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Research Article

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Boundedness of fractional sublinear operators on weighted grand Herz-Morrey spaces with variable exponents

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Abstract: This article aims to delve deeper into the weighted grand variable Herz-Morrey spaces, and try to establish the boundedness of fractional sublinear operators and their multilinear commutators within this framework. The results are still new even in the unweighted setting, extending previous research. As applications, the corresponding boundedness estimates for the commutators of maximal operator and Riesz potential operator are established.

Keywords: fractional sublinear operator, multilinear commutators, grand Herz-Morrey space, weight

MSC 2020: 42B25, 42B35, 46E30

1 Introduction

The main purpose of this article is to introduce and investigate the weighted grand variable Herz-Morrey spaces. As a class of classical operators in harmonic analysis, sublinear operators only need to meet size conditions compared with other operators. The boundedness of sublinear operators can be applied to several important operators, such as the Hardy-Littlewood maximal operators and Calderón-Zygmund operators.

On the Euclidean space $\mathbb{R}^n$, let $f \in L^1(\mathbb{R}^n)$, and define the sublinear operator T by

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \mathrm{d}y, \quad x \notin \operatorname{supp} f.$$

The commutator generated by a locally integrable function b and the operator T is defined by

$$[b, T](f) \coloneqq bT(f) - T(bf),$$

for suitable functions f. It is well known that the commutator plays an important role in harmonic analysis and has important applications in the estimates of the solutions for the elliptic equations and some nonlinear partial differential equations (PDEs).

Let $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)$, $b_j \in \mathrm{BMO}(\mathbb{R}^n)$, $j \in \{1, 2, \ldots, m\}$, $m \in \mathbb{N}$, $x \notin \operatorname{supp} f$, and define the multilinear commutators of sublinear operators $T^{\boldsymbol{b}}$ by

$$T^{\boldsymbol{b}}(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} |b_j(x) - b_j(y)| K(x, y) f(y) \mathrm{d}y,$$

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where $K(x, y)$ is the integral kernel of the operator T [1].

As we all know, the variable function spaces have found their applications in fluid dynamics, image processing, PDEs, variational calculus, and harmonic analysis [2–6]. Due to the celebrated work of Kováçik and Rákosník [7] on Lebesgue spaces with variable exponent, there has been an increasing interest in extending Lebesgue spaces with variable exponent.

In 2009, Izuki [8] introduced Herz-Morrey spaces with variable exponent and obtained the boundedness of vector-valued sublinear operators satisfying a size condition on Herz-Morrey spaces with variable exponent. In 2010, Izuki [9] defined Herz spaces with variable exponent and application to wavelet characterization. In the same year, Izuki [10] established the boundedness of sublinear operators and their commutators on Herz spaces with variable exponent. Based on these works, in 2016, Izuki and Noi [11] introduced weighted Herz spaces with variable exponent and established the boundedness of fractional integrals on weighted Herz spaces with variable exponent. It is precisely because of Izuki's significant contribution to the weighted variable exponent Herz-Morrey space that there is growing interest in extending the Herz-Morrey space using variable exponents.

In 2018, Wang and Shu [12] generalized Izuki's result for the weighted Herz-Morrey space with variable exponent $M\dot{K}^{\alpha,\lambda}_{p,q(\cdot)}(\omega)$ and proved the boundedness of some sublinear operators. In recent years, the introduction and discussion of the grand function spaces have further enriched the theoretical study of the function space. The grand function spaces is a generalization of the classical function space and have wide applications in the field of PDE. Recently, Zhang et al. [13] introduced the weighted grand Herz-Morrey spaces with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q(\cdot)}(\omega)$, and they obtained the boundedness of sublinear operators.

Inspired by Wang and Shu [12] and Zhang et al. [13], the aim of this article is to introduce the weighted grand variable Herz-Morrey spaces with variable exponents and establish the boundedness of fractional sublinear operators and their multilinear commutators on the weighted grand variable Herz-Morrey spaces with variable exponents.

To be precise, this article is organized as follows.

In Section 2, we recall some notions, notations, and basic properties on the variable exponent Lebesgue spaces and the weighted variable exponent Lebesgue space $L^{q(\cdot)}(\omega)$. Then, we recall the definition of the homogeneous weighted grand of Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q(\cdot)}(\omega)$ and show some related properties. Finally, the main results of this article are given. As applications, the corresponding boundedness estimates for the commutators of maximal operator and Riesz potential operator are established.

In Section 3, via borrowing some ideas from those used in the proofs of [12, Theorem 3.1] and [13, Theorem 4.2], we show that fractional sublinear operators are bounded on homogeneous weighted grand Herz-Morrey spaces with variable exponents. Then, based on the theory of variable exponent and the generalization of bounded mean oscillation (BMO) norm, we show that the boundedness of multilinear commutators of fractional sublinear operators $T^{\boldsymbol{b}}_{\gamma}$ from $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)$ to $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_2(\cdot)}(\omega)$ (see Theorem 2.6).

Finally, we make some conventions on notation. Throughout the whole article, we denote $\chi_k = \chi_{R_k}$, $R_k = B_k \backslash B_{k-1}$, and $B_k = x \in \mathbb{R}^n : |x| \le 2^k$ for $k \in \mathbb{Z}$. We denote by C a positive constant, which is independent of the main parameters, but it may vary from line to line. (More precisely, let C be a variable independent constant, that can represent different values on different rows.) The symbol $D \lesssim F$ means that $D \le CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \approx F$.

2 Preliminary and main results

In this section, we recall some notions, notations, and basic properties on the variable exponent Lebesgue spaces and the weighted variable exponent Lebesgue space $L^{q(\cdot)}(\omega)$.

