Research Article

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A characterization of the translational hull of a weakly type B semigroup with E-properties

https://doi.org/10.1515/math-2025-0145
received September 6, 2024; accepted March 31, 2025

Abstract: Recall that weakly type B semigroups are generalized inverse semigroups on semi-abundant semigroups. The main aim of this article is to prove that the translational hull of a weakly type B semigroup is still a semigroup of the same type. Some characterizations and properties on the translational hull of weakly type B semigroups are obtained. As an application, some characterizations of the translational hull on weakly type B semigroups with E-properties are given.

Keywords: weakly type B semigroup, translational hull, proper weakly type B semigroup, E-reflexive weakly type B semigroup

MSC 2020: 20M10, 06F05

1 Introduction

In recent years, many scholars gradually pay attention to the study of generalized regular semigroups. Usually, generalized regular semigroups can be considered by using generalized Green's relations. Fountain [1] introduced the concept of an abundant semigroup by using generalized Green's relations $\mathcal{L}^*$ and $\mathcal{R}^*$. Since then, the study of various classes of abundant semigroups has attracted the attention of many scholars at home and abroad [2–7]. Lawson [8] defined semi-abundant semigroups by generalized Green's relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$. The intersection of $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ is denoted by $\widetilde{\mathcal{H}}$. Following Lawson [8], a semigroup S is called a *left [resp., right] semi-abundant semigroup* if each $\widetilde{\mathcal{R}}$-class [resp., each $\widetilde{\mathcal{L}}$-class] of S contains an idempotent. A left [resp., right] semi-abundant semigroup is said to be *left [resp., right] semi-adequate* if its idempotents commute. A semigroup is *semi-abundant [resp., semi-adequate]* if it is both left and right semi-abundant [resp., semi-adequate]. As usual, $E(S)$ denotes the idempotents set of S. A left semi-adequate semigroup is called a *weakly left type B semigroup*, if it satisfies the following conditions:

(WB1) $\widetilde{\mathcal{R}}$ is a left congruence on S;
(WB2) $(\forall e, f \in E(S^1), a \in S)\ (aef)^+ = (ae)^+(af)^+$;
(WB3) $(\forall e \in E(S), a \in S)\ e \leqslant a^+ \Rightarrow (\exists f \in E(S^1)), e = (af)^+$.

Dually, we can define the concept of a weakly right type B semigroup. A *weakly type B semigroup* is both a weakly right type B semigroup and a weakly left type B semigroup. As usual, we denote by a^+ [resp., a^*] an idempotent $\widetilde{\mathcal{R}}$-[resp., $\widetilde{\mathcal{L}}$-] related to a.

Let S be a semigroup. For all $a, b \in S$, the mapping λ [resp., ρ] is called a *left [resp., right] translation* if $\lambda(ab) = (\lambda a)b$ [resp., $(ab)\rho = a(b\rho)$]. λ and ρ are *linked*, if $a(\lambda b) = (a\rho)b$. The linked left and right translational

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pair (λ, ρ) is a *bitranslation* of S. We denote all left [resp., right] translations of S by $\wedge(S)$ [resp., $I(S)$]. The set of all bitranslations of S is denoted by $\Omega(S)$, and $\Omega(S)$ is called *the translational hull* of S. The operation on the translational hull $\Omega(S)$ is as follows:

$$(\lambda_1, \rho_1)(\lambda_2, \rho_2) = (\lambda_1\lambda_2, \rho_1\rho_2).$$

Ponizovski [9] studied the translational hull of an inverse semigroup. Since then, a lot of scholars have extended the translational hull of an inverse semigroup to the cases of generalized inverse semigroups and have obtained some interesting results. In 1985, Fountain and Lawson [10] proved that the translational hull of an adequate semigroup is again an adequate semigroup. On this basis, Guo and Shum [11] showed that the translational hull of a type A semigroup is type A. In 2011, Li and Wang [12] verified that the translational hull of a type B semigroup is type B. As we all know, weakly type B semigroups are generalized inverse semigroups on semi-abundant semigroups. Naturally, one would ask whether the translational hull of a weakly type B semigroup is still a semigroup of the same type? In this article, we shall give a positive answer to this problem.

2 Preliminaries

In this section, we first review briefly the basic facts and concepts connected to semi-abundant semigroups. For more details, we refer the reader to [9] and [13,14].

Lemma 2.1. [14] *Let S be a semigroup, and suppose that $E \subseteq E(S)$ is a subsemilattice of S with $a, b \in S$. Then, the following statements are equivalent*:
(1) $a\widetilde{\mathcal{L}}b$ [$a\widetilde{\mathcal{R}}b$];
(2) $(\forall e \in E)\ ae = a \Leftrightarrow be = b$, [$(\forall e \in E)\ ea = a \Leftrightarrow eb = b$].

Lemma 2.2. [14] *Let S be a semigroup, and suppose that $E \subseteq E(S)$ is a subsemilattice of S with $a \in S$, $e \in E$. Then, the following statements are equivalent*:
(1) $a\widetilde{\mathcal{L}}e$ [$a\widetilde{\mathcal{R}}e$];
(2) $ae = a$, $(\forall f \in E)\ a = af \Rightarrow e = ef$ [$a = ea$, $(\forall f \in E)\ a = fa \Rightarrow e = fe$].

Evidently, the relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ are generalizations of Greens * relations $\mathcal{L}^*$ and $\mathcal{R}^*$, respectively. The elements a and b of a semigroup S are $\widetilde{\mathcal{R}}$-related if and only if a and b have the same set of idempotent left identities. The relation $\widetilde{\mathcal{L}}$ is defined dually. Generally, $\mathcal{L}^*$ and $\mathcal{R}^*$ are right and left congruences on a semigroup S, respectively. However, $\widetilde{\mathcal{L}}$ is not a right congruence and $\widetilde{\mathcal{R}}$ is not a left congruence.

Remark 2.1. In Lemmas 2.1 and 2.2, if we replace "Let S be a semigroup" by "Let S be a weakly type B semigroup," then $E = E(S)$.

Lemma 2.3. *Let S be a weakly type B semigroup and $a, b \in S$. Then, the following statements are true*:
(1) $a\widetilde{\mathcal{L}}b \Leftrightarrow a^* = b^*$, $a\widetilde{\mathcal{R}}b \Leftrightarrow a^+ = b^+$;
(2) $(ab)^* = (a^*b)^*$, $(ab)^+ = (ab^+)^+$;
(3) $aa^* = a = a^+a$.

Proof. Since $E(S)$ is a semilattice and $\widetilde{\mathcal{L}}$ [resp., $\widetilde{\mathcal{R}}$] is a right [resp., left] congruence on S, we have that (1) and (2) are true.

(3) It is clear. □

Definition 2.1. Let S be a weakly type B semigroup. Define μ_L, μ_R, and μ on S as follows:

$$(\forall a, b \in S)(a, b) \in \mu_L \Leftrightarrow (\forall e \in E(S)), (ea, eb) \in \widetilde{\mathcal{L}},$$
$$(\forall a, b \in S)(a, b) \in \mu_R \Leftrightarrow (\forall e \in E(S)), (ae, be) \in \widetilde{\mathcal{R}},$$
$$\mu = \mu_L \cap \mu_R.$$

In Definition 2.1, S is a weakly type B semigroup. We have that $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ are right congruence and left congruence on S, respectively. The following lemma shows that μ_L [resp., μ] is the largest congruence on a weakly type B semigroup contained in $\widetilde{\mathcal{L}}$ [resp., $\widetilde{\mathcal{H}}$].

