F}_q$-rational points as follows:

$$T(l_1,l_2,l_3) := \{(v_1,\ldots,v_{n_3})\in(\mathbb{F}_q)^{n_3}: \text{(4.9) holds}\}. \tag{4.17}$$

Let $\mathscr{T}(0)$ be the empty set when l_1, l_2, and l_3 are not all zero, and let $\mathscr{T}(0)$ denote the set consisting of the zero vector of dimension n_3 when $l_1=l_2=l_3=0$. For any integer $1\le n\le n_3$, let $\mathscr{T}(n)$ denote the subset of $T(l_1,l_2,l_3)$ in which the vector holds exactly n nonzero components.

Lemma 4.3. *Let $n_1<n_2<n_3$. Then each of the following assertions is true*:

(*i*) *$\mathscr{T}(n)$ is the subset of $T(l_1,l_2,l_3)$ in which the vector holds exactly the first n nonzero components.*

(*ii*) *For any integer n with $0<n<n_1$ or $n_1<n<n_2$ or $n_2<n<n_3$, $\mathscr{T}(n)=\varnothing$.*

(*iii*) *$T(l_1,l_2,l_3)=\mathscr{T}(0)\cup\mathscr{T}(n_1)\cup\mathscr{T}(n_2)\cup\mathscr{T}(n_3)$.*

Proof. (i) First, recall that

$$v_i = x_1^{d_{i1}} \dots x_{n_1}^{d_{in_1}} \quad \text{for } i \in \langle n_1 \rangle,$$
$$v_i = x_1^{d_{i1}} \dots x_{n_2}^{d_{in_2}} \quad \text{for } i \in \langle n_2 \rangle \backslash \langle n_1 \rangle,$$
$$v_i = x_1^{d_{i1}} \dots x_{n_3}^{d_{in_3}} \quad \text{for } i \in \langle n_3 \rangle \backslash \langle n_2 \rangle.$$

Because the set of x_i variables appearing in v_i for $1 \leq i \leq n_1$ is contained in the set of the x_i variables appearing in v_j when $j \geq i$, and the set of x_i variables appearing in $v_{i'}$ for $n_1 + 1 \leq i' \leq n_2$ is also contained in the set of the x_i variables appearing in $v_{j'}$ when $j' \geq i'$, it follows that if $v_i = 0$ for any integer i with $1 \leq i \leq n_1$, then $v_j = 0$ when $j \geq i$. If $v_{i'} = 0$ for any integer i' with $n_1 + 1 \leq i' \leq n_2$, then $v_{j'} = 0$ when $j' \geq i'$. Part (i) is proved.

(ii) Since $v_1, \dots, v_{n_1}$ are simultaneously zero or simultaneously nonzero, and $v_{n_1+1}, \dots, v_{n_2}$ are simultaneously zero or simultaneously nonzero, and $v_{n_2+1}, \dots, v_{n_3}$ are simultaneously zero or simultaneously nonzero, part (ii) follows immediately.

(iii) By parts (i) and (ii), we obtain the following disjoint unions:

$$T(l_1, l_2, l_3) = \bigcup_{n=0}^{n_3} \mathscr{T}(n) = \mathscr{T}(0) \cup \mathscr{T}(n_1) \cup \mathscr{T}(n_2) \cup \mathscr{T}(n_3)$$

as desired. □

Lemma 4.4. *Each of the following assertions is true*:
(*i*) *If* $l_2 \neq 0$ *or* $l_3 \neq 0$, *then* $\mathscr{T}(n_1) = \varnothing$.
(*ii*) *If* $l_3 \neq 0$, *then* $\mathscr{T}(n_2) = \varnothing$.

Proof. (i) Assume that $\mathscr{T}(n_1) \neq \varnothing$. Then by Lemma 4.3 (i), we have

$$v_1 \neq 0, \dots, v_{n_1} \neq 0 \quad \text{and} \quad v_{n_1+1} = \dots = v_{n_3} = 0.$$

Hence, $l_2 = l_3 = 0$. This is impossible. Part (i) is proved.