To begin with, we recall some basic definitions and results on the variable exponent Lebesgue spaces. We can refer to the monographs [14,15] for more information. Let $q(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a measurable function, and the *variable Lebesgue space* $L^{q(\cdot)}(\mathbb{R}^n)$ denotes the set of measurable functions f on $\mathbb{R}^n$ such that, for some $\lambda > 0$,

$$\varrho_{q(\cdot)}(f/\lambda) := \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{q(x)} \mathrm{d}x < \infty.$$

This set becomes a Banach function space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} := \inf\{\lambda > 0 : \varrho_{q(\cdot)}(f/\lambda) \le 1\}.$$

Given an open set $\Omega \subseteq \mathbb{R}^n$, the space $L^{q(\cdot)}_{\mathrm{loc}}(\Omega)$ is defined by

$$L^{q(\cdot)}_{\mathrm{loc}}(\Omega) = \{f : f \in L^{q(\cdot)}(K), \text{ for all compact subsets } K \subset \Omega\}.$$

For conciseness, we denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $q(x)$ on $\mathbb{R}^n$ with range in $[1, \infty)$ such that

$$1 < q_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} q(x) \le q(x) \le q_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} q(x) < \infty. \tag{2.1}$$

An important tool we need is the boundedness of the Hardy-Littlewood maximal operator on variable exponent function spaces. Let $\mathcal{B}(\mathbb{R}^n)$ denote the set of all functions $q(\cdot)$ that satisfy $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and make the Hardy-Littlewood maximal operator M bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, namely,

$$\mathcal{B}(\mathbb{R}^n) := \{q(\cdot) \in \mathcal{P}(\mathbb{R}^n) : M \text{ is bounded on } L^{q(\cdot)}(\mathbb{R}^n)\},$$

where M is the Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{x\in\mathbb{R}^n, r>0} r^{-n} \int_{B(x,r)} |f(y)|\mathrm{d}y.$$

A real-valued measurable function $q(\cdot) : \mathbb{R}^n \to (0, \infty)$ is called locally log-Hölder continuous if there exists a constant $C_{\log} > 0$ such that

$$|q(x) - q(y)| \le \frac{C_{\log}}{-\log(|x-y|)}, \quad \forall x, y \in \mathbb{R}^n, |x-y| < \frac{1}{2}. \tag{2.2}$$

If, for some $q_\infty \in (0, \infty)$ and $C_{\log} > 0$, there holds

$$|q(x) - q(0)| \le \frac{C_{\log}}{\log(e + \frac{1}{|x|})}, \quad \forall x \in \mathbb{R}^n, \tag{2.3}$$

$$|q(x) - q_\infty| \le \frac{C_{\log}}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n, \tag{2.4}$$

then we say $q(\cdot)$ is log-Hölder continuous at the origin (or has a log decay at the origin) and at infinity (or has a log decay at infinity), respectively. The function $q(\cdot)$ is global log-Hölder continuous if $q(\cdot)$ are both locally log-Hölder continuous (2.2) and log-Hölder continuous at infinity (2.4).

The set $\mathcal{P}_0(\mathbb{R}^n)$ consists of all measurable function $q(\cdot)$ satisfying $q_- > 0$ and $q_+ < \infty$. By $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, we denote the class of exponents $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, which satisfy conditions (2.3) and (2.4), respectively. $\mathcal{P}^{\log}(\mathbb{R}^n)$ is the set of functions $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying conditions (2.3) and (2.4), with $q_\infty := \lim_{|x|\to\infty} q(x)$. In particular, we note that if $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $1 < q_- \le q_+ < \infty$, then the Hardy-Littlewood maximal operator M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, namely, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ [15–17].

Lemma 1. ([7, Theorem 2.1]) (generalized Hölder's inequality) *Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $f \in L^{q(\cdot)}(\mathbb{R}^n)$, and $g \in L^{q'(\cdot)}(\mathbb{R}^n)$, the generalized Hölder's inequality holds in the form*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \le r_q \|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{q'(\cdot)}(\mathbb{R}^n)}, \tag{2.5}$$

where $r_q = 1 + 1/q_- - 1/q_+$.

Definition 1. Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and ω be a nonnegative measurable function on $\mathbb{R}^n$. Then, the weighted variable exponent Lebesgue space $L^{q(\cdot)}(\omega)$ is the set of all complex-valued measurable functions f such that $f\omega \in L^{q(\cdot)}$. The space $L^{q(\cdot)}(\omega)$ is a Banach space equipped with the norm

$$\|f\|_{L^{q(\cdot)}(\omega)} \coloneqq \|f\omega\|_{L^{q(\cdot)}}.$$

Lemma 2. [18, Definition 2] *Let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and a positive measurable function ω is said to be in $A_{q(\cdot)}$, if exists a positive constant C for all balls B in $\mathbb{R}^n$ such that*

$$\frac{1}{B}\|\omega\chi_B\|_{L^{q(\cdot)}}\|\omega^{-1}\chi_B\|_{L^{q'(\cdot)}} \le C.$$

The variable Muckenhoupt $A_{q(\cdot)}$ was introduced by Cruz-Uribe et al. [19]. It is easy to see that if $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\omega \in A_{q(\cdot)}$, then $\omega^{-1} \in A_{q'(\cdot)}$.

Lemma 3. [11, Lemma 6] *Let $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $\omega \in A_{q(\cdot)}$, then there exist constants $\delta_1, \delta_2 \in (0, 1)$ and $C > 0$, such that for all balls in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{q(\cdot)}(\omega)}}{\|\chi_B\|_{L^{q(\cdot)}(\omega)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1},$$
$$\frac{\|\chi_S\|_{L^{q'(\cdot)}(\omega^{-1})}}{\|\chi_B\|_{L^{q'(\cdot)}(\omega^{-1})}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

A locally integrable function b is called a BMO function, if it satisfies

$$\|b\|_* \coloneqq \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B|}\int_B |b(y) - b_B|\mathrm{d}y < \infty,$$

where and in the sequel B is ball-centered at x and radius of r, $b_B = \frac{1}{|B|}\int_B b(t)\mathrm{d}t$.

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ with j different elements. For any $\sigma \in C_j^m$, the complementary sequence $\sigma^c = \{1, \dots, m\}\backslash\sigma$. For $\boldsymbol{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$ with $1 \le j \le m$, we denote

$$\begin{aligned}
&\boldsymbol{b}_\sigma \coloneqq (b_{\sigma(1)}, \dots, b_{\sigma(j)}),\\
&[b(x) - b(y)]_\sigma \coloneqq [b_{\sigma(1)}(x) - b_{\sigma(1)}(y)]\dots[b_{\sigma(j)}(x) - b_{\sigma(j)}(y)],\\
&[(b)_B - b(y)]_\sigma \coloneqq [(b_{\sigma(1)})_B - b_{\sigma(1)}(y)]\dots[(b_{\sigma(j)})_B - b_{\sigma(j)}(y)],\\
&\|\boldsymbol{b}_\sigma\|_* = \|b_{\sigma(1)}\|_* \cdots \|b_{\sigma(j)}\|_* \quad \text{for } b_{\sigma(i)} \in \mathrm{BMO}(\mathbb{R}^n).
\end{aligned}$$

In particular,

$$\|\boldsymbol{b}\|_* = \|b_1\|_* \cdots \|b_m\|_*.$$

Lemma 4. [13, Lemma 2.4] *Let $\omega \in A_{q(\cdot)}$, $q(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $b_i \in \mathrm{BMO}(\mathbb{R}^n)$, $i = 1, 2, \dots, m$, $k > j (k, j \in \mathbb{N})$. Then, we have*

$$\sup_{B\subset\mathbb{R}^n} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\omega)}} \left\|\prod_{i=1}^m (b_i - (b_i)_B)\chi_B\right\|_{L^{q(\cdot)}(\omega)} \approx \prod_{i=1}^m \|b_i\|_*$$

and

$$\left\|\prod_{i=1}^m \big(b_i - (b_i)_{B_j}\big)\chi_{B_k}\right\|_{L^{q(\cdot)}(\omega)} \lesssim (k - j)^m \prod_{i=1}^m \|b_i\|_* \|\chi_{B_k}\|_{L^{q(\cdot)}(\omega)}.$$

Now, we recall the definition of weighted grand of Herz-Morrey spaces with variable exponent.