Lemma 2.4. *Let S be a weakly type B semigroup. Then, the following statements are true*:
(1) *μ_L is the largest congruence on S contained in $\widetilde{\mathcal{L}}$*;
(2) *μ_R is the largest congruence on S contained in $\widetilde{\mathcal{R}}$*;
(3) *μ is the largest congruence on S contained in $\widetilde{\mathcal{H}}$*.

Proof. (1) First, we prove that μ_L is a congruence on S. It is clear that μ_L is an equivalence relation on S. Note that S is a weakly type B semigroup. We have that $\widetilde{\mathcal{L}}$ is a right congruence on S, and so μ_L is a right congruence on S. Next, we verify that μ_L is a left congruence on S. To see it, suppose that $a, b, c \in S, e, f \in E(S), (a, b) \in \mu_L$, and $f\widetilde{\mathcal{L}}ec$. Since $\widetilde{\mathcal{L}}$ is a right congruence, we have $(fb, ecb) \in \widetilde{\mathcal{L}}$ and $(fa, eca) \in \widetilde{\mathcal{L}}$. Again, since $(a, b) \in \mu_L$, we obtain $(fa, fb) \in \widetilde{\mathcal{L}}$ from the definition of μ_L. Hence, $(eca, ecb) \in \widetilde{\mathcal{L}}$, and so $(ca, cb) \in \mu_L$. This means that μ_L is a congruence on S. Now, let $a, b \in S$ and $(a, b) \in \mu_L, e, f, g \in E(S)$, and $f\widetilde{\mathcal{R}}a\widetilde{\mathcal{L}}e, b\widetilde{\mathcal{L}}g$. We have $fa\widetilde{\mathcal{L}}fb$ from the definition of μ_L. But, $a = fa$, we obtain $fa\widetilde{\mathcal{L}}e$, and so $fb\widetilde{\mathcal{L}}e$. Note that $bg = b$ implies $fbg = fb$. We obtain $eg = e$ from Lemma 2.2. Similarly, $ge = g$, i.e., $e\mathcal{L}g$ implies $e\widetilde{\mathcal{L}}g$, and so $(a, b) \in \widetilde{\mathcal{L}}$. Therefore, $\mu_L \subseteq \widetilde{\mathcal{L}}$.

Finally, we prove that μ_L is the largest congruence on S contained in $\widetilde{\mathcal{L}}$. Suppose ν is a congruence that included in $\widetilde{\mathcal{L}}$. Let $a, b \in S$, $(a, b) \in \nu$. Then, for all $e \in E(S)$, we have $(ea, eb) \in \nu$, and so $(ea, eb) \in \widetilde{\mathcal{L}}$. Therefore, $(a, b) \in \mu_L$. To sum up, (1) holds.

(2) It follows from the dual of (1).

(3) It follows directly from (1) and (2). □

Definition 2.2. Let S be a weakly type B semigroup and $a \in S, (\lambda, \rho) \in \Omega(S)$. Define the mappings $\lambda^*, \lambda^+, \rho^*, \rho^+$ from S to itself as follows:

$$\lambda^* a = (\lambda a^+)^* a, \quad \lambda^+ a = (a^+\rho)^+ a, \quad a\rho^* = a(\lambda a^*)^*, \quad a\rho^+ = a(a^*\rho)^+.$$

Definition 2.3. Let S be a weakly type B semigroup. Define a relation σ on S as follows:

$$(a, b) \in \sigma \Leftrightarrow (\exists\ e \in E) eae = ebe.$$

By Definition 2.3, it is easily seen that σ is a congruence on S. We call S *proper* if $\sigma \cap \widetilde{\mathcal{R}} = \iota_S$ and $\sigma \cap \widetilde{\mathcal{L}} = \iota_S$, where ι_S is the identity relation on S.

Example 2.1. Let $S = \{[x]_{2\times2} | x \in \mathbb{N}\} \cup \{[\frac{1}{2}]_{2\times2}\}$, where $\mathbb{N}$ is the set of all non-negative integers and $[x]_{2\times2}$ denotes a matrix as follows:

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix},$$

for all $x \in \mathbb{N} \cup \{\frac{1}{2}\}$. It is clear that S is a semi-abundant semigroup with respect to the general matrix multiplication, and that

$$E(S) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right\}.$$

In fact, the $\widetilde{\mathcal{L}}$-classes and $\widetilde{\mathcal{R}}$-classes of S are both $S\backslash\{[0]_{2\times2}\}$ and $\{[0]_{2\times2}\}$. On the other hand, $E(S)$ is a semilattice, and for all $a \in \mathbb{N}$, $a \neq 0$, $[a]^*_{2\times2} = [a]^+_{2\times2} = [\frac{1}{2}]_{2\times2}$. It is routine to check that S satisfies conditions **(WB1)**–**(WB3)** and its dual conditions. Therefore, S is a weakly type B semigroup.

Definition 2.4. [15] Let S be a semigroup. Then, S is left E-unitary, if for all $a \in S$, $e \in E(S)$ such that $ae \in E(S)$ implies $a \in E(S)$. Dually, we can define right E-unitary. A semigroup S is E-unitary if S is both left and right E-unitary.

Lemma 2.5. *Let S be a proper weakly type B semigroup. Then, the following statements are true*:
(1) S *is* E-*unitary*;
(2) $E(S) = 1\sigma$;
(3) σ *is the least unipotent congruence on* S.

Proof. (1) Let $e \in E(S)$, $a \in S$ such that $ea \in E(S)$. Then,

$$(eae)a = ea = eaa^* = eeaa^* = eaea^* = (eae)a^*,$$

i.e., $(eae)a = (eae)a^*$. By multiplying it on the right by (eae), we have $(eae)a(eae) = (eae)a^*(eae)$, where $(eae) \in E(S)$. Hence, $(a, a^*) \in \sigma$ from the definition of σ. Thus, $a(\sigma \cap \widetilde{\mathcal{L}})a^*$. Note that S is a proper weakly type B semigroup. We have $a = a^* \in E(S)$, i.e., S is left E-unitary. Dually, S is right E-unitary.

(2) It is clear that $E(S) \subseteq 1\sigma$ from the definition of σ. Now, we show that $1\sigma \subseteq E(S)$. Let $a \in 1\sigma$, i.e., $(a, 1) \in \sigma$. By the definition of σ, there is $f \in E(S)$ such that $faf = f1f = f \in E(S)$. We have $a \in E(S)$ since S is E-unitary from (1), i.e., $1\sigma \subseteq E(S)$. Therefore, $E(S) = 1\sigma$.

