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Research Article

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On superstability of derivations in Banach algebras

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Abstract: In this article, we consider some types of derivations in Banach algebras. In detail, we investigate the question of whether the superstability can be achieved under some conditions for some types of derivations, such as Jordan derivations, generalized Lie 2-derivations, and generalized Lie derivations.

Keywords: Banach algebra, Jordan derivations, generalized Lie derivations, superstability

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1 Introduction

Let $\mathcal{A}$ be an algebra, not necessarily with unit e, with center $Z(\mathcal{A})$. For all $x, y \in \mathcal{A}$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will write the anticommutator $xy + yx$. A *derivation* is an additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in \mathcal{A}$. A *Jordan derivation* is an additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying $\delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y)$ for all $x, y \in \mathcal{A}$. This concept has been generalized in many ways. For example, a *Lie derivation* defined as an additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathcal{A}$. In addition, an additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called *linear* if $\delta(tx) = t\delta(x)$ for all $x \in \mathcal{A}$ and all $t \in \mathbb{C}$.

A type of stability was studied by Bake et al. [1]. Indeed, they proved that if a function is approximately exponential, then it is either a true exponential function or bounded. Then the exponential functional equation is said to be *superstable*. It was the first result concerning the *superstability phenomenon* of functional equations. Baker [2] generalized this result as follows: *Let* $(S, \cdot)$ *be an arbitrary semigroup, and let f map S into the field* $\mathbb{C}$. *Assume that f is an approximately exponential function, i.e., there exists a nonnegative number* $\varepsilon > 0$ *such that*

$$\|f(x \cdot y) - f(x)f(y)\| \le \varepsilon,$$

for all $x, y \in S$. *Then f is either bounded or exponential.*

Later, the superstability for derivations between operator algebras was investigated by Šemrl [3]. Badora presented the stability concerning derivations in [4]. The study of the stability mentioned earlier has its origin in the famous talk of Ulam [5]. Hyers [6] had answered affirmatively the question of Ulam for Banach spaces. After then, many authors have generalized Hyers' result; see, for example, [7–9]. A great amount of subsequent studies of stability to various functional equations involving derivations are still being done. The main objective of the present article is to investigate some types of derivations in Banach algebras. We first try

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to prove some theorems related to superstability of these derivations in Banach algebras. In addition, we considered the continuity of these derivations.

2 Jordan derivations

In this article, we set $\mathbb{T}_\varepsilon \coloneqq \{e^{i\theta} : 0 \le \theta \le \varepsilon\}$ for a given $\varepsilon > 0$. As an example of a Jordan derivation in Banach algebra, we can consider the following:

Let $\mathcal{A} = M_2(\mathbb{C})$ be the Banach algebra of all 2×2 upper triangle matrices over the complex field $\mathbb{C}$. We define a map $\delta : \mathcal{A} \to \mathcal{A}$ by

$$\delta\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

Then, we see that δ is a Jordan derivation.

Next, we introduce the following lemma necessary to prove theorem on Jordan derivations.

Lemma 2.1. *Let $\mathcal{L}$ be a linear mapping on a Banach algebra $\mathcal{A}$ and let τ and δ be mappings such that*

$$\mathcal{L}(x \circ y) = \tau(x) \circ y + x \circ \delta(y) \quad \text{for all } x, y \in \mathcal{A}. \tag{2.1}$$

If $\mathcal{A}$ is semiprime, then τ and δ satisfy

$$\tau(tx) - t\tau(x) \in Z(\mathcal{A}), \quad \tau(x + y) - \tau(x) - \tau(y) \in Z(\mathcal{A}) \quad (x, y \in \mathcal{A}, t \in \mathbb{C}),$$
$$\delta(tx) - t\delta(x) \in Z(\mathcal{A}), \quad \delta(x + y) - \delta(x) - \delta(y) \in Z(\mathcal{A}) \quad (x, y \in \mathcal{A}, t \in \mathbb{C}).$$

In particular, if $\mathcal{A}$ has a unit, the mappings τ and δ are linear.