Definition 2. [13, Definition 3.1] Let $0 \le \lambda < \infty$, $k \in \mathbb{Z}$, $1 \le p < \infty$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, and $\theta > 0$. We define the homogeneous weighted grand Herz-Morrey space with variable exponents is denoted by

$$M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega) = \{f \in L_{\mathrm{loc}}^{q(\cdot)}(\mathbb{R}^n \backslash \{0\}, \omega) : \|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)} < \infty\},$$

where

$$\|f\|_{M\dot{K}_{p,q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)} \coloneqq \sup_{\varepsilon>0} \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} = \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{p(1+\varepsilon)}} \|f\|_{M\dot{K}_{p(1+\varepsilon),q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}.$$

The non-homogeneous weighted grand Herz-Morrey space with variable exponents is denoted by

$$MK_{p,q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega) = \{f \in L_{\mathrm{loc}}^{q(\cdot)}(\mathbb{R}^n, \omega) : \|f\|_{MK_{p,q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)} < \infty\},$$

where

$$\|f\|_{MK_{p,q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)} \coloneqq \sup_{\varepsilon>0} \sup_{k_0 \in \mathbb{N}_0} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} = \sup_{\varepsilon>0} \varepsilon^{\frac{\theta}{p(1+\varepsilon)}} \|f\|_{MK_{p(1+\varepsilon),q(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}.$$

Let $0 < \gamma < n$. Then, the fractional sublinear operator T_γ is defined by

$$T_\gamma f(x) \coloneqq \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} \mathrm{d}y.$$

Let $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)$, $b_j \in \mathrm{BMO}(\mathbb{R}^n)$, $j \in \{1, 2, \ldots, m\}$, $m \in \mathbb{N}$, $x \notin \operatorname{supp} f$, and define the multilinear commutators of fractional sublinear operators $T_\gamma^{\boldsymbol{b}}$ by

$$T_\gamma^{\boldsymbol{b}}(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m |b_j(x) - b_j(y)| K(x,y) f(y) \mathrm{d}y, \tag{2.6}$$

where $K(x,y)$ is the integral kernel of the operator T [1]. We are now ready to state the main results of the article, Theorems 1 and 2. Theorem 1 is a generalization of theorems [12, Theorem 3.2] and [13, Theorem 4.2]. Motivated by [12] and [13], we show that fractional sublinear operators are bounded on homogeneous weighted grand Herz-Morrey spaces.

Theorem 1. *Let* $1 < p < \infty$, $q_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ $(i = 1,2.)$, $\omega \in A_{q(\cdot)}$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ *such that* $-n\delta_1 < \alpha(0)$, $\alpha_\infty < n\delta_2 - \gamma$, *where* $0 < \delta_1, \delta_2 < 1$ *are the constants in Lemma* 3. *Let* $\frac{1}{q_2(\cdot)} = \frac{1}{q_1(\cdot)} - \frac{\gamma}{n}$ *and suppose that fractional sublinear operator* T_γ *satisfies the size conditions*

$$|T_\gamma f(x)| \lesssim \|f\|_{L^1(\mathbb{R}^n)}/|x|^{n-\gamma}, \tag{2.7}$$

when $\operatorname{supp} f \subseteq R_k$ *and* $|x| \ge 2^{k+1}$ *with* $k \in \mathbb{Z}$ *and*

$$|T_\gamma f(x)| \lesssim 2^{-k(n-\gamma)} \|f\|_{L^1(\mathbb{R}^n)}, \tag{2.8}$$

when $\operatorname{supp} f \subseteq R_k$ *and* $|x| \le 2^{k-2}$ *with* $k \in \mathbb{Z}$. *Then, if* T_γ *is bounded from* $L^{q_1(\cdot)}(\omega)$ *to* $L^{q_2(\cdot)}(\omega)$, T_γ *is bounded from* $M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$ *to* $M\dot{K}_{p,q_2(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$.

Corollary 1. *Let* p, $\alpha(\cdot)$ *and* $q_i(\cdot)$, $i = 1,2$ *as in Theorem* 1. *If a sublinear operator* T_γ *satisfies the condition*

$$|T_\gamma f(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\gamma}} \mathrm{d}y, \quad x \notin \operatorname{supp} f, \tag{2.9}$$

for any integrable function f with compact support, and T_γ *is bounded from* $L^{q_1(\cdot)}(\omega)$ *to* $L^{q_2(\cdot)}(\omega)$*, then* T_γ *is bounded from* $M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$ *to* $M\dot{K}_{p,q_2(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$.

Then, we introduce multilinear commutators of $T_\gamma^{\boldsymbol{b}}$ *that are bounded from* $M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$ *to* $M\dot{K}_{p,q_2(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$*, which generalizes the previous work of Zhang et al.* [13, Theorem 5.1] *on homogeneous weighted grand Herz-Morrey spaces.*

Theorem 2. *Let* $\boldsymbol{b} = (b_1, b_2, \dots, b_m)$, $b_j \in \mathrm{BMO}(\mathbb{R}^n)$, $j \in \{1, 2, \dots, m\}$, $m \in \mathbb{N}$. *For* $1 < p < \infty$, $q_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ $(i = 1, 2\dots)$, $\omega \in A_{q(\cdot)}$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, *such that* $-n\delta_1 < \alpha(0), \alpha_\infty < n\delta_2 - \gamma$, *where* $0 < \delta_1, \delta_2 < 1$ *are the constants in Lemma* 3. *Suppose that sublinear operator* T_γ *is satisfying the size conditions* (2.7) *and* (2.8). *If* $T_\gamma^{\boldsymbol{b}}$ *is bounded from* $L^{q_1(\cdot)}(\omega)$ *to* $L^{q_2(\cdot)}(\omega)$*, then* $T_\gamma^{\boldsymbol{b}}$ *is bounded from* $M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$ *to* $M\dot{K}_{p,q_2(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$.