(3) Clearly, σ is a congruence on S. Now, we prove that σ is a unipotent congruence on S. Let $c\sigma \in E(S/\sigma)$, $c\sigma c^2$. Note that $c^+\sigma 1$. We have $cc^+\sigma c^2$ since σ is a congruence. We obtain $c^2\widetilde{\mathcal{R}}cc^+$ since $\widetilde{\mathcal{R}}$ is a left congruence on S. Again, since S is a proper weakly type B semigroup, we have $c^2 = cc^+$, and so $c^4 = c^2$, i.e., $c^2 \in E(S)$. From (2), we have $c\sigma c^2\sigma 1$. Thus, $c \in 1\sigma = E(S)$, i.e., $c\sigma = 1\sigma$. Therefore, σ is a unipotent congruence on S. Finally, we verify that σ is the least unipotent congruence on S. Let $(a, b) \in \sigma$ and ρ be an arbitrary unipotent congruence on S. Then, there is $f \in E(S)$ such that $faf = fbf$. Hence, $f\rho a\rho f\rho = f\rho b\rho f\rho$. Note that ρ is a unipotent congruence. We have $f\rho = 1\rho$. Hence, $1\rho a\rho 1\rho = 1\rho b\rho 1\rho$. It means that $(a, b) \in \rho$. This completes the proof. □

3 Translational hull of a weakly type B semigroup

In this section, we characterize the translational hull of a weakly type B semigroup, and then, we prove that the translational hull of a weakly type B semigroup is still a semigroup of the same type. In particular, we obtain some properties of such hull of a weakly type B semigroup by using our characterizations.

Lemma 3.1. *Let S be a semi-abundant semigroup, λ_1, λ_2* [resp., ρ_1, ρ_2] *be left [resp., right] translations. Then, the following statements are true*:
(1) $\lambda_1 = \lambda_2$, *if and only if* $\lambda_1 e = \lambda_2 e$ *for all* $e \in E(S)$;
(2) $\rho_1 = \rho_2$, *if and only if* $e\rho_1 = e\rho_2$ *for all* $e \in E(S)$.

Proof. (1) Necessity. It is clear.

Sufficiency. Let $a \in S$, $e \in E(S)$ and $(a, e) \in \widetilde{\mathcal{R}}$. Then,

$$\lambda_1 a = \lambda_1(ea) = (\lambda_1 e)a = (\lambda_2 e)a = \lambda_2(ea) = \lambda_2 a,$$

i.e., $\lambda_1 = \lambda_2$.

(2) It is a dual of (1). □

Proposition 3.2. *Let S be a weakly type B semigroup. Then, the following statements are true*:
(1) $(\forall e \in E(S))\ e\rho^* = \lambda^* e = (\lambda e)^* \in E(S)$;
(2) $(\forall e \in E(S))\ e\rho^+ = \lambda^+ e = (\lambda e)^+ \in E(S)$;
(3) $(\lambda^*, \rho^*), (\lambda^+, \rho^+) \in E(\Omega(S))$;
(4) $(\lambda^*, \rho^*)\widetilde{\mathcal{L}}(\lambda, \rho)\widetilde{\mathcal{R}}(\lambda^+, \rho^+)$;
(5) $E(S) = \{(\lambda, \rho) \in \Omega(S) | \lambda E(S) \cup E(S)\rho \subseteq E(S)\}$.

Proof. (1) Note that $E(S)$ is a semilattice since S is a weakly type B semigroup. We have that $e\rho^* = e(\lambda e)^* = (\lambda e)^* e = \lambda^* e$ for all $e \in E(S)$ from Definition 2.2. On the other hand,

$$(e\rho^*)^2 = (e\rho^*)(e\rho^*) = e(\lambda e)^* e(\lambda e)^* = e(\lambda e)^* = (e\rho^*) \in E(S).$$

Clearly, $\lambda e\widetilde{\mathcal{L}}(\lambda e)^*$. We have $\lambda e = (\lambda e)e\widetilde{\mathcal{L}}(\lambda e)^* e = \lambda^* e$ since $\widetilde{\mathcal{L}}$ is a right congruence. Again, since S is $\widetilde{\mathcal{L}}$ unipotent, we have $\lambda^* e = (\lambda e)^*$. It means that (1) holds.

(2) It follows from the dual of (1).

(3) First, we prove that $(\lambda^*, \rho^*), (\lambda^+, \rho^+) \in \Omega(S)$. To see it, let $a, b \in S$. Then, by Definition 2.2,

$$\begin{aligned} \lambda^*(ab) &= (\lambda(ab)^+)^* ab \\ &= (\lambda a^+(ab)^+)^* ab \\ &= ((\lambda a^+)(ab)^+)^* ab = (\lambda a^+)^*(ab)^+ ab \\ &= (\lambda a^+)^* ab = (\lambda^* a)b, \end{aligned}$$

i.e., λ^* is a left translation. Similarly, λ^+ is a left translation, and ρ^* and ρ^+ are right translations. On the other hand, by (2), we have

$$a^*(\lambda^+ b^+) = (\lambda^+ b^+)a^* = \lambda^+(b^+ a^*) = (b^+ a^*)\rho^+ = b^+(a^*\rho^+) = (a^*\rho^+)b^+.$$

Therefore, λ^* and ρ^* are linked, i.e., $(\lambda^*, \rho^*) \in \Omega(S)$. Similarly, $(\lambda^+, \rho^+) \in \Omega(S)$.

Now, we verify that $\psi(S) = \{(\lambda, \rho) \in \Omega(S) | \lambda E(S) \cup E(S)\rho \subseteq E(S)\}$ is a set of idempotents of $\Omega(S)$. To see it, let $(\lambda, \rho) \in \psi(S)$, $e \in E(S)$. Then, $\lambda^2 e = \lambda(\lambda e) = \lambda((\lambda e)e) = \lambda(e(\lambda e)) = (\lambda e)(\lambda e) = \lambda e$. Hence, $\lambda^2 = \lambda$ from Lemma 3.1. Similarly, $\rho = \rho^2$. Thus, $\psi(S)$ is a set of idempotents. It is easy to check that $(\lambda^*)^2 = \lambda^*$ and $(\rho^*)^2 = \rho^*$. Therefore, $(\lambda^*, \rho^*) \in E(\Omega(S))$. Dually, $(\lambda^+, \rho^+) \in E(\Omega(S))$.

(4) We first show that $(\lambda, \rho)(\lambda^*, \rho^*) = (\lambda, \rho)$ for all $(\lambda, \rho) \in \Omega(S)$. To see it, let $e \in E(S)$. Then,

$$\lambda\lambda^* e = \lambda(e\rho^*) = \lambda(ee\rho^*) = (\lambda e)(e\rho^*) = (\lambda e)(\lambda e)^* = (\lambda e),$$

i.e., $\lambda\lambda^* = \lambda$. Dually, $\rho\rho^* = \rho$. Therefore, $(\lambda, \rho)(\lambda^*, \rho^*) = (\lambda, \rho)$.