(ii) Suppose that $\mathscr{T}(n_2) \neq \varnothing$. Then from Lemma 4.3 (i), we know that

$$v_1 \neq 0, \dots, v_{n_2} \neq 0 \quad \text{and} \quad v_{n_2+1} = \dots = v_{n_3} = 0.$$

This contradicts $l_3 \neq 0$. Part (ii) is proved. □

Subsequently, we can prove the following important lemma of this section.

Lemma 4.5. *Let* $V = V(f_1, f_2, f_3)$ *be the affine algebraic variety* (1.4). *Then*

$$N(V) = \begin{cases} q^{n_3-n_1}B(n_1) + q^{n_3-n_2}B(n_2-n_1)\mathcal{N}_1K + B(n_3-n_2)\mathcal{N}_2L + \mathcal{N}_3E & \text{if } l_1 = l_2 = l_3 = 0, \\ q^{n_3-n_2}B(n_2-n_1)\mathcal{N}_1K + B(n_3-n_2)\mathcal{N}_2L + \mathcal{N}_3E & \text{if } l_1 \neq 0, l_2 = l_3 = 0, \\ B(n_3-n_2)\mathcal{N}_2L + \mathcal{N}_3E & \text{if } l_2 \neq 0, l_3 = 0, \\ \mathcal{N}_3E & \text{if } l_3 \neq 0, \end{cases} \tag{4.18}$$

where

$$K = (q-1)^{n_1-u}\prod_{i=1}^{u}\gcd(q-1, g_i^{(F_1)}), \quad L = (q-1)^{n_2-u'}\prod_{i=1}^{u'}\gcd(q-1, g_i^{(F_2)}),$$

$$E = (q-1)^{n_3-u''}\prod_{i=1}^{u''}\gcd(q-1, g_i^{(\mathscr{F})})$$

with u, u', *and* u'' *being the ranks of the matrices* F_1, F_2, *and* $\mathscr{F}$, *respectively.*

Proof. Obviously, we have

$$N(V) = \sum_{\substack{(v_1,\dots,v_{n_3})\in\mathbb{F}_q^{n_3}\\ (4.9)\text{ holds}}} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}: \begin{matrix} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}} = v_i,\ i\in\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}} = v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}} = v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{matrix}\right\}\right|. \tag{4.19}$$

It follows from (4.17), (4.19), and Lemma 4.3 (iii) that

$$\begin{aligned} N(V) &= \sum_{(v_1,\dots,v_{n_3})\in T(l_1,l_2,l_3)} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}: \begin{matrix} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}} = v_i,\ i\in\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}} = v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}} = v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{matrix}\right\}\right| \\ &= \sum_{(v_1,\dots,v_{n_3})\in\mathcal{T}(0)\cup\mathcal{T}(n_1)\cup\mathcal{T}(n_2)\cup\mathcal{T}(n_3)} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}: \begin{matrix} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}} = v_i,\ i\in\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}} = v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}} = v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{matrix}\right\}\right|. \end{aligned} \tag{4.20}$$

Applying (4.11), we compute and obtain that

$$\begin{aligned} &\sum_{(v_1,\dots,v_{n_3})\in\mathcal{T}(n_1)} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}: \begin{matrix} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}} = v_i,\ i\in\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}} = v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}} = v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{matrix}\right\}\right| \\ &= \sum_{\substack{(v_1,\dots,v_{n_3})\in\mathbb{F}_q^{n_3}\\ (4.5)\text{ holds},\quad v_1\neq 0,\dots,v_{n_1}\neq 0\\ v_{n_1+1}=\dots=v_{n_3}=0}} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}: \begin{matrix} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}} = v_i,\ i\in\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}} = v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}} = v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{matrix}\right\}\right| \\ &= \sum_{\substack{(v_1,\dots,v_{n_1})\in(\mathbb{F}_q^*)^{n_1}\\ (4.5)\text{ holds}}} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}: \begin{matrix} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}} = v_i,\ i\in\langle n_1\rangle \\ x_{n_1+1}\dots x_{n_2} = 0 \end{matrix}\right\}\right| \\ &= q^{n_3-n_2}\times|\{(x_{n_1+1},\dots,x_{n_2})\in\mathbb{F}_q^{n_2-n_1}: x_{n_1+1}\dots x_{n_2}=0\}| \\ &\quad\times\sum_{\substack{(v_1,\dots,v_{n_1})\in(\mathbb{F}_q^*)^{n_1}\\ (4.5)\text{ holds}}} |\{(x_1,\dots,x_{n_1})\in(\mathbb{F}_q^*)^{n_1}: x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}} = v_i,\ i\in\langle n_1\rangle\}| \\ &= q^{n_3-n_2}(q^{n_2-n_1}-(q-1)^{n_2-n_1})\mathcal{N}_1(q-1)^{n_1-u}\prod_{i=1}^{u}\gcd(q-1,g_i^{(F_1)}). \end{aligned} \tag{4.21}$$