Corollary 2. *Let* p, $\alpha(\cdot)$ *and* $q(\cdot)$ *as in Theorem 2. If a fractional sublinear operator* T_γ *satisfies condition* (2.9)*, for any integrable function f with compact support, and* $T_\gamma^{\boldsymbol{b}}$ *is bounded from* $L^{q_1(\cdot)}(\omega)$ *to* $L^{q_2(\cdot)}(\omega)$*, then* $T_\gamma^{\boldsymbol{b}}$ *is bounded from* $M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$ *to* $M\dot{K}_{p,q_2(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$.

As applications, we have established the boundedness of $I_\lambda^{\boldsymbol{b}}$ from $M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$ to $M\dot{K}_{p,q_2(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$ and the boundedness of $M_\lambda^{\boldsymbol{b}}$ from $M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$ to $M\dot{K}_{p,q_2(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$. These applications are a natural extension [20].

Remark 1. For any $0 < \lambda < n$ and $x \in \mathbb{R}^n$, the Riesz potential operator I_λ is defined by

$$I_\lambda f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}} \mathrm{d}y. \tag{2.10}$$

It is known that the Riesz potential operator I_λ satisfies the size condition (2.9), and if $I_\lambda^{\boldsymbol{b}}$ is bounded from $L^{q_1(\cdot)}(\omega)$ to $L^{q_2(\cdot)}(\omega)$, then $I_\lambda^{\boldsymbol{b}}$ is bounded from $M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$ to $M\dot{K}_{p,q_2(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$.

Remark 2. For any $0 < \lambda < n$ and $x \in \mathbb{R}^n$, the fractional maximal function M_λ is defined by

$$M_\lambda f(x) := \sup_{r>0} r^{\lambda-n} \int_{\mathbb{R}^n} |f(y)| \mathrm{d}y. \tag{2.11}$$

Then, the fractional maximal function M_λ also satisfies (2.9). Moreover, we have

$$M_\lambda(f) \lesssim I_\lambda(|f|).$$

If $M_\lambda^{\boldsymbol{b}}$ is bounded from $L^{q_1(\cdot)}(\omega)$ to $L^{q_2(\cdot)}(\omega)$, then $M_\lambda^{\boldsymbol{b}}$ is bounded from $M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$ to $M\dot{K}_{p,q_2(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)$.

3 Proofs of main results

The section is devoted to the proofs of Theorems 1 and 2. To this end, we state several technical lemmas as follows.

Lemma 5. [11, Lemma 6] *Let* $\mathbb{X}$ *be a Banach function space on* $\mathbb{R}^n$. *If* $f \in \mathbb{X}$*, then we have*

$$|B| \|\chi_B\|_{\mathbb{X}}^{-1} \|\chi_B\|_{\mathbb{X}'}^{-1} \lesssim 1,$$

where $\mathbb{X}'$ *denotes the associated space of* $\mathbb{X}$.

Lemma 6. [13, Lemma 4.1] *Let $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, ω be a weight, $\lambda \in [0, \infty)$, $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $k \in \mathbb{Z}$, and $1 \le p < \infty$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q(\cdot)}(\omega)} \approx \max\left\{\sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}},\right.$$
$$\left.\times \sup_{\varepsilon>0}\sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)} + \varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\|f\chi_k\|^{p(1+\varepsilon)}_{L^{q(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\right\}.$$

Proof of Theorem 1. We show this theorem by borrowing some ideas from the proofs of [12, Theorem 3.2] and [13, Theorem 4.2]. Let $f \in M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)$, and we decompose

$$f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x).$$

From Lemma 6, we have

$$\begin{aligned}
&\|T_\gamma f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_2(\cdot)}(\omega)}\\
&\approx \max\left\{\sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\|T_\gamma(f)\chi_k\|^{p(1+\varepsilon)}_{L^{q_2(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}},\right.\\
&\quad\left.\times \sup_{\varepsilon>0}\sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)}\|T_\gamma(f)\chi_k\|^{p(1+\varepsilon)}_{L^{q_2(\cdot)}(\omega)} + \varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\|T_\gamma(f)\chi_k\|^{p(1+\varepsilon)}_{L^{q_2(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\right\}\\
&=: \max\{E, F + G\},
\end{aligned} \tag{3.1}$$

where

$$E = \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\left\|\chi_k \sum_{l=-\infty}^{\infty} T_\gamma(f\chi_l)\right\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$F = \sup_{\varepsilon>0}\sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)}\left(\left\|\chi_k \sum_{l=-\infty}^{\infty} T_\gamma(f\chi_l)\right\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$G = \sup_{\varepsilon>0}\sup_{k_0>0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\left\|\chi_k \sum_{l=-\infty}^{\infty} T_\gamma(f\chi_l)\right\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}.$$

The next is the same as the way in [13, Theorem 4.2] to be handled. Since to estimate F is essentially similar to estimate E, it suffices for us to show that E and G are bounded on homogeneous weighted grand Herz-Morrey space. It is easy to see that

$$E \lesssim \sum_{i=1}^{3} E_i, \quad G \lesssim \sum_{i=1}^{3} G_i,$$

where

$$E_1 = \sup_{\varepsilon>0}\sup_{k_0\le 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2} \|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$E_2 = \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$E_3 = \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k+2}^{\infty}\|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$G_1 = \sup_{\varepsilon>0}\sup_{k_0> 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$G_2 = \sup_{\varepsilon>0}\sup_{k_0> 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$G_3 = \sup_{\varepsilon>0}\sup_{k_0> 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=k+2}^{\infty}\|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}.$$

We first consider E_1. For a.e. $x \in R_k$ with $k \in \mathbb{Z}$ and $l \le k-2$, from size condition of T_γ, generalized Hölder's inequality and by repeating the proof of [13, Formula (4.3)], we obtain

$$|T_\gamma(f\chi_l)(x)| \lesssim 2^{-k(n-\gamma)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}. \tag{3.2}$$

By this, Lemmas 2, 3, [11, Formula (14)], and 5, we obtain

$$\begin{aligned}\|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)} &\lesssim 2^{-k(n-\gamma)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}\|\chi_k\|_{L^{q_2(\cdot)}(\omega)}\\ &\lesssim 2^{-k(n-\gamma)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}|B_k|\|\chi_k\|^{-1}_{L^{q'_2(\cdot)}(\omega^{-1})}\\ &\lesssim 2^{k\gamma}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}\|\chi_l\|^{-1}_{L^{q'_2(\cdot)}(\omega^{-1})}\frac{\|\chi_l\|_{L^{q'_2(\cdot)}(\omega^{-1})}}{\|\chi_k\|_{L^{q'_2(\cdot)}(\omega^{-1})}}\\ &\lesssim 2^{k\gamma}2^{(l-k)n\delta_2}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}\|\chi_l\|^{-1}_{L^{q'_2(\cdot)}(\omega^{-1})}\\ &\lesssim 2^{k\gamma}2^{(l-k)n\delta_2}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}2^{l(n-\gamma)}\|\chi_l\|^{-1}_{L^{q_2(\cdot)}(\omega)}\|\chi_l\|^{-1}_{L^{q'_2(\cdot)}(\omega^{-1})}\\ &\lesssim 2^{(l-k)(n\delta_2-\gamma)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}.\end{aligned} \tag{3.3}$$