Now, suppose that $(\lambda_2, \rho_2) \in E(\Omega(S))$ such that $(\lambda, \rho) = (\lambda, \rho)(\lambda_2, \rho_2)$, i.e., $\lambda = \lambda\lambda_2$, $\rho = \rho\rho_2$. Let $e \in E(S)$, $e = a_1$, $\lambda_2 e = a_2$, $\lambda a_1^+ = b_1$, and $\lambda a_2^+ = b_2$. Then,

$$\lambda e = \lambda a_1 = (\lambda a_1^+)a_1 = b_1 a_1 \quad \text{and} \quad \lambda\lambda_2 e = \lambda a_2 = (\lambda a_2^+)a_2 = b_2 a_2.$$

By $\lambda = \lambda\lambda_2$, we have $b_1 a_1 = b_2 a_2$. Clearly,

$$b_1 a_1^+ = (\lambda a_1^+)a_1^+ = \lambda a_1^+ = b_1 \quad \text{and} \quad b_2 a_2^+ = (\lambda a_2^+)a_2^+ = \lambda a_2^+ = b_2,$$

i.e.,

$$b_1^+ = (b_1 a_1^+)^+ = (b_1 a_1)^+ \quad \text{and} \quad b_2^+ = (b_2 a_2^+)^+ = (b_2 a_2)^+.$$

Hence, $b_1^+ = b_2^+$ since $b_1 a_1 = b_2 a_2$. Again,

$$b_1 = b_1^+ b_1 = b_1^+(\lambda a_1^+) = (b_1^+\rho)a_1^+ \quad \text{and} \quad b_2 = b_2^+ b_2 = b_2^+(\lambda a_2^+) = (b_2^+\rho)a_2^+.$$

Therefore,

$$(b_1^+\rho)a_1 = (b_1^+\rho)a_1^+ a_1 = b_1 a_1 \quad \text{and} \quad (b_2^+\rho)a_2 = (b_2^+\rho)a_2^+ a_2 = b_2 a_2.$$

But $b_1 a_1 = b_2 a_2$, which implies that $(b_1^+\rho)a_1 = (b_2^+\rho)a_2$. Hence, $(b_1^+\rho)^* a_1 = (b_2^+\rho)^* a_2$, and so

$$(b_1^+\rho)^* a_1^+ = ((b_1^+\rho)^* a_1)^+ = ((b_2^+\rho)^* a_2)^+ = (b_2^+\rho)^* a_2^+,$$

i.e.,

$$b_1 = (b_1^+\rho)a_1^+ = (b_1^+\rho)(b_1^+\rho)^*a_1^+ = (b_2^+\rho)(b_2^+\rho)^*a_2^+ = b_2.$$

Suppose that $b = b_1 = b_2$. Then, $b^*a_1 = b^*a_2$ and $b_1a_1 = b_2a_2$, i.e., $(\lambda a_1^+)^*a_1 = (\lambda a_2^+)^*a_2$. Hence,

$$\begin{aligned}\lambda^*e = \lambda^*a_1 &= \lambda^*(a_1^+a_1) = (\lambda^*a_1^+)a_1\\ &= (\lambda a_1^+)^*a_1 = (\lambda a_2^+)^*a_2\\ &= (\lambda^*a_2^+)a_2 = \lambda^*(a_2^+a_2) = \lambda^*a_2\\ &= \lambda^*\lambda_2 e.\end{aligned}$$

It means that $\lambda^* = \lambda^*\lambda_2$. Dually, $\rho^* = \rho^*\rho_2$. To sum up, $(\lambda,\rho) = (\lambda,\rho)(\lambda_2,\rho_2)$ implies that $(\lambda^*,\rho^*) = (\lambda^*,\rho^*)(\lambda_2,\rho_2)$. Therefore, $(\lambda,\rho)\widetilde{\mathcal{L}}(\Omega(S))(\lambda^*,\rho^*)$ from Lemma 2.2. Dually, $(\lambda,\rho)\widetilde{\mathcal{R}}(\Omega(S))(\lambda^+,\rho^+)$. This completes the proof.

(5) It is easily seen that $E(\Omega(S)) = \psi(S)$ from the proof of (4). □

Proposition 3.3. *Let S be a weakly type B semigroup and $(\lambda_1,\rho_1),(\lambda_2,\rho_2) \in \Omega(S)$. Then, the following statements are equivalent*:

(1) $(\lambda_1,\rho_1) = (\lambda_2,\rho_2)$;

(2) $\rho_1 = \rho_2$;

(3) $\lambda_1 = \lambda_2$.

Proof. It is clear that (1) ⇔ (2) and (1) ⇔ (3) are dual. Obviously, it is true that (1) implies (2). Next, we prove that (2) implies (1). To see it, let $\rho_1 = \rho_2$. Then, $f\rho_1 = f\rho_2$ for all $f \in E(S)$. Note that S is $\widetilde{\mathcal{L}}$ unipotent. We have

$$\begin{aligned}(\forall f \in E(S))f\rho_1 = f\rho_2 &\Rightarrow (f\rho_1)e = (f\rho_2)e\\ &\Rightarrow f(\lambda_1 e) = f(\lambda_2 e)\\ &\Rightarrow [f(\lambda_1 e)]^+ = [f(\lambda_2 e)]^+\\ &\Rightarrow f(\lambda_1 e)^+ = f(\lambda_2 e)^+.\end{aligned}$$

Put $f = (\lambda_1 e)^+$. Then, $(\lambda_1 e)^+ = (\lambda_1 e)^+(\lambda_2 e)^+$. Similarly, $(\lambda_2 e)^+ = (\lambda_2 e)^+(\lambda_1 e)^+$. Hence, $(\lambda_1 e)^+ = (\lambda_2 e)^+$ since $E(S)$ is a semilattice. Therefore, for all $e \in E(S)$,

$$\lambda_1 e = (\lambda_1 e)^+\lambda_1 e = ((\lambda_1 e)^+\rho_1)e = ((\lambda_1 e)^+\rho_2)e = (\lambda_2 e)^+(\lambda_2 e) = \lambda_2 e,$$

i.e., $\lambda_1 = \lambda_2$ from Lemma 3.1. This together with $\rho_1 = \rho_2$ implies that $(\lambda_1,\rho_1) = (\lambda_2,\rho_2)$. □

Next, we shall give the main result of this section.

Theorem 3.4. *Let S be a weakly type B semigroup. Then, so is $\Omega(S)$.*

Proof. It is easy to see that $\Omega(S)$ is a semi-abundant semigroup from Proposition 3.2(4). Clearly, $E(\Omega(S))$ is a semilattice. Therefore, $\Omega(S)$ is a semi-adequate semigroup. Next, we prove that $\Omega(S)$ is a weakly left type B semigroup. To see it, suppose $(\lambda_1,\rho_1),(\lambda_2,\rho_2) \in E[(\Omega(S))^1]$, $(\lambda,\rho) \in \Omega(S)$. By Proposition 3.2(5), we have $\lambda_1 e, \lambda_2 e, e\rho_1, e\rho_2 \in E(S)$ for all $e \in E(S)$. Therefore,

$$\begin{aligned}(\lambda\lambda_1)^+(\lambda\lambda_2)^+e &= (\lambda\lambda_1)^+[(\lambda\lambda_2)^+ee] = (\lambda\lambda_1)^+e(\lambda\lambda_2)^+e\\ &= e(\rho\rho_1)^+e(\rho\rho_2)^+ = (e\rho\rho_1)^+(e\rho\rho_2)^+\\ &= (e\rho(e\rho)^*\rho_1)^+(e\rho(e\rho)^*\rho_2)^+\\ &= ((e\rho)(e\rho)^*\rho_1(e\rho)^*\rho_2)^+\\ &= ((e\rho)((e\rho)^*\rho_1(e\rho)^*)\rho_2)^+\\ &= ((e\rho)((e\rho)^*(e\rho)^*\rho_1)\rho_2)^+\\ &= (e\rho\rho_1\rho_2)^+\\ &= e(\rho\rho_1\rho_2)^+\\ &= (\lambda\lambda_1\lambda_2)^+e,\end{aligned}$$

i.e., $(\lambda\lambda_1)^+(\lambda\lambda_2)^+ = (\lambda\lambda_1\lambda_2)^+$ from Lemma 3.1 (2). By Proposition 3.3, we have

$$((\lambda\lambda_1)^+(\lambda\lambda_2)^+, (\rho\rho_1)^+(\rho\rho_2)^+) = ((\lambda\lambda_1\lambda_2)^+, (\rho\rho_1\rho_2)^+).$$