Proof. It follows from (2.1) that for $t \in \mathbb{C}$ and all $x, y \in A$,

$$t\tau(x) \circ y + tx \circ \delta(y) = t\mathcal{L}(x \circ y) = \mathcal{L}(x \circ ty) = t\tau(x) \circ y + x \circ \delta(ty).$$

Hence, we have

$$x \circ (t\delta(y) - \delta(ty)) = 0 \quad \text{for all } x, y \in \mathcal{A}, \tag{2.2}$$

which gives that

$$x(t\delta(y) - \delta(ty)) = -(t\delta(y) - \delta(ty))x \quad \text{for all } x, y \in \mathcal{A}. \tag{2.3}$$

By substituting wx in place of x in equation (2.2) and using (2.3), we obtain

$$w[x, t\delta(y) - \delta(ty)] = 0, \tag{2.4}$$

that is, $[x, t\delta(y) - \delta(ty)]$ lies in the right annihilator $\mathrm{ran}(\mathcal{A})$ of $\mathcal{A}$. If $\mathcal{A}$ is semiprime, we have from (2.4) that $[x, t\delta(y) - \delta(ty)] = 0$. So we arrive at $\delta(ty) - t\delta(y) \in Z(\mathcal{A})$. If $\mathcal{A}$ has a unit, set $x = e$ in (2.2). Then $\delta(ty) = t\delta(y)$ is true.

In view of (2.1), we have

$$\mathcal{L}(x \circ (y + z)) = \tau(x) \circ y + \tau(x) \circ z + x \circ \delta(y + z). \tag{2.5}$$

Again, by (2.1) and additivity of $\mathcal{L}$, we obtain

$$\mathcal{L}(x \circ (y + z)) = \mathcal{L}(x \circ y) + \mathcal{L}(x \circ z) = \tau(x) \circ y + \tau(x) \circ z + x \circ (\delta(y) + \delta(z)). \tag{2.6}$$

By combining (2.5) and (2.6), we see that

$$x \circ (\delta(y + z) - (\delta(y) + \delta(z))) = 0. \tag{2.7}$$

By using (2.7), we obtain similar results to (2.3) and (2.4). As a result, if A is smiprime, then we have $\delta(y+z)-\delta(y)-\delta(z)\in Z(\mathcal{A})$. Moreover, if $\mathcal{A}$ contains the unit, then we have $\delta(y+z)=\delta(y)+\delta(z)$. It can be proved in a similar way for τ. The lemma is completely established. □

Bhushan et al. [10] introduce the notion of *centrally-extended Jordan derivation* (*CE-Jordan derivation*) of a ring $\mathcal{A}$ as a mapping $\delta:\mathcal{A}\to\mathcal{A}$ satisfying

$$\delta(x+y)-\delta(x)-\delta(y)\in Z(\mathcal{A})\quad \text{for all } x,y\in\mathcal{A},$$
$$\delta(x\circ y)-\delta(x)\circ y-x\circ\delta(y)\in Z(\mathcal{A})\quad \text{for all } x,y\in\mathcal{A}.$$

In [10, Lemma 3.4], it is proved that if $\mathcal{A}$ is a 2-torsion free ring with no nonzero central ideals, then every CE-Jordan derivation is additive. Due to this fact and Lemma 2.1, we obtain the following result.

Theorem 2.2. *Let $\mathcal{A}$ be a semiprime Banach algebra with no nonzero central ideals. Assume that mappings $\Phi:\mathcal{A}\times\mathcal{A}\to[0,\infty)$ and $\varphi:\mathcal{A}\times\mathcal{A}\to[0,\infty)$ satisfy*

(1) $\sigma(x,y):=\sum_{j=0}^{\infty}\frac{1}{2^j}\Phi(2^jx,2^jy)<\infty\ (x,y\in\mathcal{A})$,

(2) $\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^nx,y)=\lim_{n\to\infty}\frac{1}{2^n}\varphi(x,2^ny)=0\ (x,y\in\mathcal{A})$.