From (4.12), we can calculate and obtain

$$\begin{aligned} &\sum_{(v_1,\dots,v_{n_3})\in\mathcal{T}(n_2)} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}: \begin{matrix} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}} = v_i,\ i\in\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}} = v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}} = v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{matrix}\right\}\right| \\ &= \sum_{\substack{(v_1,\dots,v_{n_3})\in\mathbb{F}_q^{n_3}\\ (4.7)\text{ holds},\quad v_1\neq 0,\dots,v_{n_2}\neq 0\\ v_{n_2+1}=\dots=v_{n_3}=0}} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}: \begin{matrix} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}} = v_i,\ i\in\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}} = v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}} = v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{matrix}\right\}\right| \end{aligned} \tag{4.22}$$

$$
\begin{aligned}
&= \sum_{\substack{(v_1,\dots,v_{n_2})\in(\mathbb{F}_q^*)^{n_2}\\ (4.7)\text{ holds}}} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}:\begin{array}{r} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle,\\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}}=v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle\\ x_{n_2+1}\dots x_{n_3}=0\end{array}\right\}\right|\\
&= |\{(x_{n_2+1},\dots,x_{n_3})\in\mathbb{F}_q^{n_3-n_2}:x_{n_2+1}\dots x_{n_3}=0\}|\\
&\quad\times \sum_{\substack{(v_1,\dots,v_{n_2})\in(\mathbb{F}_q^*)^{n_2}\\ (4.7)\text{ holds}}} \left|\left\{(x_1,\dots,x_{n_2})\in(\mathbb{F}_q^*)^{n_2}:\begin{cases} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle,\\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}}=v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle\end{cases}\right\}\right|\\
&= (q^{n_3-n_2}-(q-1)^{n_3-n_2})\mathscr{N}_2(q-1)^{n_2-u'}\prod_{i=1}^{u'}\gcd(q-1,g_i^{(F_2)}).
\end{aligned}
$$

From (4.13), one computes and gains that

$$
\begin{aligned}
&\sum_{(v_1,\dots,v_{n_3})\in\mathscr{T}(n_3)} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}:\begin{array}{r} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}}=v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}}=v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle\end{array}\right\}\right|\\
&= \sum_{\substack{(v_1,\dots,v_{n_3})\in\mathbb{F}_q^{n_3}\\ (4.9)\text{ holds}\\ v_1\neq0,\dots,v_{n_3}\neq0}} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}:\begin{array}{r} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}}=v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}}=v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle\end{array}\right\}\right|\\
&= \sum_{\substack{(v_1,\dots,v_{n_3})\in(\mathbb{F}_q^*)^{n_3}\\ (4.9)\text{ holds}}} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}:\begin{array}{r} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}}=v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}}=v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle\end{array}\right\}\right| \qquad (4.23)\\
&= \sum_{\substack{(v_1,\dots,v_{n_3})\in(\mathbb{F}_q^*)^{n_3}\\ (4.9)\text{ holds}}} \left|\left\{(x_1,\dots,x_{n_3})\in(\mathbb{F}_q^*)^{n_3}:\begin{array}{r} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}}=v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}}=v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle\end{array}\right\}\right|\\
&= \mathscr{N}_3(q-1)^{n_3-u''}\prod_{i=1}^{u''}\gcd(q-1,g_i^{(\mathscr{F})}).
\end{aligned}
$$