Therefore,

$$\begin{aligned}[(\lambda,\rho)(\lambda_1,\rho_1)(\lambda_2,\rho_2)]^+ &= (\lambda\lambda_1\lambda_2, \rho\rho_1\rho_2)^+\\ &= [(\lambda\lambda_1\lambda_2)^+, (\rho\rho_1\rho_2)^+]\\ &= [(\lambda\lambda_1)^+(\lambda\lambda_2)^+, (\rho\rho_1)^+(\rho\rho_2)^+]\\ &= [(\lambda\lambda_1)^+, (\rho\rho_1)^+][(\lambda\lambda_2)^+, (\rho\rho_2)^+]\\ &= (\lambda\lambda_1, \rho\rho_1)^+(\lambda\lambda_2, \rho\rho_2)^+\\ &= [(\lambda,\rho)(\lambda_1,\rho_1)]^+[(\lambda,\rho)(\lambda_2,\rho_2)]^+,\end{aligned}$$

which implies that condition **(WB2)** is satisfied. Let $(\lambda_1, \rho_1) \in E(\Omega(S))$, $(\lambda, \rho) \in \Omega(S)$ such that $(\lambda_1, \rho_1) \leq (\lambda, \rho)^+$. Then, $(\lambda_1, \rho_1) \leq (\lambda, \rho)^+ = (\lambda^+, \rho^+)$ since $\Omega(S)$ is $\widetilde{\mathcal{R}}$ unipotent. Again, since S is a weakly type B semigroup, there is $f \in E(S^1)$ such that $\lambda_1 e = [(\lambda e)f]^+$, i.e., $\lambda_1 e = (\lambda\lambda_f e)^+ = (\lambda\lambda_f)^+ e$. By Lemma 3.1, we have $\lambda_1 = (\lambda\lambda_f)^+$. Thus, by Proposition 3.3,

$$(\lambda_1, \rho_1) = ((\lambda\lambda_f)^+, (\rho\rho_f)^+) = (\lambda\lambda_f, \rho\rho_f)^+ = [(\lambda,\rho)(\lambda_f,\rho_f)]^+,$$

where $(\lambda_f, \rho_f) \in E[(\Omega(S))^1]$. Therefore, condition **(WB3)** holds.

Next, we verify that $\widetilde{\mathcal{R}}$ is a left congruence on $\Omega(S)$. To see it, suppose $(\lambda, \rho), (\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ such that $(\lambda_1, \rho_1)\widetilde{\mathcal{R}}(\Omega(S))(\lambda_2, \rho_2)$. Then, $(\lambda_1^+, \rho_1^+)\widetilde{\mathcal{R}}(\Omega(S))(\lambda_2^+, \rho_2^+)$ from Proposition 3.2(3). Hence, $(\lambda_1^+, \rho_1^+) = (\lambda_2^+, \rho_2^+)$ since $\Omega(S)$ is $\widetilde{\mathcal{R}}$ unipotent, i.e., $\lambda_1^+ = \lambda_2^+$. Thus, $\lambda_1^+ e = \lambda_2^+ e$ for all $e \in E(S)$ from Lemma 3.1. Therefore,

$$\lambda_1 e\widetilde{\mathcal{R}}(S)(\lambda_1 e)^+ = \lambda_1^+ e = \lambda_2^+ e = (\lambda_2 e)^+\widetilde{\mathcal{R}}(S)\lambda_2 e.$$

Note that $\widetilde{\mathcal{R}}$ is a left congruence on S. We obtain $\lambda e\lambda_1 e\widetilde{\mathcal{R}}(S)\lambda e\lambda_2 e$, i.e., $\lambda\lambda_1 e\widetilde{\mathcal{R}}(S)\lambda\lambda_2 e$. Hence, $(\lambda\lambda_1 e)^+ = (\lambda\lambda_2 e)^+$ since S is $\widetilde{\mathcal{R}}$ unipotent, and so $(\lambda\lambda_1)^+ e = (\lambda\lambda_2)^+ e$. By Lemma 3.1, we have $(\lambda\lambda_1)^+ = (\lambda\lambda_2)^+$. Thus, by Proposition 3.3,

$$\begin{aligned}&((\lambda\lambda_1)^+, (\rho\rho_1)^+) = ((\lambda\lambda_2)^+, (\rho\rho_2)^+)\\ \Rightarrow\ &(\lambda\lambda_1, \rho\rho_1)^+ = (\lambda\lambda_2, \rho\rho_2)^+\\ \Rightarrow\ &[(\lambda,\rho)(\lambda_1,\rho_1)]^+ = [(\lambda,\rho)(\lambda_2,\rho_2)]^+,\end{aligned}$$

and so $(\lambda, \rho)(\lambda_1, \rho_1)\widetilde{\mathcal{R}}(\Omega(S))(\lambda, \rho)(\lambda_2, \rho_2)$ from Proposition 3.2(4). This means that condition **(WB1)** is satisfied. Therefore, $\Omega(S)$ is a weakly left type B semigroup. Dually, $\Omega(S)$ is a weakly right type B semigroup. This completes the proof. □

Corollary 3.5. *Let S be a weakly type B semigroup. $\mu_L^{\Omega(S)}$, $\mu_R^{\Omega(S)}$ and $\mu^{\Omega(S)}$ are μ_L, μ_R and μ on $\Omega(S)$, respectively. Then, the following statements are true*:

(1) *for all $e \in E(S)$, $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ and $(\lambda_1, \rho_1)\widetilde{\mathcal{R}}(\Omega(S))(\lambda_2, \rho_2)$ if and only if $e\rho_1\widetilde{\mathcal{R}}(S)e\rho_2$;*

(2) *for all $e \in E(S)$, $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ and $(\lambda_1, \rho_1)\widetilde{\mathcal{L}}(\Omega(S))(\lambda_2, \rho_2)$ if and only if $\lambda_1 e\widetilde{\mathcal{L}}(S)\lambda_2 e$;*

(3) *for all $e \in E(S)$, $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ and $(\lambda_1, \rho_1)\mu_R^{\Omega(S)}(\lambda_2, \rho_2)$ if and only if $e\rho_1\mu_R^S e\rho_2$;*

(4) *for all $e \in E(S)$, $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ and $(\lambda_1, \rho_1)\mu_L^{\Omega(S)}(\lambda_2, \rho_2)$ if and only if $\lambda_1 e\mu_L^S\lambda_2 e$;*

(5) *for all $e \in E(S)$, $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ and $(\lambda_1, \rho_1)\mu^{\Omega(S)}(\lambda_2, \rho_2)$ if and only if $e\rho_1\mu_R^S e\rho_2$ and $\lambda_1 e\mu_L^S\lambda_2 e$.*