Suppose that $\delta:\mathcal{A}\to\mathcal{A}$ is a mapping subjected to

$$\|\delta(x+y)-\delta(x)-\delta(y)\|\le\Phi(x,y)\quad(x,y\in\mathcal{A}),\tag{2.8}$$

$$\|\delta(x\circ y)-\delta(x)\circ y-x\circ\delta(y)\|\le\varphi(x,y)\quad(x,y\in\mathcal{A}).\tag{2.9}$$

Then δ is a Jordan derivation. Moreover, δ is a derivation.

Proof. It follows from (2.8) and the Găvruta theorem [8] that there exists a unique additive mapping $\mathcal{L}:\mathcal{A}\to\mathcal{A}$ satisfying $\|\delta(x)-\mathcal{L}(x)\|\le\frac{1}{2}\sigma(x,x)$ for all $x\in\mathcal{A}$. In this case, the mapping $\mathcal{L}$ is defined as follows:

$$\mathcal{L}(x):=\lim_{n\to\infty}\frac{\delta(2^nx)}{2^n}\quad\text{for all } x\in\mathcal{A}.\tag{2.10}$$

We then have by (2.9) that

$$\begin{aligned}\|\mathcal{L}(x\circ y)-\mathcal{L}(x)\circ y-x\circ\mathcal{L}(y)\|&=\lim_{n\to\infty}\frac{1}{2^{2n}}\|\delta(2^nx\circ 2^ny)-\delta(2^nx)\circ 2^ny-2^nx\circ\delta(2^ny)\|\\&\le\lim_{n\to\infty}\frac{1}{2^{2n}}\varphi(2^nx,2^ny)=0,\end{aligned}$$

which implies that

$$\mathcal{L}(x\circ y)=\mathcal{L}(x)\circ y+x\circ\mathcal{L}(y)\quad\text{for all } x,y\in\mathcal{A}.\tag{2.11}$$

By (2.9), we see that

$$\begin{aligned}\|\mathcal{L}(x\circ y)-\mathcal{L}(x)\circ y-x\circ\delta(y)\|&=\lim_{n\to\infty}\frac{1}{2^n}\|\delta(2^nx\circ y)-\delta(2^nx)\circ y-2^nx\circ\delta(y)\|\\&\le\lim_{n\to\infty}\frac{1}{2^n}\varphi(2^nx,y)=0.\end{aligned}$$

Hence, we obtain

$$\mathcal{L}(x\circ y)=\mathcal{L}(x)\circ y+x\circ\delta(y)\quad\text{for all } x,y\in\mathcal{A}.\tag{2.12}$$

So, by virtue of Lemma 2.1, we find that

$$\delta(x+y)-\delta(x)-\delta(y)\in Z(\mathcal{A})\quad\text{for all } x,y\in\mathcal{A}.\tag{2.13}$$

By combining (2.11) and (2.12), we obtain

$$x \circ \mathcal{L}(y) = x \circ \delta(y) \quad \text{for all } x, y \in \mathcal{A}. \tag{2.14}$$

But then, we found that

$$\begin{aligned}\|\mathcal{L}(x \circ y) - \delta(x) \circ y - x \circ \mathcal{L}(y)\| &= \lim_{n\to\infty} \frac{1}{2^n} \|\delta(x \circ 2^n y) - \delta(x) \circ 2^n y - x \circ \delta(2^n y)\| \\ &\le \lim_{n\to\infty} \frac{1}{2^n} \varphi(x, 2^n y) = 0.\end{aligned}$$

This gives that

$$\mathcal{L}(x \circ y) = \delta(x) \circ y + x \circ \mathcal{L}(y) \quad \text{for all } x, y \in \mathcal{A}. \tag{2.15}$$

Considering (2.11) and (2.15), we are lead to

$$\mathcal{L}(x) \circ y = \delta(x) \circ y \quad \text{for all } x, y \in \mathcal{A}. \tag{2.16}$$

Therefore, with the help of (2.14) and (2.15), we figure out that

$$\mathcal{L}(x \circ y) = \delta(x) \circ y + x \circ \delta(y) \quad \text{for all } x, y \in \mathcal{A}. \tag{2.17}$$

The expression (2.16) can be represented as follows:

$$(\mathcal{L}(x) - \delta(x)) \circ y = 0 \quad \text{for all } x, y \in \mathcal{A}. \tag{2.18}$$