Let $v \coloneqq n\delta_2-\gamma-\alpha(0) > 0$. From (3.3), Hölder's inequality, Fubini's theorem for series, $2^{-p(1+\varepsilon)} < 2^{-p}$, $\alpha_\infty < n\delta_2-\gamma$, and Lemma 6, we obtain that

$$\begin{aligned}E_1 &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}2^{(l-k)(n\delta_2-\gamma)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}2^{\alpha(0)l}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}2^{v(l-k)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}2^{vp(1+\varepsilon)(l-k)/2}\right)\left(\sum_{l=-\infty}^{k-2}2^{v(l-k)(p(1+\varepsilon))'/2}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right]^{\frac{1}{p(1+\varepsilon)}}\\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{k_0}\left[\sum_{l=-\infty}^{k-2}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}2^{v(l-k)p(1+\varepsilon)/2}\right]\right]^{\frac{1}{p(1+\varepsilon)}}\\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{l=-\infty}^{k_0}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\sum_{k=l+2}^{k_0}2^{v(l-k)p(1+\varepsilon)/2}\right)^{\frac{1}{p(1+\varepsilon)}}\end{aligned}$$

$$\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0} 2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\sum_{k=l+2}^{k_0}2^{\nu(l-k)p/2}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$\lesssim \|f\|_{M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}.$$

For the term E_2, T_γ is bounded from $L^{q_1(\cdot)}(\omega)$ to $L^{q_2(\cdot)}(\omega)$, $-1\le l-k\le 1$, and Lemma 6, we conclude that

$$E_2 \lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\|T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0+1} 2^{k\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0+1} 2^{(k-l)\alpha(0)p(1+\varepsilon)}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0+1} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 1,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}$$
$$\lesssim \max\Bigg\{\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$
$$\sup_{\varepsilon>0}\sup_{k_0=1,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{-1} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}+\varepsilon^{\theta}\sum_{l=0}^{k_0} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\Bigg\}$$
$$\lesssim \max\Bigg\{\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$
$$\sup_{\varepsilon>0}\sup_{k_0=1,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{-1} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}+\varepsilon^{\theta}\sum_{l=0}^{k_0} 2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\Bigg\}$$
$$\lesssim \max\Bigg\{\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}},$$
$$\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{-1} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}+\varepsilon^{\theta}\sum_{l=0}^{k_0} 2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\Bigg\}$$
$$\approx \|f\|_{M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)},$$

the penultimate inequality holds because when $l=0$, we have $2^{0\alpha(0)}=2^{0\alpha_\infty}$; and when $l=0$, we have $2^{\alpha_\infty}=C2^{\alpha(0)}$.

Now, we turn to estimate E_3. For each $k \in \mathbb{Z}$, $l \geq k+2$ and a.e. $x \in R_k$, the size condition of T_γ, and generalized Hölder's inequality, by repeating the proof of (3.2), we obtain

$$|T_\gamma(f\chi_l)(x)| \lesssim 2^{-l(n-\gamma)} \|f\chi_l\|_{L^{q_1(\cdot)}(\omega)} \|\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}. \tag{3.4}$$

Combining this, Lemma 3 and [11, Formula (14)], by an estimate similar to that of (3.3), we have

$$\|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)} \lesssim 2^{(k-l)n\delta_1} \|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}. \tag{3.5}$$

Splitting E_3 by means of Minkowski's inequality, we deduce

$$\begin{aligned} E_3 &\lesssim \sup_{\varepsilon>0} \sup_{k_0\leq 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\sum_{l=k+2}^{-1} \|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\quad + \sup_{\varepsilon>0} \sup_{k_0\leq 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)} \left(\sum_{l=0}^{\infty} \|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &=: E_{31} + E_{32}. \end{aligned}$$

For the convenience of proof, define $d := n\delta_1 + \alpha(0) > 0$. Combining (3.5), Hölder's inequality, $l > k+2$, Fubini's theorem for series, $2^{-p(1+\varepsilon)} < 2^{-p}$, $-n\delta_1 < \alpha(0)$, and Lemma 6, by an estimate similar to that of E_1, we obtain that

$$\begin{aligned} E_{31} &\lesssim \sup_{\varepsilon>0} \sup_{k_0\leq 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left[\varepsilon^\theta \sum_{k=-\infty}^{k_0} \left(\sum_{l=k+2}^{-1} 2^{k\alpha(0)} 2^{(k-l)n\delta_1} \|f\chi_l\|_{L^{q_1(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right]^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \sup_{k_0\leq 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left[\varepsilon^\theta \sum_{k=-\infty}^{k_0} \left(\sum_{l=k+2}^{-1} 2^{l\alpha(0)} \|f\chi_l\|_{L^{q_1(\cdot)}(\omega)} 2^{(n\delta_1+\alpha(0))(k-l)} \right)^{p(1+\varepsilon)} \right]^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \sup_{k_0\leq 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left[\varepsilon^\theta \sum_{k=-\infty}^{k_0} \left(\sum_{l=k+2}^{-1} 2^{l\alpha(0)} \|f\chi_l\|_{L^{q_1(\cdot)}(\omega)} 2^{d(k-l)} \right)^{p(1+\varepsilon)} \right]^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \sup_{k_0\leq 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{l=-\infty}^{k_0} 2^{\alpha(0)lp(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)} \sum_{k=-\infty}^{l-2} 2^{d(k-l)p/2} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \sup_{k_0\leq 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \sum_{l=-\infty}^{k_0} 2^{l\alpha(0)p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}. \end{aligned}$$