On the other hand, we can calculate and obtain

$$
\begin{aligned}
&\sum_{(v_1,\dots,v_{n_3})\in\mathscr{T}(0)} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}:\begin{array}{r} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}}=v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}}=v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle\end{array}\right\}\right|\\
&\quad= |\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}:x_1\dots x_{n_1}=0\}| \qquad (4.24)\\
&\quad= q^{n_3-n_1}\times|\{(x_1,\dots,x_{n_1})\in\mathbb{F}_q^{n_1}:x_1\dots x_{n_1}=0\}|\\
&\quad= q^{n_3-n_1}\times(q^{n_1}-(q-1)^{n_1}).
\end{aligned}
$$

If $l_1=l_2=l_3=0$, then using (4.20) to (4.24), we deduce that

$$
N(V)=\sum_{(v_1,\dots,v_{n_3})\in\mathscr{T}(0)\cup\mathscr{T}(n_1)\cup\mathscr{T}(n_2)\cup\mathscr{T}(n_3)} \left|\left\{(x_1,\dots,x_{n_3})\in\mathbb{F}_q^{n_3}:\begin{array}{r} x_1^{d_{i1}}\dots x_{n_1}^{d_{in_1}}=v_i,\ i\in\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_2}^{d_{in_2}}=v_i,\ i\in\langle n_2\rangle\backslash\langle n_1\rangle\\ x_1^{d_{i1}}\dots x_{n_3}^{d_{in_3}}=v_i,\ i\in\langle n_3\rangle\backslash\langle n_2\rangle\end{array}\right\}\right|
$$

$$= \sum_{(v_1,\ldots,v_{n_3})\in\mathscr{T}(0)} \left|\left\{(x_1,\ldots,x_{n_3})\in\mathbb{F}_q^{n_3} : \begin{array}{r} x_1^{d_{i1}}\ldots x_{n_1}^{d_{in_1}} = v_i,\, i\in\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_2}^{d_{in_2}} = v_i,\, i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_3}^{d_{in_3}} = v_i,\, i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{array}\right\}\right|$$
$$+ \sum_{(v_1,\ldots,v_{n_3})\in\mathscr{T}(n_1)} \left|\left\{(x_1,\ldots,x_{n_3})\in\mathbb{F}_q^{n_3} : \begin{array}{r} x_1^{d_{i1}}\ldots x_{n_1}^{d_{in_1}} = v_i,\, i\in\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_2}^{d_{in_2}} = v_i,\, i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_3}^{d_{in_3}} = v_i,\, i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{array}\right\}\right|$$
$$+ \sum_{(v_1,\ldots,v_{n_3})\in\mathscr{T}(n_2)} \left|\left\{(x_1,\ldots,x_{n_3})\in\mathbb{F}_q^{n_3} : \begin{array}{r} x_1^{d_{i1}}\ldots x_{n_1}^{d_{in_1}} = v_i,\, i\in\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_2}^{d_{in_2}} = v_i,\, i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_3}^{d_{in_3}} = v_i,\, i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{array}\right\}\right|$$
$$+ \sum_{(v_1,\ldots,v_{n_3})\in\mathscr{T}(n_3)} \left|\left\{(x_1,\ldots,x_{n_3})\in\mathbb{F}_q^{n_3} : \begin{array}{r} x_1^{d_{i1}}\ldots x_{n_1}^{d_{in_1}} = v_i,\, i\in\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_2}^{d_{in_2}} = v_i,\, i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_3}^{d_{in_3}} = v_i,\, i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{array}\right\}\right|$$
$$= q^{n_3-n_1}\times(q^{n_1}-(q-1)^{n_1}) + q^{n_3-n_2}(q^{n_2-n_1}-(q-1)^{n_2-n_1})\mathscr{N}_1K$$
$$+ (q^{n_3-n_2}-(q-1)^{n_3-n_2})\mathscr{N}_2L + \mathscr{N}_3E,$$

with K, L, and E being defined as in (4.18).