Proof. (1) Necessity. Note that $\Omega(S)$ is a weakly type B semigroup from Theorem 3.4. Let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ and $(\lambda_1, \rho_1)\widetilde{\mathcal{R}}(\Omega(S))(\lambda_2, \rho_2)$. Then, $(\lambda_1^+, \rho_1^+)\widetilde{\mathcal{R}}(\Omega(S))(\lambda_2^+, \rho_2^+)$ from Proposition 3.2(4). Hence, $(\lambda_1^+, \rho_1^+) = (\lambda_2^+, \rho_2^+)$ since $\Omega(S)$ is $\widetilde{\mathcal{R}}$ unipotent, i.e., $\rho_1^+ = \rho_2^+$. Thus, $e\rho_1^+ = e\rho_2^+$ for all $e \in E(S)$ from Lemma 3.1. Therefore, $e\rho_1\widetilde{\mathcal{R}}(S)(e\rho_1)^+ = e\rho_1^+ = e\rho_2^+ = (e\rho_2)^+\widetilde{\mathcal{R}}(S)e\rho_2$.

Sufficiency. Suppose that $e\rho_1\widetilde{\mathcal{R}}(S)e\rho_2$ for all $e \in E(S)$. Then, $(e\rho_1)^+ = (e\rho_2)^+$ since S is $\widetilde{\mathcal{R}}$ unipotent, i.e., $e\rho_1^+ = e\rho_2^+$. Hence, by Lemma 3.1, $\rho_1^+ = \rho_2^+$, and so $(\lambda_1^+, \rho_1^+) = (\lambda_2^+, \rho_2^+)$ from Proposition 3.3. Therefore, $(\lambda_1, \rho_1)\widetilde{\mathcal{R}}(\Omega(S))(\lambda_2, \rho_2)$.

(2) It follows from the dual of (1).

(3) Necessity. Let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ such that $(\lambda_1, \rho_1)\mu_R^{\Omega(S)}(\lambda_2, \rho_2)$. Then, for all $f \in E(S)$,

$$(\lambda_1, \rho_1)(\lambda_f, \rho_f)\mu_R^{\Omega(S)}(\lambda_2, \rho_2)(\lambda_f, \rho_f),$$

where $(\lambda_f, \rho_f) \in E(\Omega(S))$. Hence,

$$(\lambda_1, \rho_1)(\lambda_f, \rho_f)\widetilde{\mathcal{R}}(\Omega(S))(\lambda_2, \rho_2)(\lambda_f, \rho_f),$$

i.e.,

$$(\lambda_1\lambda_f, \rho_1\rho_f)\widetilde{\mathcal{R}}(\Omega(S))(\lambda_2\lambda_f, \rho_2\rho_f).$$

Thus, by (1), for all $e \in E(S)$, $e\rho_1\rho_f\widetilde{\mathcal{R}}(S)e\rho_2\rho_f$, and so $e\rho_1 f\widetilde{\mathcal{R}}(S)e\rho_2 f$. By the definition of μ_R, we have $e\rho_1\mu_R(S)e\rho_2$.

Sufficiency. Suppose that $e\rho_1\mu_R^S e\rho_2$ for all $e \in E(S)$. Then, for all $(\lambda, \rho) \in E(\Omega(S))$, $f \in E(S)$,

$$(e\rho_1)(\lambda f)\mu_R^S(e\rho_2)(\lambda f),$$

i.e., $e(\rho_1\rho)f\mu_R^S(e\rho_2\rho)f$. Hence, $e(\rho_1\rho)f\widetilde{\mathcal{R}}(S)(e\rho_2\rho)f$, and so $e\rho_1\rho\mu_R^S e\rho_2\rho$ from the definition of μ_R. By (1), $(\lambda_1\lambda, \rho_1\rho)\widetilde{\mathcal{R}}(\Omega(S))(\lambda_2\lambda, \rho_2\rho)$, i.e., $(\lambda_1, \rho_1)(\lambda, \rho)\widetilde{\mathcal{R}}(\Omega(S))(\lambda_1, \rho_1)(\lambda, \rho)$. Therefore, $(\lambda_1, \rho_1)\mu_R^{\Omega(S)}(\lambda_2, \rho_2)$.

(4) It follows from the dual of (3).

(5) It follows directly from (3) and (4) □

Proposition 3.6. *Let S be a weakly type B semigroup, and $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$. Then, the following statements are equivalent*:

(1) $(\lambda_1, \rho_1)\sigma_{\Omega(S)}(\lambda_2, \rho_2)$;

(2) $(\forall e \in E(S))$, $e\rho_1\sigma_S e\rho_2$;

(3) $(\forall e \in E(S))$, $\lambda_1 e\sigma_S\lambda_2 e$.

Proof. Obviously, (1) ⇔ (2) and (1) ⇔ (3) are dual. We only need to prove that (1) ⇔ (2).

(1) ⇒ (2). Let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ such that $(\lambda_1, \rho_1)\sigma_{\Omega(S)}(\lambda_2, \rho_2)$. Then, by Definition 2.3, there is $(\lambda, \rho) \in E(\Omega(S))$ such that

$$(\lambda, \rho)(\lambda_1, \rho_1)(\lambda, \rho) = (\lambda, \rho)(\lambda_2, \rho_2)(\lambda, \rho),$$

i.e.,

$$(\lambda\lambda_1\lambda, \rho\rho_1\rho) = (\lambda\lambda_2\lambda, \rho\rho_2\rho).$$

Hence, $\rho\rho_1\rho = \rho\rho_2\rho$, and so $e\rho\rho_1\rho = e\rho\rho_2\rho$ for all $e \in E(S)$ from Lemma 3.1, i.e., $e\rho\rho_1(e\rho\rho_1)^*\rho = e\rho\rho_2(e\rho\rho_2)^*\rho$. Hence, $[e\rho\rho_1(e\rho\rho_1)^*\rho]^* = [e\rho\rho_2(e\rho\rho_2)^*\rho]^*$. Obviously, for all $e \in E(S)$, $e\rho, (e\rho\rho_1)^*\rho, (e\rho\rho_2)^*\rho \in E(S)$. Again, since S is $\widetilde{\mathcal{R}}$ unipotent, we obtain $(e\rho\rho_1)^*(e\rho\rho_1)^*\rho = (e\rho\rho_2)^*(e\rho\rho_2)^*\rho$, i.e., $(e\rho\rho_1)^*\rho = (e\rho\rho_2)^*\rho$. Hence, for all $e \in E(S)$, $[(e\rho\rho_1)^*\rho(e\rho)] \in E(S)$,

$$\begin{aligned} e\rho\rho_1\rho = e\rho\rho_2\rho &\Rightarrow (e\rho\rho_1)\rho = (e\rho\rho_2)\rho \\ &\Rightarrow (e\rho\rho_1)(e\rho\rho_1)^*\rho = (e\rho\rho_2)(e\rho\rho_2)^*\rho \\ &\Rightarrow (e \cdot e\rho)\rho_1(e\rho\rho_1)^*\rho = (e \cdot e\rho)\rho_2(e\rho\rho_2)^*\rho \\ &\Rightarrow (e\rho)(e\rho_1)(e\rho\rho_1)^*\rho = (e\rho)(e\rho_2)(e\rho\rho_2)^*\rho \\ &\Rightarrow [(e\rho\rho_1)^*\rho(e\rho)]e\rho_1[(e\rho\rho_1)^*\rho(e\rho)] \\ &= [(e\rho\rho_1)^*\rho(e\rho)]e\rho_2[(e\rho\rho_2)^*\rho(e\rho)]. \end{aligned}$$

This means that $e\rho_1\sigma_S e\rho_2$ from the definition of σ.