By replacing y by yw in (2.18) and using (2.18), we obtain

$$[\mathcal{L}(x) - \delta(x), y]w = 0 \quad \text{for all } x, y, w \in \mathcal{A}.$$

That is, $[\mathcal{L}(x) - \delta(x), y]$ lies in the left annihilator $\mathrm{lan}(\mathcal{A})$ of $\mathcal{A}$. The semiprimeness of $\mathcal{A}$ implies that $[\mathcal{L}(x) - \delta(x), y] = 0$. So we have

$$[\mathcal{L}(x), y] = [\delta(x), y] \quad \text{for all } x, y \in \mathcal{A}. \tag{2.19}$$

By applying (2.17) and (2.19), we yield that $[\delta(x \circ y) - \delta(x) \circ y - x \circ \delta(y), z] = 0$, that is,

$$\delta(x \circ y) - \delta(x) \circ y - x \circ \delta(y) \in Z(\mathcal{A}) \quad \text{for all } x, y \in \mathcal{A}. \tag{2.20}$$

Then, due to (2.13) and (2.20), we see that the mapping δ is a CE-Jordan derivation. So, based on [10, Lemma 3.4], δ is additive, hence the relation (2.10) forces to $\delta = \mathcal{L}$. Therefore, by referring (2.17), we conclude that

$$\delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y) \quad \text{for all } x, y \in \mathcal{A}.$$

Consequently, δ is a Jordan derivation. Moreover, since $\mathcal{A}$ is semiprime, δ is a derivation [11]. This proves the theorem. □

To prove the next corollary, we will use the following well-known results [12,13]. *Note that semisimple algebras are semiprime and that any linear derivation on a semisimple Banach algebra is continuous.*

Corollary 2.3. *Let $\mathcal{A}$ be a semisimple Banach algebra with no nonzero central ideals. Assume that mappings $\Phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ and $\varphi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfy the assumptions of Theorem* 2.2. *Suppose that $\delta : \mathcal{A} \to \mathcal{A}$ is a mapping such that*

$$\|\delta(tx + ty) - t\delta(x) - t\delta(y)\| \le \Phi(x, y) \quad (x, y \in \mathcal{A},\ t \in \mathbb{T}_\varepsilon), \tag{2.21}$$

together with (2.9). *Then δ is a continuous derivation.*

Proof. We first consider $t = 1$ in (2.21). According to Theorem 2.2, we see that δ is a derivation.

Inequality (2.21) yields that for all $x \in \mathcal{A}$ and all $t \in \mathbb{T}_\varepsilon$,

$$\|\mathcal{L}(tx) - t\mathcal{L}(x)\| = \lim_{n\to\infty} \frac{1}{2^n}\|\delta(2^n tx) - 2t\delta(2^{n-1}x)\| \le \lim_{n\to\infty} \frac{1}{2^n}\Phi(2^{n-1}x, 2^{n-1}x) = 0,$$

where the mapping $\mathcal{L}$ is defined as (2.10). Hence, we have $\mathcal{L}(tx) = t\mathcal{L}(x)$. Then the mapping $\mathcal{L}$ is linear (refer to [14]). Therefore, since $\delta = \mathcal{L}$, the derivation δ is also linear. The semisimplicity of $\mathcal{A}$ ensures that δ is continuous. This completes the proof of the corollary. □

3 Generalized Lie 2-derivations and generalized Lie derivations

We first prove Lemma 3.1 for the proof of Theorem 3.2 concerning generalized Lie 2-derivations.

Lemma 3.1. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ be a linear mapping. Suppose that $\tau : \mathcal{A} \to \mathcal{A}$ and $\delta : \mathcal{A} \to \mathcal{A}$ are mappings such that*

$$\mathcal{L}([x, y]) = [\tau(x), y] + [x, \delta(y)] \quad \text{for all } x, y \in \mathcal{A}. \tag{3.1}$$

If $Z(\mathcal{A}) = \{0\}$ holds, then the mappings τ and δ are linear.