We conclude that if $l_1\neq 0$, $l_2=l_3=0$, then $\mathscr{T}(0)=\varnothing$. By (4.20) to (4.23), we have

$$N(V) = \sum_{(v_1,\ldots,v_{n_3})\in\mathscr{T}(n_1)\cup\mathscr{T}(n_2)\cup\mathscr{T}(n_3)} \left|\left\{(x_1,\ldots,x_{n_3})\in\mathbb{F}_q^{n_3} : \begin{array}{r} x_1^{d_{i1}}\ldots x_{n_1}^{d_{in_1}} = v_i,\, i\in\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_2}^{d_{in_2}} = v_i,\, i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_3}^{d_{in_3}} = v_i,\, i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{array}\right\}\right|$$
$$= q^{n_3-n_2}(q^{n_2-n_1}-(q-1)^{n_2-n_1})\mathscr{N}_1K + (q^{n_3-n_2}-(q-1)^{n_3-n_2})\mathscr{N}_2L + \mathscr{N}_3E$$

with K, L, and E being defined as in (4.18).

If $l_2\neq 0$, $l_3=0$, then $\mathscr{T}(0)=\varnothing$. Meanwhile, Lemma 4.4 (i) ensures that $\mathscr{T}(n_1)=\varnothing$. Substituting (4.22) and (4.23) into (4.20) tells us that

$$N(V) = \sum_{(v_1,\ldots,v_{n_3})\in\mathscr{T}(n_2)\cup\mathscr{T}(n_3)} \left|\left\{(x_1,\ldots,x_{n_3})\in\mathbb{F}_q^{n_3} : \begin{array}{r} x_1^{d_{i1}}\ldots x_{n_1}^{d_{in_1}} = v_i,\, i\in\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_2}^{d_{in_2}} = v_i,\, i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_3}^{d_{in_3}} = v_i,\, i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{array}\right\}\right|$$
$$= (q^{n_3-n_2}-(q-1)^{n_3-n_2})\mathscr{N}_2L + \mathscr{N}_3E$$

as required, where L and E are given as in (4.18).

If $l_3\neq 0$, then $\mathscr{T}(0)=\varnothing$. Further, from Lemma 4.4, one has $\mathscr{T}(n_1)=\mathscr{T}(n_2)=\varnothing$. By (4.20) and (4.23), we have

$$N(V) = \sum_{(v_1,\ldots,v_{n_3})\in\mathscr{T}(n_3)} \left|\left\{(x_1,\ldots,x_{n_3})\in\mathbb{F}_q^{n_3} : \begin{array}{r} x_1^{d_{i1}}\ldots x_{n_1}^{d_{in_1}} = v_i,\, i\in\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_2}^{d_{in_2}} = v_i,\, i\in\langle n_2\rangle\backslash\langle n_1\rangle \\ x_1^{d_{i1}}\ldots x_{n_3}^{d_{in_3}} = v_i,\, i\in\langle n_3\rangle\backslash\langle n_2\rangle \end{array}\right\}\right|$$
$$= \mathscr{N}_3(q-1)^{n_3-u''}\prod_{i=1}^{u''}\gcd(q-1, g_i^{(\mathscr{F})}).$$

as expected.

This concludes the proof of Lemma 4.5. □

Proof of Theorem 1.2. Taking determinants of both sides of (4.2) to (4.4), one can deduce that

$$\det(M_i)\det(F_i)\det(W_i) = g_1^{(F_i)} \dots g_{n_i}^{(F_i)} \quad \text{for all } i \in \{1,2\}$$

and

$$\det(M_3)\det(\mathscr{F})\det(W_3) = g_1^{(\mathscr{F})} \dots g_{n_3}^{(\mathscr{F})}.$$