(2) ⇒ (1). Suppose that $e\rho_1\sigma_S e\rho_2$ for all $e \in E(S)$. Then, there is $f \in E(S)$ such that $f(e\rho_1)f = f(e\rho_2)f$. Hence, $(ef\rho_1)f = (ef\rho_2)f$, and so $(e\rho_f\rho_1)\rho_f = (e\rho_f\rho_2)\rho_f$, i.e., $e\rho_f\rho_1\rho_f = e\rho_f\rho_2\rho_f$. By Lemma 3.1, $\rho_f\rho_1\rho_f = \rho_f\rho_2\rho_f$. Therefore, $(\lambda_f\lambda_1\lambda_f, \rho_f\rho_1\rho_f) = (\lambda_f\lambda_2\lambda_f, \rho_f\rho_2\rho_f)$ from Proposition 3.3, i.e.,

$$(\lambda_f, \rho_f)(\lambda_1, \rho_1)(\lambda_f, \rho_f) = (\lambda_f, \rho_f)(\lambda_2, \rho_2)(\lambda_f, \rho_f),$$

where $(\lambda_f, \rho_f) \in E(\Omega(S))$. By the definition of σ, we have $(\lambda_1, \rho_1)\sigma_{\Omega(S)}(\lambda_2, \rho_2)$. This completes the proof. □

4 Translational hulls of weakly type B semigroups with E-properties

In this section, we characterize some properties of the translational hull of weakly type B semigroups with E-properties. In particular, we prove that the translational hull of an E-reflexive weakly type B semigroup is still a semigroup of the same type.

Theorem 4.1. *Let S be a proper weakly type B semigroup. Then, so is $\Omega(S)$.*

Proof. Obviously, $\Omega(S)$ is a weakly type B semigroup from Theorem 3.4. Next, we prove that $\Omega(S)$ is proper. To see it, let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ such that $(\lambda_1, \rho_1)[\widetilde{\mathcal{L}}_{\Omega(S)} \cap \sigma_{\Omega(S)}](\lambda_2, \rho_2)$. Then, $\lambda_1 e\widetilde{\mathcal{L}}(S)\lambda_2 e$ and $\lambda_1 e\sigma_S\lambda_2 e$ for all $e \in E(S)$ from Corollary 3.5 and Proposition 3.6. Hence, $\lambda_1 e[\widetilde{\mathcal{L}}_S \cap \sigma_S]\lambda_2 e$. Note that S is a proper weakly type B semigroup. We have $\lambda_1 e = \lambda_2 e$, i.e., $\lambda_1 = \lambda_2$ from Lemma 3.1. Therefore, $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$ from the Proposition 3.3, i.e., $\widetilde{\mathcal{L}}_{\Omega(S)} \cap \sigma_{\Omega(S)} = \iota_{\Omega(S)}$. Dually, we have $\widetilde{\mathcal{R}}_{\Omega(S)} \cap \sigma_{\Omega(S)} = \iota_{\Omega(S)}$. This completes the proof. □

Theorem 4.2. *Let S be a weakly type B semigroup. Then, the following statements are true*:
(1) *if S is primitive* (i.e., for all $0 \neq e, f \in E(S)$, $e \leq f \Rightarrow e = f$), *then so is* $\Omega(S)$;
(2) *if S is E-unitary, then so is* $\Omega(S)$.

Proof. (1) Clearly, $\Omega(S)$ is a weakly type B semigroup from Theorem 3.4. Next, we prove that $\Omega(S)$ is primitive. To see it, let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in E(\Omega(S))$ such that $(\lambda_1, \rho_1) \leq (\lambda_2, \rho_2)$. Then,

$$(\lambda_1, \rho_1) = (\lambda_1, \rho_1)(\lambda_2, \rho_2) = (\lambda_2, \rho_2)(\lambda_1, \rho_1).$$

Hence, $\rho_1 = \rho_1\rho_2 = \rho_2\rho_1$, and so $e\rho_1 = e\rho_1\rho_2 = e\rho_2\rho_1$ for all $e \in E(S)$, i.e.,

$$e\rho_1 = (ee\rho_1)\rho_2 = (ee\rho_2)\rho_1 = e\rho_2\rho_1.$$

It is clear that $E(S)$ is a semilattice and $e\rho_1, e\rho_2 \in E(S)$. Thus, $e\rho_1 = (e\rho_1)(e\rho_2) = (e\rho_2)(e\rho_1)$. That is, $e\rho_1 \leq e\rho_2$. This means that $e\rho_1 = e\rho_2$ since S is primitive. By Lemma 3.1, we have $\rho_1 = \rho_2$. Therefore, $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$ from Proposition 3.3.

(2) We only need to verify that $\Omega(S)$ is E-unitary. Suppose $(\lambda_1, \rho_1) \in E(\Omega(S))$, $(\lambda, \rho) \in \Omega(S)$, and $(\lambda, \rho)(\lambda_1, \rho_1) \in E(\Omega(S))$, i.e., $(\lambda\lambda_1, \rho\rho_1) \in E(\Omega(S))$. Note that $\lambda_1 e, \lambda\lambda_1 e \in E(S)$ for all $e \in E(S)$. We have $\lambda\lambda_1 e = \lambda\lambda_1 ee = (\lambda e)(\lambda_1 e) \in E(S)$. Hence, $\lambda e \in E(S)$ since S is E-unitary, and so $\lambda e = (\lambda e)(\lambda e) = \lambda(e\lambda e) = \lambda(\lambda ee) = \lambda^2 e$. By Lemma 3.1, $\lambda = \lambda^2$. Therefore, by Proposition 3.3,

$$(\lambda, \rho)^2 = (\lambda, \rho)(\lambda, \rho) = (\lambda^2, \rho^2) = (\lambda, \rho) \in E(\Omega(S)),$$

i.e., $\Omega(S)$ is right E-unitary. Similarly, $\Omega(S)$ is left E-unitary. This completes the proof. □

Theorem 4.3. *Let S be a fundamental weakly type B semigroup. Then, so is $\Omega(S)$.*

Proof. By Theorem 3.4, $\Omega(S)$ is a weakly type B semigroup. Next, we prove that $\Omega(S)$ is fundamental. Suppose $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$ such that $(\lambda_1, \rho_1)\mu_{\Omega(S)}(\lambda_2, \rho_2)$. Then, by Corollary 3.5, for all $e, f \in E(S)$,

$$(\lambda_1 e)\mu_L^S(\lambda_2 e) \quad \text{and} \quad (f\rho_1)\mu_R^S(f\rho_2).$$

Hence,

$$f(\lambda_1 e)\mu_L^S f(\lambda_2 e) \quad \text{and} \quad (f\rho_1)e\mu_R^S(f\rho_2)e,$$

i.e.,

$$(f\rho_1)e\mu_L^S(f\rho_2)e \quad \text{and} \quad (f\rho_1)e\mu_R^S(f\rho_2)e.$$