Proof. For all $x, y \in \mathcal{A}$ and all $t \in \mathbb{C}$,

$$t[\tau(x), y] + [x, \delta(ty)] = \mathcal{L}([x, ty]) = t\mathcal{L}([x, y]) = t[\tau(x), y] + [x, t\delta(y)].$$

Then δ satisfies the equation $[x, \delta(ty) - t\delta(y)] = 0$, that is,

$$\delta(tx) - t\delta(x) \in Z(\mathcal{A}) \quad (x \in \mathcal{A},\ t \in \mathbb{C}).$$

We now have from (3.1) that

$$\mathcal{L}([x, y + z]) = [\tau(x), y + z] + [x, \delta(y + z)]. \tag{3.2}$$

On the other hand, if we calculate in a different way, we obtain

$$\mathcal{L}([x, y + z]) = \mathcal{L}([x, y]) + \mathcal{L}([x, z]) = [\tau(x), y + z] + [x, \delta(y) + \delta(z)]. \tag{3.3}$$

By combining (3.2) and (3.3) gives $[x, \delta(y + z) - \delta(y) - \delta(z)] = 0$, which forces to

$$\delta(x + y) - \delta(x) - \delta(y) \in Z(\mathcal{A}) \quad (x, y \in \mathcal{A}).$$

It can be similarly proved that τ also satisfies the following relation:

$$\tau(tx) - t\tau(x) \in Z(\mathcal{A}), \quad \tau(x + y) - \tau(x) - \tau(y) \in Z(\mathcal{A}) \quad (x, y \in \mathcal{A},\ t \in \mathbb{C}).$$

Therefore, if $Z(\mathcal{A}) = \{0\}$, then τ and δ are linear, which is the assertion of Lemma. □

A *generalized Lie 2-derivation $\delta_0 : \mathcal{A} \to \mathcal{A}$ associated with the Lie derivation δ_1* means that A generalized Lie 2-derivation $\delta_0 : \mathcal{A} \to \mathcal{A}$ is a linear mapping satisfying $\delta_0([x, y]) = [\delta_0(x), y] + [x, \delta_1(y)]$ for all $x, y \in \mathcal{A}$, where $\delta_1 : \mathcal{A} \to \mathcal{A}$ is a Lie derivation, that is, δ_1 is a linear mapping such that $\delta_1([x, y]) = [\delta_1(x), y] + [x, \delta_1(y)]$ for all $x, y \in \mathcal{A}$, see to paper [15,16].

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra. Assume that mappings $\Phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$, $\varphi_0 : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ and $\varphi_1 : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfy*

(1) $\sigma(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\Phi(2^j x, 2^j y) < \infty$ $(x, y \in \mathcal{A})$,

(2) $\lim_{n\to\infty} \frac{1}{2^n}\varphi_k(2^n x, y) = \lim_{n\to\infty} \frac{1}{2^n}\varphi_k(x, 2^n y) = 0$ $(k = 0, 1;\ x, y \in \mathcal{A})$.

Suppose that $\delta_0 : \mathcal{A} \to \mathcal{A}$ and $\delta_1 : \mathcal{A} \to \mathcal{A}$ are mappings such that for each $k = 0, 1$,

$$\|\delta_k(tx + ty) - t\delta_k(x) - t\delta_k(y)\| \le \Phi(x, y) \quad (x, y \in \mathcal{A},\ t \in \mathbb{T}_\varepsilon) \tag{3.4}$$

and the following inequalities

$$\|\delta_0([x,y]) - [\delta_0(x),y] - [x,\delta_1(y)]\| \le \varphi_0(x,y) \quad (x,y \in \mathcal{A}), \tag{3.5}$$

$$\|\delta_1([x,y]) - [\delta_1(x),y] - [x,\delta_1(y)]\| \le \varphi_1(x,y) \quad (x,y \in \mathcal{A}). \tag{3.6}$$

If $Z(\mathcal{A}) = \{0\}$, *then* δ_0 *is a generalized Lie 2-derivation associated with the Lie derivation* δ_1.