Since

$$\det(M_i) = \pm 1, \quad \det(W_i) = \pm 1 \quad \text{for all } i \in \{1,2,3\}$$

and $\det(F_1)|\det(F_2)|\det(\mathscr{F})$, the condition $\gcd(q-1, \det(\mathscr{F})) = 1$ guarantees that

$$\gcd(q-1, g_j^{(F_1)}) = 1 \quad \text{for all } j \in \langle n_1 \rangle,$$
$$\gcd(q-1, g_j^{(F_2)}) = 1 \quad \text{for all } j \in \langle n_2 \rangle,$$

and

$$\gcd(q-1, g_j^{(\mathscr{F})}) = 1 \quad \text{for all } j \in \langle n_3 \rangle.$$

So (4.6), (4.8), and (4.10) are all satisfied. It follows from Lemma 4.5 that

$$N(V) = \begin{cases} q^{n_3-n_1}B(n_1) + q^{n_3-n_2}B(n_2-n_1)\mathscr{N}_1 + B(n_3-n_2)\mathscr{N}_2 + \mathscr{N}_3 & \text{if } l_1 = l_2 = l_3 = 0, \\ q^{n_3-n_2}B(n_2-n_1)\mathscr{N}_1 + B(n_3-n_2)\mathscr{N}_2 + \mathscr{N}_3 & \text{if } l_1 \neq 0, l_2 = l_3 = 0, \\ B(n_3-n_2)\mathscr{N}_2 + \mathscr{N}_3 & \text{if } l_2 \neq 0, l_3 = 0, \\ \mathscr{N}_3 & \text{if } l_3 \neq 0. \end{cases}$$

From Lemmas 2.1 and 2.2, one derives that

$$\mathscr{N}_1 = \sum_{\substack{(v_1,\dots,v_{n_1}) \in (\mathbb{F}_q^*)^{n_1} \\ (4.5) \text{ holds}}} 1 = \begin{cases} \frac{q-1}{q}A(n_1-1) & \text{if } l_1 = 0, \\ \frac{1}{q}A(n_1) & \text{if } l_1 \neq 0, \end{cases} \tag{4.25}$$

$$\mathscr{N}_2 = \sum_{\substack{(v_1,\dots,v_{n_2}) \in (\mathbb{F}_q^*)^{n_2} \\ (4.7) \text{ holds}}} 1 = \begin{cases} \frac{(q-1)^2}{q^2}A(n_1-1)A(n_2-n_1-1) & \text{if } l_1 = l_2 = 0, \\ \frac{q-1}{q^2}A(n_1)A(n_2-n_1-1) & \text{if } l_1 \neq 0, l_2 = 0, \\ \frac{q-1}{q^2}A(n_1-1)A(n_2-n_1) & \text{if } l_1 = 0, l_2 \neq 0, \\ \frac{1}{q^2}A(n_1)A(n_2-n_1) & \text{if } l_1 \neq 0, l_2 \neq 0, \end{cases} \tag{4.26}$$

and

$$\mathcal{N}_3 = \sum_{\substack{(v_1,\dots,v_{n_3})\in(\mathbb{F}_q^*)^{n_3} \\ \text{such that (4.9) holds}}} 1$$

$$= \begin{cases} \frac{(q-1)^3}{q^3}A(n_1-1)A(n_2-n_1-1)A(n_3-n_2-1) & \text{if } l_1=l_2=l_3=0, \\ \frac{(q-1)^2}{q^3}A(n_1)A(n_2-n_1-1)A(n_3-n_2-1) & \text{if } l_1\neq 0, l_2=l_3=0, \\ \frac{(q-1)^2}{q^3}A(n_2-n_1)A(n_1-1)A(n_3-n_2-1) & \text{if } l_1=0, l_2\neq 0, l_3=0, \\ \frac{q-1}{q^3}A(n_1)A(n_2-n_1)A(n_3-n_2-1) & \text{if } l_1\neq 0, l_2\neq 0, l_3=0, \\ \frac{(q-1)^2}{q^3}A(n_1-1)A(n_2-n_1-1)A(n_3-n_2) & \text{if } l_1=l_2=0, l_3\neq 0, \\ \frac{q-1}{q^3}A(n_1-1)A(n_2-n_1)A(n_3-n_2) & \text{if } l_1=0, l_2\neq 0, l_3\neq 0, \\ \frac{q-1}{q^3}A(n_1)A(n_2-n_1-1)A(n_3-n_2) & \text{if } l_1\neq 0, l_2=0, l_3\neq 0, \\ \frac{1}{q^3}A(n_1)A(n_2-n_1)A(n_3-n_2) & \text{if } l_1\neq 0, l_2\neq 0, l_3\neq 0. \end{cases} \tag{4.27}$$