To deal with E_{32}, by (3.5), $k_0 \leq 0$, Hölder's inequality, $-n\delta_1 < \alpha(0)$, $\alpha_\infty < n\delta_2 - \gamma$, and note that $h := n\delta_1 + \alpha_\infty > 0$, and Lemma 6, we obtain

$$\begin{aligned} E_{32} &\lesssim \sup_{\varepsilon>0} \sup_{k_0\leq 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left[\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k(\alpha(0)+n\delta_1)p(1+\varepsilon)} \left(\sum_{l=0}^{\infty} 2^{-ln\delta_1} \|f\chi_l\|_{L^{q_1(\cdot)}(\omega)} \right)^{p(1+\varepsilon)} \right]^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \sup_{k_0\leq 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \left(\sum_{l=0}^{\infty} 2^{l\alpha_\infty} \|f\chi_l\|_{L^{q_1(\cdot)}(\omega)} 2^{-lh} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \sup_{k_0\leq 0, k_0\in\mathbb{Z}} 2^{-k_0\lambda} \left(\varepsilon^\theta \left(\sum_{l=0}^{\infty} 2^{l\alpha_\infty p(1+\varepsilon)} \|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)} \right) \left(\sum_{l=0}^{\infty} 2^{-lh(p(1+\varepsilon))'} \right)^{p(1+\varepsilon)/(p(1+\varepsilon))'} \right)^{\frac{1}{p(1+\varepsilon)}} \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}, \end{aligned}$$

which together with the estimate of E_{31} implies that

$$E_3 \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}.$$

Thus,

$$E \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}$$

holds true.

We now prove that $G \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}$.

To prove $G \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}$, we borrow some ideas from those used in the proof of [13, Theorem 4.2].

Splitting G_1 by means of Minkowski's inequality, we deduce

$$\begin{aligned} G_1 &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=-\infty}^{-1} \|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}} \\ &\quad + \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=0}^{k-2} \|\chi_k T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}} \\ &=: G_{11} + G_{12}. \end{aligned}$$

For the term G_{11}, by (3.3) and using the fact that $e := n\delta_2 - \gamma - \alpha_\infty > 0$, Hölder's inequality, $v = n\delta_2 - \alpha(0) - \gamma > 0$, and Lemma 6, we obtain that

$$\begin{aligned} G_{11} &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=-\infty}^{-1} \|f\chi_l\|_{L^{q_1(\cdot)}(\omega)} 2^{(l-k)(n\delta_2-\gamma)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=0}^{k_0} 2^{-kep(1+\varepsilon)}\left(\sum_{l=-\infty}^{-1} \|f\chi_l\|_{L^{q_1(\cdot)}(\omega)} 2^{l(n\delta_2-\gamma)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \left(\sum_{l=-\infty}^{-1} \|f\chi_l\|_{L^{q_1(\cdot)}(\omega)} 2^{l(n\delta_2-\gamma)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \left(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)} 2^{lv}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \left(\sum_{l=-\infty}^{-1} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)\left(\sum_{l=-\infty}^{-1} 2^{lv(p(1+\varepsilon))'}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right]^{\frac{1}{p(1+\varepsilon)}} \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}. \end{aligned}$$

For the term G_{12}, by (3.3), Hölder's inequality, $e := n\delta_2 - \gamma - \alpha_\infty > 0$, $l \le k-2$, Fubini's theorem for series, $2^{-p(1+\varepsilon)} < 2^{-p}$, and Lemma 6, by an estimate similar to that of E_1, we have

$$G_{12} \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}.$$

For G_2, from T_γ bounded from $L^{q_1(\cdot)}(\omega)$ to $L^{q_2(\cdot)}(\omega)$,$-1 \le l-k \le 1$, $2^{-\alpha_\infty} = C2^{-\alpha(0)}$, and Lemma 6, we conclude that

$$G_2 \lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1} \|T_\gamma(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}$$

$$
\begin{aligned}
&\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^{\theta}\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\|(f\chi_l)\|_{L^{q_1(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-1}^{k_0+1}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \sup_{\varepsilon>0}\sup_{k_0>1,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-1}^{k_0}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \sup_{\varepsilon>0}\sup_{k_0>1,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}2^{-1\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}+\varepsilon^{\theta}\sum_{l=0}^{k_0}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \sup_{\varepsilon>0}\sup_{k_0>1,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}2^{-1\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}+\varepsilon^{\theta}\sum_{l=0}^{k_0}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}2^{-1\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}+\varepsilon^{\theta}\sum_{l=0}^{k_0}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \max\left\{\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{k_0}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}},\right.\\
&\left.\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-\infty}^{-1}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}+\varepsilon^{\theta}\sum_{l=0}^{k_0}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\right\}\\
&\approx \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}.
\end{aligned}
$$

For G_3, combining (3.5), Hölder's inequality, $l \ge k+2$, Fubini's theorem for series, $2^{-p(1+\varepsilon)} < 2^{-p}$, $-n\delta_1 < \alpha(0)$, $\alpha_\infty < n\delta_2 - \gamma$, and note that $h := n\delta_1 + \alpha_\infty > 0$, $k > 0$, and Lemma 6, by an estimate similar to that of E_1, we obtain that

$$G_3 \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}.$$

Therefore, combining the estimates for E and G, we deduce that

$$\|T_\gamma f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_2(\cdot)}(\omega)} \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)},$$

which implies that T_γ is bounded from $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)$ to $M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_2(\cdot)}(\omega)$, which holds true and hence completes the proof of Theorem 1. □

Proof of Theorem 2. We show this theorem by borrowing some ideas from the proofs of Theorem 1 and [13, Theorem 5.1]. Let $f \in M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)$, and we decompose

$$f(x) = \sum_{l=-\infty}^{\infty} f(x)\chi_l(x).$$

From Lemma 6, we have

$$
\begin{aligned}
&\|T^{\boldsymbol{b}}_\gamma f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_2(\cdot)}(\omega)}\\
&\quad\approx \max\left\{\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\|T^{\boldsymbol{b}}_\gamma(f)\chi_k\|^{p(1+\varepsilon)}_{L^{q_2(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}},\right.
\end{aligned}
$$

$$\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)p(1+\varepsilon)}\|T_\gamma^{\boldsymbol{b}}(f)\chi_k\|_{L^{q_2(\cdot)}(\omega)}^{p(1+\varepsilon)}+\varepsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\|T_\gamma^{\boldsymbol{b}}(f)\chi_k\|_{L^{q_2(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}}\Bigg\}$$

$$=: \max\{A, N+S\},$$

where

$$A=\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\left(\left\|\chi_k\sum_{l=-\infty}^{\infty}T_\gamma^{\boldsymbol{b}}(f\chi_l)\right\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$N=\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{-1}2^{k\alpha(0)p(1+\varepsilon)}\left(\left\|\chi_k\sum_{l=-\infty}^{\infty}T_\gamma^{\boldsymbol{b}}(f\chi_l)\right\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$S=\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\left(\left\|\chi_k\sum_{l=-\infty}^{\infty}T_\gamma^{\boldsymbol{b}}(f\chi_l)\right\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}.$$