Proof. Employing the same method as in the proof of Corollary 2.3, for each $k = 0, 1$, there exists a unique linear mapping $\mathcal{L}_k : \mathcal{A} \to \mathcal{A}$ satisfying $\|\delta_k(x) - \mathcal{L}_k(x)\| \le \frac{1}{2}\sigma(x,x)$ for all $x \in \mathcal{A}$, where the mapping $\mathcal{L}_k$ is defined as (2.10).

It follows from (3.5) that

$$\begin{aligned}\|\mathcal{L}_0([x,y]) - [\mathcal{L}_0(x),y] - [x,\mathcal{L}_1(y)]\| &= \lim_{n\to\infty}\frac{1}{2^{2n}}\|\delta_0([2^n x, 2^n y]) - [\delta_0(2^n x), 2^n y] - [2^n x, \delta_1(2^n y)]\| \\ &\le \lim_{n\to\infty}\frac{1}{2^{2n}}\varphi_0(2^n x, 2^n y) = 0.\end{aligned}$$

This means that

$$\mathcal{L}_0([x,y]) = [\mathcal{L}_0(x),y] + [x,\mathcal{L}_1(y)] \quad \text{for all } x,y \in \mathcal{A}. \tag{3.7}$$

Moreover, we have by (3.5) that

$$\begin{aligned}\|\mathcal{L}_0([x,y]) - [\mathcal{L}_0(x),y] - [x,\delta_1(y)]\| &= \lim_{n\to\infty}\frac{1}{2^{n}}\|\delta_0([2^n x, y]) - [\delta_0(2^n x), y] - [2^n x, \delta_1(y)]\| \\ &\le \lim_{n\to\infty}\frac{1}{2^{n}}\varphi_0(2^n x, y) = 0,\end{aligned}$$

which implies that

$$\mathcal{L}_0([x,y]) = [\mathcal{L}_0(x),y] + [x,\delta_1(y)] \quad \text{for all } x,y \in \mathcal{A}. \tag{3.8}$$

It follows from (3.5) that

$$\begin{aligned}\|\mathcal{L}_0([x,y]) - [\delta_0(x),y] - [x,\mathcal{L}_1(y)]\| &= \lim_{n\to\infty}\frac{1}{2^{n}}\|\delta_0([x, 2^n y]) - [\delta_0(2^n x), 2^n y] - [2^n x, \delta_1(2^n y)]\| \\ &\le \lim_{n\to\infty}\frac{1}{2^{n}}\varphi_0(x, 2^n y) = 0.\end{aligned}$$

Hence, we have

$$\mathcal{L}_0([x,y]) = [\delta_0(x),y] + [x,\mathcal{L}_1(y)] \quad \text{for all } x,y \in \mathcal{A}. \tag{3.9}$$

Expressions (3.8), (3.9), and Lemma 3.1 ensure that δ_0 and δ_1 are linear. Then by (2.10), we are lead to $\delta_0 = \mathcal{L}_0$ and $\delta_1 = \mathcal{L}_1$. Thus, we obtain by (3.7) that

$$\delta_0([x,y]) = [\delta_0(x),y] + [x,\delta_1(y)] \quad \text{for all } x,y \in \mathcal{A}.$$

The following equation $\delta_1([x,y]) = [\delta_1(x),y] + [x,\delta_1(y)]$ can be proved in a similar way to the one proved above. Hence, δ_1 is a Lie derivation. Therefore, δ_0 is a generalized Lie 2-derivation associated with the Lie derivation δ_1. This completes the proof. □

We consider the following property regarding Lie derivations.

Theorem 3.3. *Let* $\mathcal{A}$ *be a semisimple Banach algebra. Assume that mappings* $\Phi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ *and* $\varphi : \mathcal{A} \times \mathcal{A} \to [0,\infty)$ *satisfy the assumptions of Theorem* 3.2. *Suppose that* $\delta : \mathcal{A} \to \mathcal{A}$ *is a mapping subjected to the inequality* (2.21) *and*

$$\|[\delta([x,y]) - [\delta(x),y] - [x,\delta(y)]\| \le \Phi(x,y)$$

for all $x,y \in \mathcal{A}$. *If* $Z(\mathcal{A}) = \{0\}$, *then* δ *is a continuous Lie derivation.*