Hence, by (4.25) to (4.27), the identity (1.8) follows immediately. This completes the proof of Theorem 1.2. □

5 Examples

In this section, we supply two examples to demonstrate the validity of Theorems 1.1 and 1.2.

Example 5.1. We use Theorem 1.1 to compute the number $N(V)$ of rational points on the following variety over $\mathbb{F}_7$:

$$\begin{cases} f_1(x_1,\dots,x_5) = x_1x_2x_3 + x_1x_2^2x_3^3 + x_1^2x_2^2x_3 - 2, \\ f_2(x_1,\dots,x_5) = x_1^3x_2x_3^2x_4^3x_5 + x_1^2x_2^3x_3^4x_4x_5^2. \end{cases}$$

Clearly, we have $b_1 = 2, b_2 = 0, q = 7, q-1 = 6, m_1 = 3, m_2 = 5$, and

$$\mathcal{E} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 1 & 2 \end{pmatrix}.$$

Since $\det(\mathcal{E}) = -5$, one derives that $\gcd(q-1, \det(\mathcal{E})) = 1$. By Theorem 1.1, we can calculate and obtain that

$$N(V) = \frac{6^3-(-1)^3}{7}\cdot(7^2-6^2) + \frac{6^3-(-1)^3}{7}\cdot\frac{6^2+(-1)^2\times 6}{7} = 589.$$

Example 5.2. We use Theorem 1.2 to compute the number $N(V)$ of rational points on the variety over $\mathbb{F}_{17}$ determined by

$$\begin{cases} f_1(x_1,\dots,x_7) = x_1x_2^2x_3^3x_4^4x_5^5 + x_1^2x_2^5x_3^4x_4^5x_5^3 + x_1^3x_2^4x_3^2x_4^3x_5^2 + x_1^2x_2^3x_3^5x_4^2x_5 + x_1^2x_2^6x_3^3x_4^2x_5^2 - 1, \\ f_2(x_1,\dots,x_7) = x_1^2x_2^2x_3^3x_4^3x_5^5x_6^3 + x_1^2x_2^2x_3^4x_4^5x_5^3x_6^5x_7^3 - 2, \\ f_2(x_1,\dots,x_7) = x_1^2x_2^2x_3^4x_4^5x_5^3x_6^5x_7^3x_8 - 3. \end{cases}$$

Clearly, we have $l_1 = 1$, $l_2 = 2$, $l_3 = 3$, $q = 17$, $q - 1 = 16$, $n_1 = 5$, $n_2 = 7$, $n_3 = 8$, and

$$\mathscr{F} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 \\ 2 & 5 & 4 & 5 & 3 & 0 & 0 & 0 \\ 3 & 4 & 2 & 3 & 2 & 0 & 0 & 0 \\ 2 & 3 & 5 & 2 & 1 & 0 & 0 & 0 \\ 2 & 6 & 3 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 3 & 3 & 5 & 3 & 0 & 0 \\ 2 & 2 & 4 & 5 & 3 & 5 & 3 & 0 \\ 2 & 2 & 4 & 5 & 3 & 5 & 3 & 1 \end{pmatrix}.$$

Since $\det(\mathscr{F}) = 2889$, we deduce that $\gcd(q-1, \det(\mathscr{F})) = 1$. By Theorem 1.2, we can calculate and obtain

$$N(V) = \frac{1}{17^3}((17-1)^5 - (-1)^5)((17-1)^2 - (-1)^2)((17-1) - (-1)) = 925215.$$

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