The next is the same as the way the (3.1) is handled. Since to estimate N is essentially similar to estimate A, it suffices for us to show that A and S are bounded on homogeneous grand weighted Herz-Morrey spaces. It is easy to see that

$$A\lesssim\sum_{i=1}^{3}A_i,\quad S\lesssim\sum_{i=1}^{3}S_i,$$

where

$$A_1=\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k T_\gamma^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$A_2=\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\|\chi_k T_\gamma^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$A_3=\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=-\infty}^{k_0}2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k+2}^{\infty}\|\chi_k T_\gamma^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$S_1=\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=-\infty}^{k-2}\|\chi_k T_\gamma^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$S_2=\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=k-1}^{k+1}\|\chi_k T_\gamma^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}},$$

$$S_3=\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta\sum_{k=0}^{k_0}2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=k+2}^{\infty}\|\chi_k T_\gamma^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}.$$

First, we consider A_1. For a.e. $x\in R_k$ with $k\in\mathbb{Z}$, and $l\le k-2$, from size condition of T, (2.6), and generalized Hölder's inequality, we have

$$|T_\gamma^{\boldsymbol{b}}(f\chi_l)(x)|\lesssim 2^{-k(n-\gamma)}\int_{R_l}\prod_{j=1}^{m}|b_j(x)-b_j(y)||f(y)|\mathrm{d}y$$

$$
\begin{aligned}
&= 2^{-k(n-\gamma)}\int_{R_l}\prod_{j=1}^{m}|b_j(x)-(b_j)_{B_l}+(b_j)_{B_l}-b_j(y)||f(y)|\mathrm{d}y\\
&\lesssim 2^{-k(n-\gamma)}\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}|[b(x)-(b)_{B_l}]_\sigma|\int_{R_l}|[b(y)-(b)_{B_l}]_{\sigma^c}||f(y)|\mathrm{d}y \qquad (3.6)\\
&\lesssim 2^{-k(n-\gamma)}\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}|[b(x)-(b)_{B_l}]_\sigma|\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|[b(y)-(b)_{B_l}]_{\sigma^c}\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}.
\end{aligned}
$$

By Lemmas 4, 2, 3, [11, Formula (14)], and 5, we obtain

$$
\begin{aligned}
&\|\chi_k T_\gamma^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\\
&\lesssim \|\boldsymbol{b}\|_* 2^{-k(n-\gamma)}(k-l)^m\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}\|\chi_k\|_{L^{q_2(\cdot)}(\omega)}\\
&\lesssim \|\boldsymbol{b}\|_* 2^{k\gamma}(k-l)^m\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}\|\chi_k\|^{-1}_{L^{q'_2(\cdot)}(\omega^{-1})}\\
&= \|\boldsymbol{b}\|_* 2^{k\gamma}(k-l)^m\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}\|\chi_l\|^{-1}_{L^{q'_2(\cdot)}(\omega^{-1})}\frac{\|\chi_l\|_{L^{q'_2(\cdot)}(\omega^{-1})}}{\|\chi_k\|_{L^{q'_2(\cdot)}(\omega^{-1})}} \qquad (3.7)\\
&\lesssim \|\boldsymbol{b}\|_*(k-l)^m 2^{k\gamma}2^{(l-k)n\delta_2}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}\|\chi_l\|^{-1}_{L^{q'_2(\cdot)}(\omega^{-1})}\\
&\lesssim \|\boldsymbol{b}\|_*(k-l)^m 2^{k\gamma}2^{(l-k)n\delta_2}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}2^{l(n-\gamma)}\|\chi_l\|^{-1}_{L^{q_2(\cdot)}(\omega)}\|\chi_l\|^{-1}_{L^{q'_2(\cdot)}(\omega^{-1})}\\
&\lesssim \|\boldsymbol{b}\|_*(k-l)^m 2^{(l-k)(n\delta_2-\gamma)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}.
\end{aligned}
$$

From (3.7), Hölder's inequality, $\nu := n\delta_2-\alpha(0)-\gamma>0$, Fubini's theorem for series, $2^{-p(1+\varepsilon)}<2^{-p}$, and repeating the proof of [13, Theorem 5.1] with some slight modifications, we obtain that

$$
\begin{aligned}
A_1 &\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg[\varepsilon^\theta\sum_{k=-\infty}^{k_0}\left(\sum_{l=-\infty}^{k-2}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}2^{\nu p(1+\varepsilon)(l-k)/2}\right)\\
&\quad\times\left(\sum_{l=-\infty}^{k-2}(k-l)^{m(p(1+\varepsilon))'}2^{\nu(l-k)(p(1+\varepsilon))'/2}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\Bigg]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{l=-\infty}^{k_0}2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\sum_{k=l+2}^{k_0}2^{\nu(l-k)p/2}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{l=-\infty}^{k_0}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}.
\end{aligned}
$$

For A_2, $T_\gamma^{\boldsymbol{b}}$ is bounded from $L^{q_1(\cdot)}(\omega)$ to $L^{q_2(\cdot)}(\omega)$, $2^{0\alpha(0)}=2^{0\alpha_\infty}$, $2^{\alpha_\infty}=C2^{\alpha(0)}$ and Lemma 6; similar to the estimation for E_2, we conclude that

$$
A_2 \lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 1,k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\varepsilon^\theta\sum_{l=-\infty}^{k_0}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}} \lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}.
$$

For A_3, by repeating the proof of [13, Formula (5.3)], we obtain

$$
|T_\gamma^{\boldsymbol{b}}(f\chi_l)(x)| \lesssim 2^{-l(n-\gamma)}\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}|[b(x)-(b)_{B_l}]_\sigma|\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\|[b(y)-(b)_{B_l}]_{\sigma^c}\chi_l\|_{L^{q'_1(\cdot)}(\omega^{-1})}. \qquad (3.8)
$$

This together with Lemmas 4, 3 and [11, Formula (14)] further implies that

$$
\|\chi_k T_\gamma^{\boldsymbol{b}}f\chi_l\|_{L^{q_2(\cdot)}(\omega)} \lesssim \|\boldsymbol{b}\|_*(l-k)^m 2^{(k-l)n\delta_1}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}. \qquad (3.9)
$$

The next is the same as the way the E_3 is handled. Splitting A_3 by means of Minkowski's inequality, we have

$$
\begin{aligned}
A_3 &\lesssim \sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=k+2}^{-1}\|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\quad+\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p(1+\varepsilon)}\left(\sum_{l=0}^{\infty}\|\chi_k T^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&=: A_{31}+A_{32}.
\end{aligned}
$$