Proof. By similar approach to the proof of Corollary 2.3 and Theorem 3.2, we can see that δ is a Lie derivation. Also, it is known that every Lie derivation on a semisimple Banach algebra $\mathcal{A}$ is continuous if $Z(\mathcal{A}) = 0$ (refer to [17]). Therefore, we complete the proof. □

The concept of generalized Lie derivation can be found in [15,16]: A linear mapping $\delta_0 : \mathcal{A} \to \mathcal{A}$ is called *a generalized Lie derivation with the associated derivation* δ_1 if there exists a derivation $\delta_1 : \mathcal{A} \to \mathcal{A}$ such that

$$\delta_0([x, y]) = \delta_0(x)y - \delta_0(y)x + x\delta_1(y) - y\delta_1(x) \quad \text{for all } x, y \in \mathcal{A}. \tag{3.10}$$

For an example of a generalized Lie derivation in Banach algebra, we have the following:

Let $\mathcal{A} = M_2(\mathbb{C})$ be the Banach algebra of all 2×2 upper triangle matrices over the complex field $\mathbb{C}$. We define a map $\delta_0 : \mathcal{A} \to \mathcal{A}$ by

$$\delta_0\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & 2b \\ 0 & 0 \end{bmatrix}.$$

Then, we see that δ_0 is a generalized Lie derivation with an associated derivation δ_1, where δ_1 is a map defined by

$$\delta_1\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

Note that the definition of the generalized Lie derivation should not be confused with the concept of the generalized Lie 2-derivation. Now we consider superstability of generalized Lie derivation.

Theorem 3.4. *Let $\mathcal{A}$ be either a semiprime Banach algebra or a unital Banach algebra. Assume that mappings $\Phi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$, $\varphi_0 : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ and $\varphi_1 : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ satisfy the assumptions of Theorem* 3.2. *If $\delta_0 : \mathcal{A} \to \mathcal{A}$ and $\delta_1 : \mathcal{A} \to \mathcal{A}$ are mappings such that the inequality* (3.4) *and*

$$\|\delta_0([x, y]) - \delta_0(x)y + \delta_0(y)x - x\delta_1(y) + y\delta_1(x)\| \le \varphi_0(x, y) \quad (x, y \in \mathcal{A}), \tag{3.11}$$

$$\|\delta_1(xy) - \delta_1(x)y - x\delta_1(y)\| \le \varphi_1(x, y) \quad (x, y \in \mathcal{A}) \tag{3.12}$$

are fulfilled, then δ_0 is a generalized Lie derivation associated with the derivation δ_1.

Proof. As we did in the proof of Corollary 2.3, for each $k = 0,1$, there exists a unique linear mapping $\mathcal{L}_k : \mathcal{A} \to \mathcal{A}$ satisfying $\|\delta_k(x) - \mathcal{L}_k(x)\| \le \frac{1}{2}\sigma(x, x)$ for all $x \in \mathcal{A}$. Here, the mapping $\mathcal{L}_k$ is defined as (2.10).

It follows from (3.11) that

$$\begin{aligned} &\|\mathcal{L}_0([x, y]) - \delta_0(x)y + \mathcal{L}_0(y)x - x\mathcal{L}_1(y) + y\delta_1(x)\| \\ &= \lim_{n\to\infty} \frac{1}{2^n}\|\delta_0([x, 2^n y]) - 2^n\delta_0(x)y + \delta_0(2^n y)x - x\delta_1(2^n y) + 2^n y\delta_1(x)\| \\ &\le \lim_{n\to\infty} \frac{1}{2^n}\varphi_0(x, 2^n y) = 0. \end{aligned}$$

Hence, we figure out that

$$\mathcal{L}_0([x, y]) = \delta_0(x)y - \mathcal{L}_0(y)x + x\mathcal{L}_1(y) - y\delta_1(x) \quad \text{for all } x, y \in \mathcal{A}. \tag{3.13}$$