Thus, $(f\rho_1)e\mu^S(f\rho_2)e$. Note that S is a fundamental weakly type B semigroup. We have $(f\rho_1)e = (f\rho_2)e$. On the other hand, S is $\widetilde{\mathcal{R}}$ unipotent, and we have

$$(f\rho_1)e = (f\rho_2)e \Rightarrow [(f\rho_1)e]^* = [(f\rho_2)e]^* \Rightarrow (f\rho_1)^*e = (f\rho_2)^*e.$$

Let $e = (f\rho_1)^*$. Then, $(f\rho_1)^* = (f\rho_2)^*(f\rho_1)^*$. Similarly, $(f\rho_2)^* = (f\rho_1)^*(f\rho_2)^*$. Again, since $E(S)$ is a semilattice, we obtain $(f\rho_2)^* = (f\rho_1)^*$. Hence, for all $e, f \in E(S)$, $(f\rho_1)e = (f\rho_2)e$, i.e., $(f\rho_1)(f\rho_1)^* = (f\rho_2)(f\rho_2)^*$, and so $f\rho_1 = f\rho_2$. Thus, $\rho_1 = \rho_2$ from Lemma 3.1. By Proposition 3.3, we have $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$. This shows that $\mu^{\Omega(S)} = \iota_{\Omega(S)}$. This completes the proof. □

Definition 4.1. Let S be a weakly type B semigroup. S is called E-reflexive, if for all $e \in E(S)$, $x, y \in S$ and $exy \in E(S)$ implies $eyx \in E(S)$.

Example 4.1. Let $\mathbb{N}$ be a set of all non-negative integers. Put

$$S = \left\{ \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \middle| \ x \in \mathbb{N} \right\}.$$

It is easy to check that S is a semi-abundant semigroup with respect to the general matrix multiplication for all $x \in \mathbb{N}$ and that

$$E(S) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

In fact, $\widetilde{\mathcal{L}}$-classes and $\widetilde{\mathcal{R}}$-classes of S are both

$$S \backslash \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

On the other hand, $E(S)$ is a semilattice, and for all $a \in \mathbb{N}$, $a \neq 0$,

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}^* = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is routine to check that S satisfies conditions **(WB1)**−**(WB3)** and its dual conditions. Therefore, S is a weakly type B semigroup.

Suppose that

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}, \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} \in S, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} \in E(S),$$

i.e.,

$$\begin{bmatrix} xy & 0 \\ 0 & xy \end{bmatrix} \in E(S).$$

By the multiplication of $\mathbb{N}$, we have $xy = yx$, and so

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} yx & 0 \\ 0 & yx \end{bmatrix} = \begin{bmatrix} xy & 0 \\ 0 & xy \end{bmatrix} \in E(S).$$

On the other hand, it is easy to observe that

$$\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}x&0\\0&x\end{bmatrix}\begin{bmatrix}y&0\\0&y\end{bmatrix}\in E(S)\Leftrightarrow\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}y&0\\0&y\end{bmatrix}\begin{bmatrix}x&0\\0&x\end{bmatrix}\in E(S).$$

Therefore, S is an E-reflexive weakly type B semigroup.

The following theorem is the main result of this section.

Theorem 4.4. *Let S be an E-reflexive weakly type B semigroup. Then, so is $\Omega(S)$.*

Proof. First, we denote "$a \in E(S)$, which implies $b \in E(S)$" by "$a \twoheadrightarrow b$."

By Theorem 3.4, $\Omega(S)$ is a weakly type B semigroup. Next, we prove that $\Omega(S)$ is E-reflexive. To see it, let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$, $(\lambda, \rho) \in E(\Omega(S))$ such that $(\lambda, \rho)(\lambda_1, \rho_1)(\lambda_2, \rho_2) \in E(\Omega(S))$, i.e., $(\lambda\lambda_1\lambda_2, \rho\rho_1\rho_2) \in E(\Omega(S))$. By Proposition 3.2, we have λe, $e\rho$, $\lambda\lambda_1\lambda_2 e$, $e\rho\rho_1\rho_2 \in E(S)$ for all $e \in E(S)$. Hence, for all $e_1, e, f \in E(S)$,

$$\begin{aligned}
e\rho\rho_1\rho_2 &\twoheadrightarrow (e\rho\rho_1\rho_2)e_1 = (e\rho\rho_1)(\lambda_2 e_1)\\
&\twoheadrightarrow e\rho\rho_1(\lambda_2 e_1)e_1 = e\rho\rho_1 e_1(\lambda_2 e_1)\\
&\twoheadrightarrow (e\rho)(\lambda_2 e_1)(\lambda_1 e_1) \quad (\text{since } S \text{ is E-reflexive})\\
&\twoheadrightarrow (e\rho)f(\lambda_2 e_1)(\lambda_1 e_1) = (e\rho)(f\rho_2)e_1(\lambda_1 e_1) = (e\rho)(f\rho_2)(\lambda_1 e_1)\\
&\twoheadrightarrow (e\rho)((e\rho)^*\rho_2)(\lambda_1 e_1) \quad (f = (e\rho)^*)\\
&\quad = (e\rho\rho_2)(\lambda_1 e_1) = (e\rho\rho_2\rho_1)e_1\\
&\twoheadrightarrow (e\rho\rho_2\rho_1)^*(e\rho\rho_2\rho_1) \quad (e_1 = (e\rho\rho_2\rho_1)^*)\\
&\quad = e\rho\rho_2\rho_1,
\end{aligned}$$

i.e., $f\rho\rho_2\rho_1 \in E(S)$ for all $f \in E(S)$. Similarly, for all $f \in E(S)$, $\lambda\lambda_2\lambda_1 f \in E(S)$. By Proposition 3.2, we have $(\lambda\lambda_2\lambda_1, \rho\rho_2\rho_1) \in E(\Omega(S))$. This means that $(\lambda, \rho)(\lambda_2, \rho_2)(\lambda_1, \rho_1) \in E(\Omega(S))$. Therefore, $\Omega(S)$ is an E-reflexive weakly type B semigroup. This completes the proof. □

5 Conclusion remarks

This article extends the theory of translational hulls of inverse semigroups to the case of weakly type B semigroups. It is well known that the translational hull of a certain class of semigroups is not necessarily of the same type. Therefore, investigating whether the translational hull of a class of semigroups remains of the same type is one of the main direction of research within the theory of translational hulls of semigroups. In this article, we not only prove that the translational hull of a weakly type B semigroup is still a weakly type B semigroup, but also demonstrate that the translational hull of a weakly type B semigroup with the E-property remains a semigroup of the same type. This is a highlight of this article. The conclusions drawn in this article provide some references for studying the translational hulls of some generalized inverse semigroups. In future work, the idempotents and abundance of the translational hulls of weakly type B semigroups are worthy of further investigation.

Funding information: This work was supported by the NNSF(CN) (No. 11961026) and the JiangXi Educational Department Natural Science Foundation of China (No. GJJ2200634).

Author contributions: The authors equally conceived the study, participated in its design and coordination, drafted the manuscript, and read and approved the final manuscript.

Conflict of interest: The authors state no conflict of interests.

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