For the term A_{31}, by (3.9), Hölder's inequality, $d = n\delta_1 + \alpha(0) > 0$, Fubini's theorem for series, $2^{-p(1+\varepsilon)} < 2^{-p}$, Lemma 6, and repeating the proof of [13, Theorem 5.1] with some slight modifications, we obtain that

$$
\begin{aligned}
A_{31} &\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{-1} 2^{\alpha(0)l}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}(l-k)^m 2^{d(k-l)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=-\infty}^{k_0}\left(\sum_{l=k+2}^{-1} 2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)} 2^{dp(1+\varepsilon)(k-l)/2}\right)\right.\\
&\quad\times\left.\left(\sum_{l=k+1}^{-1}(l-k)^{m(p(1+\varepsilon))'}2^{d(k-l)(p(1+\varepsilon))'/2}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{l=-\infty}^{k_0} 2^{\alpha(0)lp(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\sum_{k=-\infty}^{l-2}2^{d(k-l)p/2}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^\theta \sum_{l=-\infty}^{k_0} 2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}.
\end{aligned}
$$

For the term A_{32}, applying (3.9), Hölder's inequality, $h = n\delta_1 + \alpha_\infty > 0$, and Lemma 6, we deduce that

$$
\begin{aligned}
A_{32} &\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \sum_{k=-\infty}^{k_0} 2^{k(\alpha(0)+n\delta_1)p(1+\varepsilon)}\left(\sum_{l=0}^{\infty}(l-k)^m 2^{-ln\delta_1}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \left(\sum_{l=0}^{\infty}2^{l\alpha_\infty}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}(l-k)^m 2^{-lh}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\sup_{\varepsilon>0}\sup_{k_0\le 0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^\theta \left(\sum_{l=0}^{\infty}2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|^{p(1+\varepsilon)}_{L^{q_1(\cdot)}(\omega)}\right)\right.\\
&\quad\times\left.\left(\sum_{l=0}^{\infty}(l-k)^{m(p(1+\varepsilon))'}2^{-lh(p(1+\varepsilon))'}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}.
\end{aligned}
$$

Therefore, combining the estimates for A_{31} and A_{32} deduces that

$$
A_3 \lesssim \|\boldsymbol{b}\|_*\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda,\theta}_{p,q_1(\cdot)}(\omega)}.
$$

Thus,

$$A \lesssim \|\boldsymbol{b}\|_{*}\|f\|_{M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}$$

holds true.

Next, we consider S_1. The next is the same as the way the E_3 is handled. Splitting S_1 by means of Minkowski's inequality, we deduce

$$\begin{aligned}S_1 &\lesssim \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^{\theta}\sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=-\infty}^{-1}\|\chi_k T_\gamma^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\quad + \sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^{\theta}\sum_{k=0}^{k_0} 2^{k\alpha_\infty p(1+\varepsilon)}\left(\sum_{l=0}^{k-2}\|\chi_k T_\gamma^{\boldsymbol{b}}(f\chi_l)\|_{L^{q_2(\cdot)}(\omega)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&=: S_{11} + S_{12}.\end{aligned}$$

To show S_{11} using (3.7) and $e = n\delta_2 - \alpha_\infty - \gamma > 0$, Hölder's inequality, and using the fact that $v = n\delta_2 - \alpha(0) - \gamma > 0$, Lemma 6 and by repeating the proof of [13, Theorem 5.1] with regular modifications, we obtain that

$$\begin{aligned}S_{11} &\lesssim \|\boldsymbol{b}\|_{*}\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^{\theta}\sum_{k=0}^{k_0} 2^{k(\alpha_\infty-n\delta_2-\gamma)p(1+\varepsilon)} \times \left(\sum_{l=-\infty}^{-1}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}(k-l)^m 2^{l(n\delta_2-\gamma)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_{*}\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^{\theta}\sum_{k=0}^{k_0} 2^{-kep(1+\varepsilon)} \times \left(\sum_{l=-\infty}^{-1}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}(k-l)^m 2^{l(n\delta_2-\gamma)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_{*}\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}(k-l)^m 2^{l(n\delta_2-\gamma)}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_{*}\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1}2^{l\alpha(0)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}(k-l)^m 2^{lv}\right)^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_{*}\sup_{\varepsilon>0}\sup_{k_0>0,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left[\varepsilon^{\theta}\left(\sum_{l=-\infty}^{-1}2^{l\alpha(0)p(1+\varepsilon)}\|f\chi_l\|_{L^{q_1(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)\left(\sum_{l=-\infty}^{-1}(k-l)^{m(p(1+\varepsilon))'}2^{lv(p(1+\varepsilon))'}\right)^{p(1+\varepsilon)/(p(1+\varepsilon))'}\right]^{\frac{1}{p(1+\varepsilon)}}\\
&\lesssim \|\boldsymbol{b}\|_{*}\|f\|_{M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}.\end{aligned}$$

For the term S_{12}, by Hölder's inequality, $e = n\delta_2 - \alpha_\infty - \gamma > 0$, $k > 0$, Fubini's theorem for series, $2^{-p(1+\varepsilon)} < 2^{-p}$, and Lemma 6, by an estimate similar to that of A_1, we have

$$S_{12} \lesssim \|\boldsymbol{b}\|_{*}\|f\|_{M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}.$$

To deal with S_2, in the view of the $L^{q_1(\cdot)}(\omega)$ to $L^{q_2(\cdot)}(\omega)$ boundedness of $T_\gamma^{\boldsymbol{b}}$, $-1 \le l - k \le 1$, $2^{-\alpha_\infty} = C2^{-\alpha(0)}$, and Lemma 6, similar to the estimation for G_2, we obtain

$$S_2 \lesssim \|\boldsymbol{b}\|_{*}\sup_{\varepsilon>0}\sup_{k_0>1,k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\varepsilon^{\theta}\sum_{l=-1}^{k_0} 2^{l\alpha_\infty p(1+\varepsilon)}\|f\chi_l\|_{L^{q(\cdot)}(\omega)}^{p(1+\varepsilon)}\right)^{\frac{1}{p(1+\varepsilon)}} \lesssim \|\boldsymbol{b}\|_{*}\|f\|_{M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}.$$

The estimate for S_3 can be obtained by similar way to A_{31} and using the fact that $h = n\delta_1 + \alpha_\infty > 0$, $k > 0$. From the estimates of A and S, we see that

$$\|T_\gamma^{\boldsymbol{b}}f\|_{M\dot{K}_{p,q_2(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)} \lesssim \|\boldsymbol{b}\|_{*}\|f\|_{M\dot{K}_{p,q_1(\cdot)}^{\alpha(\cdot),\lambda,\theta}(\omega)}$$

and hence completes the proof of Theorem 2. □

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