On the other hand, in view of (3.12), we see that

$$\begin{aligned} \|\mathcal{L}_1(xy) - \mathcal{L}_1(x)y - x\delta_1(y)\| &= \lim_{n\to\infty} \frac{1}{2^n}\|\delta_1(2^n xy) - \delta_1(2^n x)y - 2^n x\delta_1(y)\| \\ &\le \lim_{n\to\infty} \frac{1}{2^n}\varphi_1(2^n x, y) = 0, \end{aligned}$$

which implies that

$$\mathcal{L}_1(xy) = \mathcal{L}_1(x)y + x\delta_1(y) \quad \text{for all } x, y \in \mathcal{A}.$$

Then we have by [18, Lemma 2.1] that δ_1 is linear. Considering (2.10) gives $\delta_1 = \mathcal{L}_1$ and then, δ_1 is a derivation.

Equation (3.13) can be expressed as follows:

$$\mathcal{L}_0([x,y]) = \delta_0(x)y - \mathcal{L}_0(y)x + x\delta_1(y) - y\delta_1(x) \quad \text{for all } x,y \in \mathcal{A}. \tag{3.14}$$

Hence, by linearity of δ_1, we obtain

$$\mathcal{L}_0([tx,y]) = \delta_0(tx)y - t\mathcal{L}_0(y)x + tx\delta_1(y) - ty\delta_1(x) \tag{3.15}$$

for all $x, y \in \mathcal{A}$ and all $t \in \mathbb{C}$. Furthermore, due to (3.14), we have

$$\mathcal{L}_0([tx,y]) = t\mathcal{L}_0([x,y]) = t\delta_0(x)y - t\mathcal{L}_0(y)x + tx\delta_1(y) - ty\delta_1(x) \tag{3.16}$$

for all $x, y \in \mathcal{A}$ and all $t \in \mathbb{C}$. By combining (3.15) and (3.16), we are lead to

$$(\delta_0(tx) - t\delta_0(x))y = 0 \quad (x,y \in \mathcal{A}, t \in \mathbb{C}). \tag{3.17}$$

This means that $\delta_0(tx) - t\delta_0(x)$ lies in the left annihilator $\mathrm{lan}(\mathcal{A})$ of $\mathcal{A}$. If $\mathcal{A}$ is semiprime, expression (3.17) ensures that $\delta_0(tx) = t\delta_0(x)$. If $\mathcal{A}$ has a unit, set $y = e$ in (3.17). Then we arrive at $\delta_0(tx) = t\delta_0(x)$.

But then, we have by (3.14) and by additivity of δ_1, that

$$\mathcal{L}_0([x+z,y]) = \delta_0(x+z)y - \mathcal{L}_0(y)x - \mathcal{L}_0(y)z + x\delta_1(y) + z\delta_1(y) - y\delta_1(x) - y\delta_1(z). \tag{3.18}$$

Another way to compute this is

$$\begin{aligned}\mathcal{L}_0([x+z,y]) &= \mathcal{L}_0([x,y]) + \mathcal{L}_0([z,y]) \\ &= \delta_0(x)y - \mathcal{L}_0(y)x + x\delta_1(y) - y\delta_1(x) + \delta_0(z)y - \mathcal{L}_0(y)z + z\delta_1(y) - y\delta_1(z).\end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) yields

$$(\delta_0(x+z) - \delta_0(x) - \delta_0(z))y = 0 \quad \text{for all } x,y \in \mathcal{A}. \tag{3.20}$$

That is, $\delta_0(x+z) - \delta_0(x) - \delta_0(z)$ lies in the left annihilator $\mathrm{lan}(\mathcal{A})$ of $\mathcal{A}$. If $\mathcal{A}$ is semiprime, then by (3.20), δ satisfies $\delta_0(x+z) = \delta_0(x) + \delta_0(z)$. If $\mathcal{A}$ has a unit, let $y = e$ in (3.20). Then $\delta_0(x+z) = \delta_0(x) + \delta_0(z)$.

Thus, δ_0 is linear, hence we are forced to conclude that $\delta_0 = \mathcal{L}_0$. Therefore, expression (3.14) guarantees that δ_0 satisfies (3.10). Consequently, δ_0 is a generalized Lie derivation with the associated derivation δ_1. The theorem is proved. □

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